Çankaya University Journal of Science and Engineering
Volume 16, No. 1 (2019) 063-080
Date Received: Feb 25, 2019
Date Accepted: May 15, 2019

The Bilinear Hardy-Littlewood Maximal Function and Littlewood-Paley Square Function on Weighted Variable Exponent Wiener Amalgam Space

Öznur Kulak[1]

[1]*Department of Banking and Finance, Görele Applied Sciences Academy, Giresun University,Turkey*
e-mail: oznur.kulak@giresun.edu.tr

Abstract: The space "weighted variable exponent Wiener amalgam" whose local component is "variable exponent Lorentz space" is considered. Then boundedness of the "bilinear Hardy-Littlewood maximal function" and "Littlewood-Paley square function" is discussed on this space.

Keywords: The bilinear Hardy-Littlewood maximal function1, the bilinear Littlewood-Paley Square function 2.

---

1. Introduction

Wiener amalgam spaces are defined by N. Wiener firstly [25]. Many researchers worked on these spaces like [2],[6],[8],[10],[13],[14],[15]. Some of characterizations of these spaces has been given in [11]. In [12], a kind of generalization of $W(L^{p(x)}, L_w^q)$ has been given. Many researchers considered boundedness of bilinear "Hardy-Littlewood maximal function" and "Littlewood-Paley square function", [1], [3], [18], [19], [20], [21], [22], [23].

The main purpose of this paper, by using similar technics in [2], [11] and [12], to define a new amalgam space $W\left(L^{p(.),q(.)}, L_\omega^r\right)$. Later to discuss boundedness of these operators on $W\left(L^{p(.),q(.)}, L_\omega^r\right)$.

2. Preliminaries

The space $C_c^\infty(\mathbb{R})$consists of infinitely differentiable complex-valued functions with compact supported on $(\mathbb{R})$. $L^p(\mathbb{R})$, $(1 \le p \le \infty)$ denotes "usual Lebesgue space". Weight function ω on $\mathbb{R}$ is local integrable and non-negative continuous. If $\omega_1(x) \le C\omega_2(x)$, $(x \in \mathbb{R})$ and for some $C > 0$, we say that $\omega_1 \prec \omega_2$. For $1 \le p \le \infty$, we set $L_w^p(\mathbb{R}) = \{f: fw \in L_w^p(\mathbb{R})\}$, [8], [9].

The function λ_f is called distribution function and given by

$$\lambda_f(y) = \mu(\{x \in \mathbb{R}: |f(x)| > y\}) = \int_{\{x\in\mathbb{R}:|f(x)|>y\}} d\mu(x), [16].$$

The function f^* is called rearrangement function and is given by

$$f^*(t) = inf\{y > 0: \lambda_f(y) > t\}, t \ge 0, [16].$$

Also, the function f^{**} is said the average function and defined by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds, [16].$$

Following notations are given by

$$p_- = \inf_{x\in[0,l]} p(x),\ p^+ = \sup_{x\in[0,l]} p(x),\ 0 < l \le \infty$$

Let $P_a = \{p: a < p_- \le p^+ < \infty\}, a \in \mathbb{R}$.

In this work, special case a=1 will be used. $\wp([0,l])$ consists of $p \in L^\infty([0,l])$ where $p(\infty) = \lim_{x\to\infty} p(x)$ and $p(0) = \lim_{x\to 0} p(x)$ exist, we have

$$|p(x)-p(0)| \le \frac{C}{\ln\frac{1}{|x|}},\ |x| \le (1/2)\ \ (C>0),$$

$$|p(x)-p(\infty)| \le \frac{C}{\ln(e+|x|)},\ \ (C>0), \tag{1}$$

If $l=\infty$, it's enough to the inequality (1) satisfies and $p(\infty)=\lim_{x\to\infty} p(x)$ exists. Also the set $\wp_a$ is given by $\wp_a([0,l]) = P_a([0,l]) \cap \wp([0,l])$, [5]. Set $l=\mu(\Omega)$ such that $\Omega\subset\mathbb{R}$ and μ is Lebesgue measure on $\mathbb{R}$. Let $1 \le p_- \le p^+ < \infty$. The space "variable exponent Lebesgue space" $L^{p(.)}(\Omega)$ is space of measurable functions f on Ω such that $J_p(f) = \int_\Omega |f(x)|^{p(x)} dx < \infty$. The space $L^{p(.)}(\Omega)$ is a Banach with this norm $\|f\|_{L^{p(.)}} = inf\left\{\lambda > 0 : J_p\left(\frac{f}{\lambda}\right) \le 1\right\}$, [5]. Assume that $p, q \in \wp_0([0,l])$. The "variable exponent Lorentz space" $L^{p(.),q(.)}(\Omega)$ consists of all measurable functions f on Ω where

$$\rho_{p,q}(f) = \int_0^l t^{\frac{q(t)}{p(t)}-1} \left(f^*(t)\right)^{q(t)} dt < \infty, [5] \tag{2}$$

The space $L^{p(.),q(.)}(\Omega)$ is normed space with this norm

$$\|f\|_{L^{p(.),q(.)}(\Omega)} = \inf\left\{\lambda > 0 : \rho_{p,q}\left(\frac{f}{\lambda}\right) \le 1\right\} = \left\| t^{\frac{1}{p(.)}-\frac{1}{q(.)}} f^{**} \right\|_{L^{q(.)}([0,l])}$$

where $\rho_{p,q}(f) = \int_0^l t^{\frac{q(t)}{p(t)}-1} \left(f^{**}(t)\right)^{q(t)} dt$, [5].

3. The Weighted Variable Exponent Amalgam Space $W\left(L^{p(.),q(.)},L_{\omega}^{r}\right)$

3.1 Definition Let r,p(.),q(.)$\in \wp_1\left(\left[0,\infty\right]\right)$. $\left(L^{p(.),q(.)}(\mathbb{R})\right)_{loc}$ denotes the space of all measurable functions f on $\mathbb{R}$ where $f\chi_K \epsilon L^{p(.),q(.)}(\mathbb{R})$ for every compact $K \subset \mathbb{R}$.

3.2 Definition Let r,p(.),q(.)$\in \wp_1\left(\left[0,\infty\right]\right)$.$\in \wp_1\left(\left[0,\infty\right]\right)$. Fix a compact Q$\subset$ $\mathbb{R}$ and $Q^o \neq \emptyset$. Assume that ω is weight function. “Weighted variable exponent amalgam space” $W\left(L^{p(.),q(.)},L_{\omega}^{r}\right)$ is space of all $f\epsilon \left(L^{p(.),q(.)}(\mathbb{R})\right)_{loc}$ where $F_f\left(z\right)=\left\|f\chi_{z+Q}\right\|$ is in $L_{\omega}^{r}(\mathbb{R})$. The norm of this space is given by

$$\|f\|_{W\left(L^{p(.),q(.)},L_{\omega}^{r}\right)}=\left\|F_f\right\|_{r,\omega}=\left\|\left\|f\chi_{z+Q}\right\|_{L^{p(.),q(.)}}\right\|_{r,\omega}.$$

3.1 Theorem Let r,p(.),q(.)$\in \wp_1\left(\left[0,\infty\right]\right)$.

a) $W\left(L^{p(.),q(.)},L_{\omega}^{r}\right)$ is a Banach space with norm $\|.\|_{W\left(L^{p(.),q(.)},L_{\omega}^{r}\right)}$,

b) $W\left(L^{p(.),q(.)},L_{\omega}^{r}\right)$ is continuously embedded into $\left(L^{p(.),q(.)}(\mathbb{R})\right)_{loc}$,

c) The space

$$\wedge_0=\left\{f\in L^{p(.),q(.)}:supp\left(f\right) is\ compact\right\}$$

is continuously embedded into $W\left(L^{p(.),q(.)},L_{\omega}^{r}\right)$,

d) $W\left(L^{p(.),q(.)},L_{\omega}^{r}\right)$ does not depend on the particular choice of Q, [6].

Lemma 3.1 Let r,p(.),q(.)$\in \wp_1\left(\left[0,\infty\right]\right)$ and ω be weight function.

a) The space $\left(L^{p(.),q(.)}(\mathbb{R})\right)_{loc}$ is continuously embedded into $\left(L^{1}(\mathbb{R})\right)_{loc}$.

b) C_c^∞ ($\mathbb{R}$) is dense in $W\left(L^{p(.),q(.)},L_\omega^r\right)$.

Proof a) Take any $f\epsilon\left(L^{p(.),q(.)}(\mathbb{R})\right)_{loc}$. Since $L^{p(.),q(.)}(\mathbb{R})$ is Banach function space, [5] for given any compact $K\subset\mathbb{R}$, we have $\chi_K\epsilon L^{p'(.),q'(.)}(\mathbb{R})$. If we use "Hölder inequality", we write

$$\|f\|_{L_{loc}^1}=\|f\chi_K\|_1=\|f\chi_K\chi_K\|_1\leq C\|f\chi_K\|_{L^{p(.),q(.)}}\|\chi_K\|_{L^{p'(.),q'(.)}}=C\|\chi_K\|_{L^{p'(.),q'(.)}}\|f\|_{\left(L^{p(.),q(.)}\right)_{loc}}\text{, C>0.}$$

Set $C_0=C\|\chi_K\|_{L^{p'(.),q'(.)}}$. we find that

$$\|f\|_{L_{loc}^1}\leq C_0\|f\|_{\left(L^{p(.),q(.)}\right)_{loc}}.$$

b) Since $\overline{C_c}=L_\omega^r$, we have $\overline{C_c}=W\left(L^{p(.),q(.)},L_\omega^r\right)$, [4]. So we have this inclusion

$$C_c^\infty\subset C_c\subset W\left(L^{p(.),q(.)},L_\omega^r\right).$$

Now take any $f\in W\left(L^{p(.),q(.)},L_\omega^r\right)$ Then using density, we find that $g\in C_c$ where

$$\|f-g\|_{W\left(L^{p(.),q(.)},L_\omega^r\right)}<\frac{\varepsilon}{2}. \tag{3}$$

Also there exists $h\in C_c^\infty$ such that

$$\|g-h\|_\infty<\frac{\varepsilon}{2\|\chi_A\|_{W\left(L^{p(.),q(.)},L_\omega^r\right)}} \tag{4}$$

where supp(g-h)=A , by from approximation theorem, [24]. Also we we can write $g-h=k\chi_A$ such that $k\epsilon\mathbb{C}$. So we have that

$$\|g-h\|_{W\left(L^{p(.),q(.)},L_\omega^r\right)}=\left\|\|(g-h)\chi_{z+Q}\|_{L^{p(.),q(.)}}\right\|_{r,\omega}=\left\|\|k\chi_A\chi_{z+Q}\|_{L^{p(.),q(.)}}\right\|_{r,\omega}=\left\|\|k\chi_A\chi_A\chi_{z+Q}\|_{L^{p(.),q(.)}}\right\|_{r,\omega}$$

$$=\left\|\|(g-h)\chi_A\chi_{z+Q}\|_{L^{p(.),q(.)}}\right\|_{r,\omega}\leq\|g-h\|_\infty\left\|\|\chi_A\chi_{z+Q}\|_{L^{p(.),q(.)}}\right\|_{r,\omega}=\|g-h\|_\infty\|\chi_A\|_{W\left(L^{p(.),q(.)},L_\omega^r\right)}<\infty.$$

So using inequalitiy (4), we write

$$\|g-h\|_{W\left(L^{p(.),q(.)},L^r_\omega\right)} \le \|g-h\|_\infty \|\chi_A\|_{W\left(L^{p(.),q(.)},L^r_\omega\right)} < \frac{\varepsilon}{2\|\chi_A\|_{W\left(L^{p(.),q(.)},L^r_\omega\right)}} \|\chi_A\|_{W\left(L^{p(.),q(.)},L^r_\omega\right)} = \frac{\varepsilon}{2} \quad (5)$$

Then by from (3) and (5), we find that

$$\|f-h\|_{W\left(L^{p(.),q(.)},L^r_\omega\right)} \le \|f-g\|_{W\left(L^{p(.),q(.)},L^r_\omega\right)} + \|g-h\|_{W\left(L^{p(.),q(.)},L^r_\omega\right)} < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon .$$

3.1 Corollary Let r, p(.),q(.)$\in \wp_1([0,\infty])$. The space $W\left(L^{p(.),q(.)},L^r_\omega\right)$ is continuously embedded into $\left(L^1(\mathbb{R})\right)_{loc}$.

3.2 Lemma Let r, $p_i(.),q_i(.)\in\wp_1([0,\infty])$, (i=1,2). Suppose that there exist the inequalities

$$\|hk\|_{L^{p(.),q(.)}} \le C_1 \|h\|_{L^{p1(.),q1(.)}} \|k\|_{L^{p2(.),q2(.)}},\ (C_1>0)$$

for all $h\in L^{p_1(.),q_1(.)}$, $k\in L^{p_2(.),q_2(.)}$ and

$$\|hk\|_{r_3\omega_3} \le C_2 \|h\|_{r_1\omega_1} \|k\|_{r_2\omega_2},\ (C_2>0)$$

for all $h\in L^{r_1}_{\omega_1}$, $k\in L^{r_2}_{\omega_2}$. So the inequality

$$\|hk\|_{W\left(L^{p3(.),q3(.)},L^{r3}_{\omega_3}\right)} \le C\|h\|_{W\left(L^{p1(.),q1(.)},L^{r1}_{\omega_1}\right)} \|k\|_{W\left(L^{p2(.),q2(.)},L^{r2}_{\omega_2}\right)},\ (C>0)$$

holds for all $h\in W\left(L^{p_1(.),q_1(.)},L^{r_1}_{\omega_1}\right)$, $k\in W\left(L^{p_2(.),q_2(.)},L^{r_2}_{\omega_2}\right)$, [6].

3.2 Theorem Let r, $p_i(.),q_i(.)\in\wp_1([0,\infty])$, (i=1,2). If $q_1(0)\le p_1(0)$, $q_2(.)\le q_1(.)$ and $q_2(0)\ge p_2(0)$, then $W\left(L^{p_1(.),q_1(.)},L^r_\omega\right)\subset W\left(L^{p_2(.),q_2(.)},L^r_\omega\right)$ holds.

Proof Since all hypothesis of Theorem 4, in [17] are satisfied, then $L^{p_1(.),q_1(.)}(z+Q)\subset L^{p_2(.),q_2(.)}(z+Q)$. Hence by "closed graph mapping theorem", we have that $\|f\|_{L^{p2(.),q2(.)}(z+Q)} \le C\|f\|_{L^{p1(.),q1(.)}(z+Q)}$, (C>0), $f\in L^{p_1(.),q_1(.)}(z+Q)$. Let $f\in W\left(L^{p_1(.),q_1(.)},L^r_\omega\right)$. Then

$$\left\| f\chi_{z+Q} \right\|_{L^{p_2(.),q_2(.)}} \leq C\left\| f\chi_{z+Q} \right\|_{L^{p_1(.),q_1(.)}} \tag{5}$$

Since $f \in W\left(L^{p_1(.),q_1(.)}, L^r_\omega\right)$, then $\left\| f\chi_{z+Q} \right\|_{L^{p_1(.),q_1(.)}(\mathbb{R}^n)} \in L^r_\omega$. Then by (5), we have $\left\| f\chi_{z+Q} \right\|_{L^{p_2(.),q_2(.)}} \in L^r_\omega$. Hence by the solidness

$$\|f\|_{W\left(L^{p_2(.),q_2(.)}, L^r_\omega\right)} = \left\| \left\| f\chi_{z+Q} \right\|_{L^{p_2(.),q_2(.)}} \right\|_{r,\omega} \leq C \left\| \left\| f\chi_{z+Q} \right\|_{L^{p_1(.),q_1(.)}} \right\|_{r,\omega} = C\|f\|_{W\left(L^{p_1(.),q_1(.)}, L^r_\omega\right)}.$$

3.3 Theorem Let 1≤r<∞. Then the equality $W\left(L^r, L^r_\omega\right) = L^r_\omega$ holds, [13].

3.4 Theorem Let p(.),q(.),r∈ $\wp_1\left(\left[0,\infty\right]\right)$. Then the inclusion $W\left(L^{p(.),q(.)}, L^r_\omega\right).W\left(L^{p'(.),q'(.)}, L^{r'}_{\omega^{-1}}\right) \subset L^1$ and $\|fg\|_1 \leq C\|f\|_{W\left(L^{p(.),q(.)}, L^r_\omega\right)} \|g\|_{W\left(L^{p'(.),q'(.)}, L^{r'}_{\omega^{-1}}\right)}$, (C>0) holds for all $f \in W\left(L^{p(.),q(.)}, L^r_\omega\right)$, $g \in W\left(L^{p'(.),q'(.)}, L^{r'}_{\omega^{-1}}\right)$, ($p'$(.), q' (.) and r' are conjugated of p(.),q(.),r respectively).

Proof It is writen $\|fg\|_1 \leq C\|f\|_{L^{p(.),q(.)}} \|g\|_{L^{p'(.),q'(.)}}$ by [17]. So we have

$$\|fg\|_1 = \|fg\|_{W\left(L^1, L^1\right)} = \left\| \left\| fg\chi_{z+Q} \right\|_1 \right\|_1 = \left\| \left\| \left(f\chi_{z+Q}\right)\left(g\chi_{z+Q}\right) \right\|_1 \right\|_1 \leq C \left\| \left\| f\chi_{z+Q} \right\|_{L^{p(.),q(.)}} \left\| g\chi_{z+Q} \right\|_{L^{p'(.),q'(.)}} \right\|_1$$

$$= C \left\| \left\| f\chi_{z+Q} \right\|_{L^{p(.),q(.)}} \left\| g\chi_{z+Q} \right\|_{L^{p'(.),q'(.)}} \omega\omega^{-1} \right\|_1 \leq C \left\| \left\| f\chi_{z+Q} \right\|_{L^{p(.),q(.)}} \omega \right\|_r \left\| \left\| g\chi_{z+Q} \right\|_{L^{p'(.),q'(.)}} \omega^{-1} \right\|_{r'}$$

$$= C\|f\|_{W\left(L^{p(.),q(.)}, L^r_\omega\right)} \|g\|_{W\left(L^{p'(.),q'(.)}, L^{r'}_{\omega^{-1}}\right)}.$$

Theorem 3.5 a) Let r, $p_i\left(.\right), q_i\left(.\right) \in \wp_1\left(\left[0,\infty\right]\right)$, (i=1,2). If $p_1(0) \geq q_1(0)$, $q_2\left(.\right) \leq q_1\left(.\right)$ and $q_2\left(0\right) \geq p_2\left(0\right)$, $r_1 \leq r_2$, $\omega_2 \prec \omega_1$, then the inclusion

$W\left(L^{p_1(.),q_1(.)}, L^{r_1}_{\omega_1}\right) \subset W\left(L^{p_2(.),q_2(.)}, L^{r_2}_{\omega_2}\right)$ holds.

b) Let p(.)∈ $\wp_1\left(\left[0,\infty\right]\right)$. If $\omega_2 \prec \omega_1$, then the inclusions

$L^{p^+}_{\omega_1} \subset W\left(L^{p(.),p(.)}, L^{p^+}_{\omega_2}\right)$ and $W\left(L^{p(.),p(.)}, L^{p_-}_{\omega_1}\right) \subset L^{p_-}_{\omega_2}$ hold.

Proof a) Take any $f \in W\left(L^{p_1(.),q_1(.)}, L^{r_1}_{\omega_1}\right)$. From the assumptions, we write the inclusion $L^{p_1(.),q_1(.)}(z+Q) \subset L^{p_2(.),q_2(.)}(z+Q)$, [17]. So there exists the inequality

$$\left\| f\chi_{z+Q} \right\|_{L^{p_2(.),q_2(.)}} \leq C \left\| f\chi_{z+Q} \right\|_{L^{p_1(.),q_1(.)}}, \text{ (C>0)} \tag{6}$$

for z∈ ℝ. By the solidness of $L^{r_1}_{\omega_1}$ and by (6), we have

$$\left\| \left\| f\chi_{z+Q} \right\|_{L^{p_2(.),q_2(.)}} \right\|_{r_1,\omega_1} \leq C \left\| \left\| f\chi_{z+Q} \right\|_{L^{p_1(.),q_1(.)}} \right\|_{r_1,\omega_1}$$

Then

$$W\left(L^{p_1(.),q_1(.)}, L^{r_1}_{\omega_1}\right) \subset W\left(L^{p_2(.),q_2(.)}, L^{r_1}_{\omega_1}\right) \tag{7}$$

Also it's writen by [7] that

$$W\left(L^{p_2(.),q_2(.)}, L^{r_1}_{\omega_1}\right) \subset W\left(L^{p_2(.),q_2(.)}, L^{r_2}_{\omega_2}\right) \tag{8}$$

if and only if $l^{r_1}_{\omega_1} \subset l^{r_2}_{\omega_2}$ where $l^{r_1}_{\omega_1}$ and $l^{r_2}_{\omega_2}$ are the associated sequence spaces of $L^{r_1}_{\omega_1}$ and $L^{r_2}_{\omega_2}$ respectively. Since $r_1 \leq r_2$ and $\omega_2 \prec \omega_1$, then $l^{r_1}_{\omega_1} \subset l^{r_2}_{\omega_2}$. Therefore from by (7) and (8), we obtain that $W\left(L^{p_1(.),q_1(.)}, L^{r_1}_{\omega_1}\right) \subset W\left(L^{p_2(.),q_2(.)}, L^{r_2}_{\omega_2}\right)$.

b) It's known by Theorem 3.3, that

$$L^{p^+}_{\omega_1} = W\left(L^{p^+,p^+}, L^{p^+}_{\omega_1}\right) \tag{9}$$

Since $p(.) \leq p^+$ by using (a), we write

$$W\left(L^{p^+,p^+}, L^{p^+}_{\omega_1}\right) \subset W\left(L^{p(.),p(.)}, L^{p^+}_{\omega_2}\right). \tag{10}$$

So by using (9) and (10), we find $L^{p^+}_{\omega_1} \subset W\left(L^{p(.),p(.)}, L^{p^+}_{\omega_2}\right)$. Other inclusion is easly proved with similar technique.

3.6 Theorem Let r_1, r_2, $p_1(.)$, $p_2(.)$, $q_1(.), q_2(.) \in \wp_1([0,\infty])$. Assume that $q_i(0) < p_i(0)$, $r_1 \le r_2$, $q_i(\infty) > p_i(\infty)$, $\omega_2 \prec \omega_1$, (i=1,2). If $L^{p_1(.),q_1(.)} \subset L^{p_2(.),q_2(.)}$ then the inclusion $W\left(L^{p_1(.),q_1(.)}, L^{r_1}_{\omega_1}\right) \subset W\left(L^{p_2(.),q_2(.)}, L^{r_2}_{\omega_2}\right)$ holds.

Proof Since $L^{p_1(.),q_1(.)} \subset L^{p_2(.),q_2(.)}$ there exists C>0 such that $\|.\|_{L^{p_2(.),q_2(.)}} \le C\|.\|_{L^{p_1(.),q_1(.)}}$

by [17]. Take any $f \in W\left(L^{p_1(.),q_1(.)}, L^{r_1}_{\omega_1}\right)$. Let $Q \subset \mathbb{R}$ be a fixed compact subset. We write

$\|f\chi_{z+Q}\|_{L^{p_2(.),q_2(.)}} \le C\|f\chi_{z+Q}\|_{L^{p_1(.),q_1(.)}}$ for all $z \in \mathbb{R}$ Since $L^{r_1}_{\omega_1}$ is solid space, then

$\left\|\|f\chi_{z+Q}\|_{L^{p_2(.),q_2(.)}}\right\|_{r_1,\omega_1} \le C\left\|\|f\chi_{z+Q}\|_{L^{p_1(.),q_1(.)}}\right\|_{r_1,\omega_1}$. That means,

$$W\left(L^{p_1(.),q_1(.)}, L^{r_1}_{\omega_1}\right) \subset W\left(L^{p_2(.),q_2(.)}, L^{r_1}_{\omega_1}\right) \tag{11}$$

On the other hand since $r_1 \le r_2$ and $\omega_2 \prec \omega_1$, by [7], we have

$$W\left(L^{p_2(.),q_2(.)}, L^{r_1}_{\omega_1}\right) \subset W\left(L^{p_2(.),q_2(.)}, L^{r_2}_{\omega_2}\right) \tag{12}$$

Combining (11) and (12), we obtain $W\left(L^{p_1(.),q_1(.)}, L^{r_1}_{\omega_1}\right) \subset W\left(L^{p_2(.),q_2(.)}, L^{r_2}_{\omega_2}\right)$.

4. Boundedness of The Bilinear Hardy-littlewood Maximal Function on $W\left(L^{p(.),q(.)}, L^r_\omega\right)$

4.1 Definition The bilinear "Hardy-Littlewood maximal function" M is defined by

$$M(f,g)(x) = \sup_{t>0} \frac{1}{2t} \int_{-t}^{t} \left| f(x+y)\, g(x-y) \right| dy,\ x \in \mathbb{R},\ [1,20]$$

for all $f, g \in \left(L^1(\mathbb{R})\right)_{loc}$.

4.1 Theorem Let $1 \le p_i(.), q_i(.) < \infty$, $1 \le r_i < \infty$ and $p_i(.), q_i(.) \in \wp_1([0,\infty])$ and ω_i be weight function, (i=1,2,3). Assume that $q_3(0) \le p_3(0)$ and $\frac{1}{r_3} + \frac{1}{r'_3} = 1$. If $\frac{1}{\omega_3} \in L^{r'_3}$, then the Hardy-Littlewood maximal function

$$M : W\left(L^{p_1(.),q_1(.)}, L^{r_1}_{\omega_1}\right) \times W\left(L^{p_2(.),q_2(.)}, L^{r_2}_{\omega_2}\right) \to W\left(L^{p_3(.),q_3(.)}, L^{r_3}_{\omega_3}\right)$$

is not bounded.

Proof Since $\frac{1}{\omega_3} \in L^{r'_3}$, we have $L^{r_3}_{\omega_3} \subset L^1$. So we write

$$W\left(L^{p_3(.),q_3(.)}, L^{r_3}_{\omega_3}\right) \subset W\left(L^{p_3(.),q_3(.)}, L^1\right).$$

On the other hand since $1 \le q_3(.)$ and $q_3(0) \le p_3(0)$, we have

$$W\left(L^{p_3(.),q_3(.)}, L^1\right) \subset W\left(L^{1,1}, L^1\right) = L^1. \tag{13}$$

by Theorem 3.5 and Theorem 3.6. Then from (13), we obtain

$$W\left(L^{p_3(.),q_3(.)}, L^{r_3}_{\omega_3}\right) \subset L^1. \tag{14}$$

Take the indicator functions χ_A, χ_B where $A, B \subset \mathbb{R}$ is a compact subset. By Theorem 2.1, $\chi_A \in W\left(L^{p_1(.),q_1(.)}, L^{r_1}_{\omega_1}\right)$ and $\chi_B \in W\left(L^{p_2(.),q_2(.)}, L^{r_2}_{\omega_2}\right)$. Also by Theorem 1 in [1], we know that the function M is unbounded on $L^1 \times L^1$. If $f, g \in L^1$ are not identically zero, then the function M is not integrable on $\mathbb{R}$. That means $M\left(\chi_A,\chi_B\right)$ is not in L^1. Hence by (14), $M\left(\chi_A,\chi_B\right)\notin W\left(L^{p_3(.),q_3(.)},L^{r_3}_{\omega_3}\right)$. This completes proof.

Theorem 4.2 Let $r_i, r, p_i(.), q_i(.), t(.), s(.)\in\wp_1\left(\left[0,\infty\right]\right)$, (i=1,2) and ω,ω_1,ω_2 be weight functions. Assume that $r_1 \le q_1(.)$, $q_2(0)\le p_2(0)$, $r_2\le q_2(.)$, $q_1(0)\le p_1(0)$, $s(.)\le r$, $s(0)\ge t(0)$ and $\frac{1}{r}+\frac{1}{r'}=1$. If $\frac{1}{\omega}\notin L^{r'}$ and bilinear "Hardy-Littlewood maximal function" $M: L^{r_1}_{\omega_1}\times L^{r_2}_{\omega_2}\to L^r_\omega$ is bounded, then bilinear "Hardy-Littlewood maximal function" $M: W\left(L^{p_1(.),q_1(.)},L^{r_1}_{\omega_1}\right)\times W\left(L^{p_2(.),q_2(.)},L^{r_2}_{\omega_2}\right)\to W\left(L^{t(.),s(.)},L^r_\omega\right)$

is bounded.

Proof Since $M: L^{r_1}_{\omega_1}\times L^{r_2}_{\omega_2}\to L^r_\omega$ is bounded, there exists the inequality

$$\left\|M(f,g)\right\|_{r,\omega}\le C_1\left\|f\right\|_{r_1,\omega_1}\left\|g\right\|_{r_2,\omega_2},\ C\square>0 \tag{15}$$

for all $f\in L^{r_1}_{\omega_1}$ and $g\in L^{r_2}_{\omega_2}$. Also since by Theorem 3.3, $W\left(L^{r,r},L^r_\omega\right)=L^r_\omega$, and $r_1\le q_1(.)$, $q_2(0)\le p_2(0)$, $r_2\le q_2(.)$, $q_1(0)\le p_1(0)$, $s(.)\le r$, $s(0)\ge t(0)$ then by 3.5 Theorem

$$W\left(L^{p_1(.),q_1(.)},L^{r_1}_{\omega_1}\right)\subset W\left(L^{r_1,r_1},L^{r_1}_{\omega_1}\right)=L^{r_1}_{\omega_1},$$

$$W\left(L^{p_2(.),q_2(.)},L^{r_2}_{\omega_2}\right)\subset W\left(L^{r_2,r_2},L^{r_2}_{\omega_2}\right)=L^{r_2}_{\omega_2} \tag{16}$$

and

$$W\left(L^{r,r},L^r_\omega\right)=L^r_\omega\subset W\left(L^{t(.),s(.)},L^r_\omega\right)$$

Then boundedness of these unit maps

$I_{W\left(L^{p_1(.),q_1(.)},L^{r_1}_{\omega_1}\right)}: W\left(L^{p_1(.),q_1(.)},L^{r_1}_{\omega_1}\right)\to L^{r_1}_{\omega_1}$, $\qquad I_{W\left(L^{p_2(.),q_2(.)},L^{r_2}_{\omega_2}\right)}: W\left(L^{p_2(.),q_2(.)},L^{r_2}_{\omega_2}\right)\to L^{r_2}_{\omega_2}$

and $I_{L^r_\omega}: L^r_\omega\to W\left(L^{t(.),s(.)},L^r_\omega\right)$. Thus we write following inequalities

$$\left\|f\right\|_{r_1,\omega_1}\le C_2\left\|f\right\|_{W\left(L^{p_1(.),q_1(.)},L^{r_1}_{\omega_1}\right)},\ C_2>0, \tag{17}$$

$$\|g\|_{r_2,\omega_2} \le C_3 \|g\|_{W\left(L^{p_2(.),q_2(.)},L^{r_2}_{\omega_2}\right)}, C_3 > 0 \tag{18}$$

and

$$\|h\|_{W\left(L^{t(.),s(.)},L^{r}_{\omega}\right)} \le C_4 \|h\|_{r,\omega}, C_4 > 0 \tag{19}$$

for all $f \in W\left(L^{p_1(.),q_1(.)},L^{r_1}_{\omega_1}\right)$, $g \in W\left(L^{p_2(.),q_2(.)},L^{r_2}_{\omega_2}\right)$ and $h \in L^{r}_{\omega}$. Take any $f \in W\left(L^{p_1(.),q_1(.)},L^{r_1}_{\omega_1}\right)$, $g \in W\left(L^{p_2(.),q_2(.)},L^{r_2}_{\omega_2}\right)$. By the inclusion (16), $f, g \in L^{r}_{\omega}$. Also by (15), we have M(f,g)$\in L^{r}_{\omega}$. Hence combining (15), (17), (18) and (19), we find

$$\|M(f,g)\|_{W\left(L^{t(.),s(.)},L^{r}_{\omega}\right)} = \left\|I_{L^{r}_{\omega}}\left(M(f,g)\right)\right\|_{W\left(L^{t(.),s(.)},L^{r}_{\omega}\right)} \le C_4 \|M(f,g)\|_{r,\omega} \le C_1 C_4 \|f\|_{r_1,\omega_1} \|g\|_{r_2,\omega_2}$$

$\le C_1 C_2 C_3 C_4 \|f\|_{W\left(L^{p_1(.),q_1(.)},L^{r_1}_{\omega_1}\right)} \|g\|_{W\left(L^{p_2(.),q_2(.)},L^{r_2}_{\omega_2}\right)}$. This completes proof.

5. Boundedness of The Bilinear Littlewood-Paley Square Function on $W\left(L^{p(.),q(.)},L^r_\omega\right)$.

5.1 Definition Let $K_n(\xi)=K(\xi-n)$ fo $n\in\mathbb{N}$ where K is supported on unit interval of $\mathbb{R}$ and "smooth bump function" on $\mathbb{R}$. "Bilinear Littlewood-Paley square function" is defined by

$$S(f,g)(x)=\left(\sum_n\left|S_n(f,g)(x)\right|^2\right)^{\frac{1}{2}}$$

where $S_n(f,g)(x)=\int_{\circ} f(x-y)g(x+y)K_n(y)dy$, [19,21] for $f,g\in S$.

Theorem 5.1 Let $r_i, r,\ p_i(.),q_i(.),t(.),s(.)\in\wp_1([0,\infty])$ and ω,ω_i be weight functions, (i=1,2). Assume that $r_1\le\frac{q_1(.)}{2}$, $q_2(0)\le p_2(0)$, $r_2\le\frac{q_2(.)}{2}$, $q_1(0)\le p_1(0)$, $\frac{s(.)}{2}\le r$, $s(0)\ge t(0)$and $\frac{1}{r}+\frac{1}{r'}=1$. If $\frac{1}{\omega}\notin L^{r'}$and bilinear "Hardy-Littlewood maximal function" $M:L^{r_1}_{\omega_1}\times L^{r_2}_{\omega_2}\to L^r_\omega$ is bounded, then bilinear "Hardy-Littlewood maximal function" $S:W\left(L^{p_1(.),q_1(.)},L^{r_1}_{\omega_1}\right)\times W\left(L^{p_2(.),q_2(.)},L^{r_2}_{\omega_2}\right)\to W\left(L^{t(.),s(.)},L^r_\omega\right)$is bounded.

Proof By from [21], we write $S(f,g)(x)\le C\left(M\left(|f|^2,|g|^2\right)(x)\right)^{\frac{1}{2}}$, (a.e $x\in\mathbb{R}$) where C>0. Using this last inequality and since $W\left(L^{t(.),s(.)},L^r_\omega\right)$ is solid space , we have

$$\|S(f,g)\|_{W\left(L^{t(.),s(.)},L^r_\omega\right)}\le C\left\|\left(M\left(|f|^2,|g|^2\right)\right)^{\frac{1}{2}}\right\|_{W\left(L^{t(.),s(.)},L^r_\omega\right)}=C\left\|\left\|\left(M\left(|f|^2,|g|^2\right)\right)^{\frac{1}{2}}\chi_{z+Q}\right\|_{L^{t(.),s(.)}}\right\|_{r,\omega}$$

$$=C\left\|\left\|t^{\frac{1}{t(.)}-\frac{1}{s(.)}}\left(\left(M\left(|f|^2,|g|^2\right)\right)^{\frac{1}{2}}\chi_{z+Q}\right)^{**}\right\|_{L^{s(.)}([0,\infty])}\right\|_{r,\omega}=C\left\|\left\|t^{\frac{2}{t(.)}-\frac{2}{s(.)}}\left(\left(M\left(|f|^2,|g|^2\right)\chi_{z+Q}\right)^{**}\right)^{\frac{1}{2}}\right\|_{L^{s(.)}([0,\infty])}\right\|_{r,\omega}$$

$$= C\left\|\left\| t^{\frac{\frac{1}{t(.)}}{2}-\frac{\frac{1}{s(.)}}{2}}\left(M\left(|f|^2,|g|^2\right)\chi_{z+Q}\right)^{**}\right\|_{L^{\frac{s(.)}{2}}([0,\infty])}\right\|_{r,\omega} = C\left\|\left\|M\left(|f|^2,|g|^2\right)\chi_{z+Q}\right\|_{L^{\frac{t(.)}{2},\frac{s(.)}{2}}}\right\|_{r,\omega}$$

$$= C\left\|M\left(|f|^2,|g|^2\right)\right\|_{W\left(L^{\frac{t(.)}{2},\frac{s(.)}{2}},L^r_\omega\right)} \tag{20}$$

So using assumptions and (20), from the theorem 4.2, we find

$$\left\|S(f,g)\right\|_{W\left(L^{t(.),s(.)},L^r_\omega\right)} \le C\left\|M\left(|f|^2,|g|^2\right)\right\|_{W\left(L^{\frac{t(.)}{2},\frac{s(.)}{2}},L^r_\omega\right)} \le CC_1\left\||f|^2\right\|_{W\left(L^{\frac{p_1(.)}{2},\frac{q_1(.)}{2}},L^{r_1}_{\omega_1}\right)}\left\||g|^2\right\|_{W\left(L^{\frac{p_2(.)}{2},\frac{q_2(.)}{2}},L^{r_2}_{\omega_2}\right)}$$

$$= CC_1\left\|\left\||f|^2\chi_{z+Q}\right\|_{L^{\frac{p_1(.)}{2},\frac{q_1(.)}{2}}}\right\|_{r_1,\omega_1}\left\|\left\||g|^2\chi_{z+Q}\right\|_{L^{\frac{p_2(.)}{2},\frac{q_2(.)}{2}}}\right\|_{r_2,\omega_2}$$

$$= CC_1\left\|\left\| t^{\frac{\frac{1}{p_1(.)}}{2}-\frac{\frac{1}{q_1(.)}}{2}}\left(|f|^2\chi_{z+Q}\right)^{**}\right\|_{L^{\frac{q_1(.)}{2}}([0,\infty])}\right\|_{r_1,\omega_1}\left\|\left\| t^{\frac{\frac{1}{p_2(.)}}{2}-\frac{\frac{1}{q_2(.)}}{2}}\left(|g|^2\chi_{z+Q}\right)^{**}\right\|_{L^{\frac{q_2(.)}{2}}([0,\infty])}\right\|_{r_2,\omega_2}$$

$$= CC_1\left\|\left\|\left(t^{\frac{1}{p_1(.)}-\frac{1}{q_1(.)}}\left(|f|\chi_{z+Q}\right)^{**}\right)^2\right\|_{L^{\frac{q_1(.)}{2}}([0,\infty])}\right\|_{r_1,\omega_1}\left\|\left\|\left(t^{\frac{1}{p_2(.)}-\frac{1}{q_2(.)}}\left(|g|\chi_{z+Q}\right)^{**}\right)^2\right\|_{L^{\frac{q_2(.)}{2}}([0,\infty])}\right\|_{r_2,\omega_2}$$

$$= CC_1\left\|\left\| t^{\frac{1}{p_1(.)}-\frac{1}{q_1(.)}}\left(|f|\chi_{z+Q}\right)^{**}\right\|_{L^{q_1(.)}([0,\infty])}\right\|_{r_1,\omega_1}\left\|\left\| t^{\frac{1}{p_2(.)}-\frac{1}{q_2(.)}}\left(|g|\chi_{z+Q}\right)^{**}\right\|_{L^{q_2(.)}([0,\infty])}\right\|_{r_2,\omega_2}$$

$$= CC_1\left\|\left\||f|\chi_{z+Q}\right\|_{L^{p_1(.),q_1(.)}}\right\|_{r_1,\omega_1}\left\|\left\||g|\chi_{z+Q}\right\|_{L^{p_2(.),q_2(.)}}\right\|_{r_2,\omega_2} = CC_1\left\|f\right\|_{W\left(L^{p_1(.),q_1(.)},L^{r_1}_{\omega_1}\right)}\left\|g\right\|_{W\left(L^{p_2(.),q_2(.)},L^{r_2}_{\omega_2}\right)}$$

for $C_1>0$.

References

[1] I. Assani, Z. Buczolih, *The (L^1,L^1) bilinear Hardy-Littlewood function and Furstenberg averages*, Rev. Mat. Iberoamericana, **26**(3), (2010), 861-890.

[2] I. Aydın, A. T. Gürkanlı, *Weighted variable exponent amalgam spaces* $W\left(L^{p(x)}, L^q_\omega\right)$, Glasnik Matematicki, **47**(67), (2012), 165-174.

[3] F. Bernicot, *L^p estimates for non smoth bilinear Littlewood-Paley square functions on* ℝ, arXiv:0811.2854, to appear in Math. Ann., (2018).

[4] T. Dobler, Wiener Amalgam Spaces on Locally Compact Groups, Master's thesis, University of Vienna, (1989).

[5] L. Ephremidze, V. Kokilashvili, S. Samko, *Fractional, maximal and singular operators in variable exponent Lorentz spaces*, Fract. Calc. Appl. Anal., **11**(4), (2008), 1-14.

[6] H. G. Feichtinger, *Banach convolution algebras of Wiener's type,* Proc. Conf. Functions, Series, Operators, Budapest, 1980, Colloquia Math. Soc. J. Bolyai, North Holland Publ. Co., Amsterdam- Oxford- New York, (1983), 509-524.

[7] H. G. Feichtinger, K. H. Gröchenig, *Banach spaces related to integrable group representations and their atomic decompositions*, I, J. Funct. Anal. **86**, (1989), 307-340.

[8] H. G. Feichtinger, A. T. Gürkanlı, *On a family of weighted convolution algebras*, Int. J. Math. Math. Sci. **13**, (1990), 517-526.

[9] R. H. Fischer, A. T. Gürkanlı, T. S. Liu, *On a family of Wiener type spaces*, Internat. J. Math. Math. Sci. **19**, (1996), 57-66.

[10] J. J. Fournier, J. Stewart, *Amalgams of L^p and l^q ,* Bull. Amer. Math. Soc. (N. S.) **13**, (1985), 1-21.

[11] A. T. Gürkanlı, I. Aydın, *On the weighted variable exponent amalgam space* $W\left(L^{p(x)}, L^q_m\right)$, Acta Mathematica Scientia, **34B**(4), (2014), 1098-1110.

[12] A. T. Gürkanlı, "The Amalgam spaces $W\left(L^{p(x)}, l^{\{p_n\}}\right)$ and boundedness of Hardy-Littlewood Maximal operators", Current Trends in Analysis and Its Applications, Springer International Publishing Switzerland, (2015).

[13] C. Heil, "An introduction to weighted Wiener Amalgams", In: Wavelets and their Applications (Chennai, 2002), Allied Publishers, New Delhi, 183-216,(2003).

[14] F. Holland, *Square -summable positive-definite functions on real line*, Linear operators Approx. II, Ser. Numer. Math. **25**, Birkhauser, Basel, (1974), 247-257.

[15] F. Holland, *Harmonic analysis on amalgams of* L^p *and* l^q*,* London math. Soc. **10**(2), (1975), 295-305.

[16] R. A. Hunt, *On L(p,q) spaces*, Extrait de L'Enseignement Mathematique T., XII, fasc.,**4**, (1966), 249-276.

[17] O. Kulak, *The inclusion theorems for variable exponent Lorentz spaces*, Turk J Math, **40**, (2016), 605-619.

[18] O. Kulak, *Boundedness of bilinear Littlewood-Paley Square function on variable Lorentz spaces*, AIP (2017), 1863, 1.

[19] M. Lacey, *On bilinear Littlewood-Paley square functions*, Publ. Mat. **40**(2), (1996), 387-396.

[20] M. T. Lacey, *The bilinear maximal fuctions map into* L^p *for* $(2/3)<p\leq 1$, Annals of Mathematics, **151**, (2000), 35-37.

[21] P. Mohanty, S. Shrivastava, *A note on the bilinear Littlewood-Paley square function*, Proceedings of the American Mathematical Society, **6**(138), (2010), 2095-2098.

[22] J. L. Rubio de Francia, *Estimates for some square functions of Littlewood-Paley type*, Publ. Sec. Mat. Univ. Autonoma Barcelona, **27**(2), (1983), 81-108.

[23] J. L. Rubio de Francia, *A Littlewood-Paley inequalty for arbitrary intervals*, Rev. Mat. Iberoamericana, **1**(2), (1985), 1-14.

[24] L. Schwartz, Mathematics for the Pyhsical Science, Addison-Wesley Publishing Company, (1966).

[25] N. Wiener, *Tauberian theorems*, Ann.Math, **33**, (1932), 1-100.