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Shift λ-Invariant Operators

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ABSTRACT. The present note is devoted to a generalization of the notion of shift invariant operators that we call it λ-invariant operators ($\lambda \geq 0$). Some properties of this new class are presented. By using probabilistic methods, three examples are delivered.

Keywords: Modulus of continuity, integral operator, convolution type operator, probabilistic distribution function.

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1. INTRODUCTION

This research is mainly motivated by the work of G. A. Anastassiou and H. H. Gonska [6]. The authors have introduced a general family of integral type operators. Sufficient conditions were given for shift invariance and also the property of global smoothness preservation was studied. Let (X, d) be a metric space of real valued functions defined on D, where $D = \mathbb{R}$ or $D = \mathbb{R}_+$. An operator L which maps X into itself is called a shift invariant operator if and only if

$$Lf_\alpha = (Lf)_\alpha \text{ for any } f \in X \text{ and } \alpha > 0,$$

where $f_\alpha(\cdot) = f(\cdot + \alpha)$.
In this note we give a generalization of the notion of shift invariant operator. Some properties of this class are presented and a general family of such operators in the space of integrable functions $L^1(\mathbb{R})$ is introduced by using the convolution product of another operators with a scaling type function. By resorting to probabilistic methods, we indicate some classical operators as shift λ-invariant, where λ is calculated in each case.
We refer to the following operators: Szász-Mirakjan, Baskakov and Weierstrass. It is honest to mention that the value of λ does not target the whole sequence, it depends on the rank of the considered term.
The general results are concentrated in Section 2 and the applications are detailed in Section 3. It is known that the shift invariant operators are useful in wavelet analysis. Along with the paper [6], the subject was developed in other papers, among which we quote [3], [4], [5]. Until now, we have built a generalization of the shift invariant operators and we proved that the new class is consistent. The applications presented reinforce the significance of the construction. The use of this class of operators could lead for generating wavelet bases type. In this direction, the conditions for multiresolution analysis can be relaxed by using shift λ-invariant operators. Thus, we can talk about quasi-wavelet functions that can serve to reconstruct certain signals. We admit that this research direction is at an early stage.

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2. Results

Firstly, we present the following informal definition.

Definition 2.1. *Let λ be a non negative number and X be a metric space of real valued functions defined on $\mathbb{R}$ or $\mathbb{R}_+$. An operator L acting on X is called a shift λ-invariant operator if*

$$|(Lf)_\alpha - Lf_\alpha| \le \lambda, \text{ for any } f \in X \text{ and } \alpha > 0. \tag{2.1}$$

Clearly, for $\lambda = 0$ we reobtain the notion of shift invariant operator.

Theorem 2.1. *Let A, B operators acting on a compact metric space X of real valued functions defined on $\mathbb{R}$ or $\mathbb{R}_+$.*
i) If A is a shift λ-invariant and B is a shift invariant, then AB is a shift λ-invariant operator.
ii) If A is a shift invariant, linear and positive operator, and B is a shift λ-invariant, then AB is a shift $\lambda\mu$-invariant operator, where $\mu = \|A\|$.
Call $\|A\| = \sup\{\|Ag\|_X : \ g \in X \text{ and } \|g\|_X \le 1\}$.

Proof. i) We take $g = Bf$ and, in concordance with the hypothesis, we can write successively

$$|(ABf)_\alpha - ABf_\alpha| = |(Ag)_\alpha - A(Bf_\alpha)| = |(Ag)_\alpha - Ag_\alpha| \le \lambda,$$

which implies the first statement of the theorem.
ii) Since B is a shift λ-invariant operator, we get

$$-\lambda \le (Bf)_\alpha - Bf_\alpha \le \lambda. \tag{2.2}$$

The operator A is linear and positive, consequently it is monotone, i.e., $Au \le Av$ for any u, v belong to X with the property $u \le v$.
A being a shift invariant operator too, relation (2.2) implies

$$|(ABf)_\alpha - ABf_\alpha| \le \lambda A e_0,$$

where $e_0(x) = 1$, $x \in \mathbb{R}$ or $x \in \mathbb{R}_+$. Because of $0 < Ae_0 \le \|A\|$, the result follows. □

Remark. Assuming $Ae_0 = e_0$, relation usually verified by linear and positive operators (so called Markov type operators), we deduce that $\mu = 1$ and Theorem 2.1 (ii) guarantees that AB becomes a shift λ-invariant operator.
In what follows, starting from a sequence of shift λ-invariant operators and using a scaling type function, we construct a sequence of integral type operators.
For each $n \in \mathbb{N}$, let l_n be a shift λ_n-operator which maps the space $L^1(\mathbb{R})$ into itself. Also, we are fixing a function $\psi \in L^1(\mathbb{R})$ such that

$$\|\psi\|_1 = \int_{\mathbb{R}} |\psi(x)| dx \ne 0.$$

For any $f \in L^1(\mathbb{R})$, the convolution of $l_n f$ with ψ is a function named $L_n f$ which belongs to $L^1(\mathbb{R})$ and is defined by

$$(L_n f)(x) = (l_n f * \psi)(x) = \int_{\mathbb{R}} (l_n f)(y)\psi(x-y)dy. \tag{2.3}$$

It is known that the convolution product $*$ enjoys the commutativity property. Let $n \in \mathbb{N}$ arbitrarily be set. On the other hand, we have the following relations

$$(L_n f)_\alpha(x) = \int_{\mathbb{R}} (l_n f)(x + \alpha - u)\psi(u)du,$$

$$(L_n f_\alpha)(x) = \int_{\mathbb{R}} (l_n f_\alpha)(x - u)\psi(u)du,$$

which allow us to write

$$\begin{aligned}|(L_nf)_\alpha(x)-(L_nf_\alpha)(x)| &\le \int_{\mathbb{R}}|(l_nf)(x+\alpha-u)-(l_nf_\alpha)(x-u)||\psi(u)|du\\ &=\int_{\mathbb{R}}|((l_nf)_\alpha-l_nf_\alpha)(x-u)||\psi(u)|du\\ &\le \lambda_n\|\psi\|_1.\end{aligned}$$

We just ended the proof of the following result.

Theorem 2.2. *Let $L_n : L^1(\mathbb{R}) \to L^1(\mathbb{R})$, $n \ge 1$, be operators defined by (2.3). Then, for each $n \in \mathbb{N}$, L_n is a shift $\lambda_n\|\psi\|_1$-invariant operator.*

We notice that if we substitute in (2.3) the function ψ by $\|\psi\|_1^{-1}\psi$, then the operator L_n becomes shift λ_n-invariant, $n \in \mathbb{N}$.
As usual, we denote by $C_B(D)$ the Banach lattice of all bounded and continuous real functions on D endowed with the sup-norm $\|\cdot\|$. Also $C_B^1(D)$ denotes the subspace of $C_B(D)$ consisting of all functions which are continuously differentiable and bounded on D. We recall the definition of the first modulus of smoothness $\omega(f;\cdot)$ associated to the bounded function $f : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$,

$$\omega(f;\delta)=\sup_{\substack{x,y\in I\\ |x-y|\le\delta}}|f(x)-f(y)|,\ \delta\ge 0. \tag{2.4}$$

At this moment we need the following result.

Theorem 2.3. ([2, *Theorem 7.3.4*]) *Let the random variable Y have distribution μ, $E(Y) := x_0$ and $Var(Y) := \sigma^2$. Consider $f \in C_B^1(\mathbb{R})$. Then*

$$|Ef(Y)-f(x_0)|=\left|\int_{\mathbb{R}}fd\mu-f(x_0)\right|\le(1.5625)\omega\left(f';\frac{\sigma}{2}\right)\sigma. \tag{2.5}$$

In the above $E(Y)$, $Var(Y)$ represent the expected value and variance of Y, respectively.
We consider the random variables X_j, $j \ge 1$, independent and identically distributed and we introduce

$$X_{j,\alpha}=X_j+\alpha,\ S_{n,\alpha}=\frac{1}{n}\sum_{j=1}^{n}X_{j,\alpha},\ n\ge 1. \tag{2.6}$$

Clearly, $S_{n,0}+\alpha = S_{n,\alpha}$. If we use the notations $E(X_{j,\alpha}) := x_{0,\alpha}$ and $Var(X_{j,\alpha}) := \sigma_\alpha^2$, by using the properties of the expectation respectively the variance, we obtain

$$E(S_{n,\alpha})=x_{0,0}+\alpha=x_{0,\alpha},\ Var(S_{n,\alpha})=\frac{\sigma_\alpha^2}{n}=\frac{\sigma_0^2}{n}.$$

From (2.5) we deduce

$$\begin{aligned}|E[f(S_{n,\alpha})]-f(x_{0,\alpha})| &= \left|\int_{\mathbb{R}}f\left(\frac{t}{n}\right)dF_{n,\alpha}(t)-f(x_{0,\alpha})\right|\\ &\le 1.5625\omega\left(f';\frac{\sigma_0}{2\sqrt{n}}\right)\frac{\sigma_0}{\sqrt{n}},\end{aligned} \tag{2.7}$$

where $F_{n,\alpha}$ is the distribution function of the random variable $S_{n,\alpha}$.
It is known that by using probabilistic methods several classical positive and linear operators have been obtained. Pioneers in this research field can be mentioned here W. Feller [7] and

D.D. Stancu [9]. A recent and up-to-date approach to this study direction concerning Markov semigroups and approximation processes can be found in [1].
As in [9], for each $n \geq 1$, we choose

$$(L_n f)(x) = E[f \circ S_{n,0}(x)] = \int_{-\infty}^{\infty} f dF_{S_{n,0}(x)}, \tag{2.8}$$

where $F_{S_{n,0}}$ is the probability distribution of the variable $S_{n,0}$. Note that $L_n f$ is a bounded function and clearly satisfies $\|L_n f\| \leq \|f\|$.
Taking into account (2.6) and (2.7) we can write successively

$$\begin{aligned} |(L_n f)_\alpha(x) - (L_n f_\alpha)(x)| &\leq |E[f(S_{n,\alpha})] - f(x_{0,\alpha})| + |E[f_\alpha(S_{n,0})] - f_\alpha(x_{0,0})| \\ &\leq \mu \left(\omega \left(f'; \frac{\sigma_0}{2\sqrt{n}} \right) + \omega \left(f'_\alpha; \frac{\sigma_0}{2\sqrt{n}} \right) \right) \frac{\sigma_0}{\sqrt{n}} \\ &= 2\mu\omega \left(f'; \frac{\sigma_0}{2\sqrt{n}} \right) \frac{\sigma_0}{\sqrt{n}}, \end{aligned}$$

where $\mu = 1.5625$. Also, based on the definition (2.4), we used the identity $\omega(f_\alpha; \cdot) = \omega(f; \cdot)$ for each $\alpha \geq 0$.
Finally, using that $\omega(f'; \cdot)$ is a non-decreasing function, the above relations lead us to the following result.

Theorem 2.4. *Let S_n and L_n be defined by (2.6) and (2.7) respectively, where $f \in C_B^1(\mathbb{R})$. Let I be an interval such that $\sup_{x \in I} \sigma_0(x) = \gamma < \infty$. The following identity*

$$|(L_n f)_\alpha(x) - (L_n f_\alpha)(x)| \leq 3.125\omega \left(f'; \frac{\gamma}{2\sqrt{n}} \right) \frac{\gamma}{\sqrt{n}}, \ x \in I, \tag{2.9}$$

holds.

In view of relation (2.1), the above theorem says that L_n operator, subject of certain conditions, is a λ_n-invariant operator, where

$$\lambda_n = 3.125\omega \left(f'; \frac{\gamma}{2\sqrt{n}} \right) \frac{\gamma}{\sqrt{n}}.$$

Here λ_n's expression is complicated, consequently it is practically unattractive. With the desire to simplify it, we add an additional condition to function f. We require that f' satisfies a Lipschitz condition with a constant M and exponent β, $f' \in Lip_M \beta$, $(M \geq 0, 0 < \beta \leq 1)$, that is

$$|f'(x_1) - f'(x_2)| \leq M|x_1 - x_2|^\beta, \ (x_1, x_2) \in I \times I.$$

The new requirement implies the continuity of f'. On the other hand, equivalent to this property is the inequality

$$\omega(f'; h) \leq M h^\beta, \ h \geq 0, \tag{2.10}$$

see, e.g., [8, *page 49*].
Considering (2.9) and (2.10), the main result of this note will be read as follows.

Theorem 2.5. *Let S_n and L_n be defined by (2.6) and (2.8) respectively, where $f \in C_B(\mathbb{R})$ is differentiable on the domain such that $f' \in Lip_M \beta$. Let I be an interval and $\sup_{x \in I} \sigma_0(x) = \gamma < \infty$. Then, for each $n \in \mathbb{N}$, L_n is a λ_n-shift invariant operator, where*

$$\lambda_n = \frac{3.125}{2^\alpha} M \left(\frac{\gamma}{\sqrt{n}} \right)^{\beta+1}. \tag{2.11}$$

3. Applications

In this section we present three examples of classical operators, both of discrete and continuous type, which verify Theorem 2.4. We are able to indicate explicitly λ_n such that L_n may become a shift λ_n-invariant operator. In the following $\mathbb{N}_0$ stands for $\{0\} \cup \mathbb{N}$.
Set

$$E_2(\mathbb{R}_+) = \left\{ f \in C(\mathbb{R}_+) : \frac{f(x)}{1+x^2} \text{ is convergent as } x \to \infty \right\},$$

representing a Banach lattice endowed with the norm

$$\|f\|_* = \sup_{x \geq 0} (1+x^2)^{-1} |f(x)|.$$

Example 3.1. *Let X_j, $j \geq 1$, be i.i. random variables having Poisson distribution, i.e., for each $k \in \mathbb{N}_0$*

$$P(X_j = k) = e^{-x} \frac{x^k}{k!}, \ x \geq 0,$$

which implies $E(X_j) = x$ and $Var(X_j) = x$. Formula (2.8) leads us to Szász-Mirakjan operators defined for $f \in E_2(\mathbb{R}_+)$ as follows

$$(L_n f)(x) \equiv (M_n f)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \ n \geq 1. \tag{3.12}$$

Further on, we consider $f \in C_B^1(\mathbb{R}_+)$ and $I = [0, a]$, $a > 0$ fixed.
Consequently we get $\gamma = \sqrt{a}$. Relation (2.9) yields

$$|(M_n f)_\alpha(x) - (M_n f_\alpha)(x)| \leq 3.125 \omega \left(f'; \frac{1}{2}\sqrt{\frac{a}{n}} \right) \sqrt{\frac{a}{n}}, \ x \in [0, a].$$

Example 3.2. *Let X_j, $j \geq 1$, be i.i. random variables following Pascal distribution, i.e., for each $k \in \mathbb{N}_0$*

$$P(X_j = k) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}, \ x \geq 0,$$

which implies $E(X_j) = x$ and $Var(X_j) = x + x^2$. Applying formula (2.8) we get Baskakov operators defined for $f \in E_2(\mathbb{R}_+)$ as follows

$$(L_n f)(x) \equiv (V_n f)(x) = \frac{1}{(1+x)^n} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left(\frac{x}{1+x}\right)^k f\left(\frac{k}{n}\right), \ n \geq 1. \tag{3.13}$$

We take $f \in C_B^1(\mathbb{R}_+)$ and $I = [0, a]$, $a > 0$ fixed. This time we have $\gamma = \sqrt{a(a+1)}$ and (2.9) yields

$$|(V_n f)_\alpha(x) - (V_n f_\alpha)(x)| \leq 3.125 \omega \left(f'; \frac{1}{2}\sqrt{\frac{a^2+a}{n}} \right) \sqrt{\frac{a^2+a}{n}}, \ x \in [0, a].$$

Example 3.3. *Assume X_j, $j \geq 1$, are i.i. continuous Gaussian random variables having the normal distribution $N(x, \sigma)$. This means the probability density function is given by*

$$\mu(t) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-(t-x)^2/(2\sigma^2)), \ t \in \mathbb{R}.$$

It is known that $S_{n,0}$ has a normal distribution too, with $E(S_{n,0}) = x$ and $Var(S_{n,0}) = \sigma^2/n$. In this case, (2.8) yields the operator

$$(L_n f)(x) = \frac{\sqrt{n}}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} f(t) \exp(-n(t-x)^2/(2\sigma^2)) dt, \ f \in C_B(\mathbb{R}).$$

For $\sigma^2 = 0.5$ it reduces to genuine Weierstrass operator W_n.
For any $f \in C_B^1(\mathbb{R})$ and $I \subseteq \mathbb{R}$ we have $\gamma = 2^{-1/2}$ and, in view of (2.9), we get

$$|(W_n f)_\alpha(x) - (W_n f_\alpha)(x)| \le 3.125\omega\left(f'; \frac{1}{2\sqrt{2n}}\right)\frac{1}{\sqrt{2n}},\ x \in I. \tag{3.14}$$

Remark. Taking into account the results (3.12), (3.13), (3.14), under the hypotheses of Theorem 2.5, we can state that the operators Szász-Mirakjan, Baskakov and Weierstrass of rank n are shift $C(\tau/n)^{(\beta+1)/2}$-invariant operators, where $C = 3.125M2^{-\beta}$ and τ is defined as follows: $\tau = a$ for the first operator, $\tau = a^2 + a$ for the second operator and $\tau = 0.5$ for the last operator.

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