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Best proximity point theorems of cyclic Meir-Keeler-Kannan-Chatterjea contractions

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Abstract

In this study, by using the Meir-Keeler mapping, cyclic Kannan contraction and cyclic Chatterjee contraction, we establish the notions of cyclic Meir-Keeler-Kannan-Chatterjea contraction $T : A \cup B \to A \cup B$ and cyclic Meir-Keeler-Kannan-Chatterjea contractive pair (T, S) of mappings $T : A \to B$ and $S : B \to A$, and then we prove some best proximity point theorems for these various types of cyclic contractions. Our results generalize or improve many recent best proximity point theorems in the literature.

Keywords: Best proximity points; Cyclic Meir-Keeler-Kannan-Chatterjea contraction; Cyclic Meir-Keeler-Kannan-Chatterjea contractive pair; Metric space.
2010 MSC: 47H10, 54C60, 54H25, 55M20.

1. Introduction and preliminaries

Throughout this article, by $\mathbb{R}^+$, we denote the set of all non-negative numbers, while $\mathbb{N}$ is the set of all natural numbers. Let us consider two nonempty subsets A, B of a metric space (X, d) and a mapping $T : A \to B$. Note that if $A \cap B = \phi$, the equation $Tx = x$ might have no solution. So, we find a point $x \in A$ such that $\min d(x, Tx)$ is minimum. If $d(x, Tx) = d(A, B) := inf\{d(a, b) : a \in A, b \in B\}$, then $d(x, Tx)$ is the global minimum value $d(A, B)$, and x is an approximate solution of the equation $Tx = x$ with the possible error. A point $x \in A$ is said to be the best proximity point of T if $d(x, Tx) = d(A, B) := inf\{d(a, b) : a \in A, b \in B\}$. The existence and approximation of best proximity points is an interesting topic in optimization theory. In [7], Eldred and Veeramani investigated the existence of best proximity points for a class of mappings called cyclic contraction.

Definition 1.1. *[7] Let A, B be nonempty subsets of a metric space (X, d). A mapping $T : A \cup B \to A \cup B$ is said to be a cyclic contraction if there exists $k \in [0, 1)$ such that*

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(1) *T is a cyclic mapping, that is, $T(A) \subset B$ and $T(B) \subset A$.*

(2) *$d(Tx, Ty) \leq kd(x, y) + (1 - k)d(A, B)$, for all $x \in A$ and $y \in B$.*

Theorem 1.2. *[7] Let A, B be nonempty closed and convex subsets of a complete metric space (X, d) and let $T : A \cup B \to A \cup B$ be a cyclic contraction. For $x_{n+1} = x_n$ for each $n \in \mathbb{N} \cup \{0\}$. Then there exists a unique $x \in A$ such that $x_{2n} \to x$ and $d(x, Tx) = d(A, B)$. Here x is called the best proximity point of T.*

In the recent years, many authors are studying the best proximity point problems for various types of cyclic contractions.(see, eg.[1]-[4],[5],[8],[10],[11],[14]).

We also recalled the following Meir-Keeler mapping (see, [9]). A function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a Meir-Keeler mapping, if ϕ satisfies the following condition:

$$\forall \eta > 0 \ \ \exists \delta > 0 \ \ \forall t \in \mathbb{R}^+ \ \ (\eta \leq t < \eta + \delta \Rightarrow \phi(t) < \eta).$$

Remark 1.3. *It is clear that if ϕ is a Meir-Keeler mapping, then we have*

$$\phi(t) < t \ \ \text{for all} \ \ t \in \mathbb{R}^+.$$

In this study, by using the Meir-Keeler mapping, cyclic Kannan contraction and cyclic Chatterjee contraction, we establish the notions of cyclic Meir-Keeler-Kannan-Chatterjea contraction $T : A \cup B \to A \cup B$ and cyclic Meir-Keeler-Kannan-Chatterjea contractive pair (T, S) of mappings $T : A \to B$ and $S : B \to A$, and then we prove some best proximity point theorems for these various types of cyclic contractions. Our results generalize or improve many recent best proximity point theorems in the literature.

2. Main Results (I)

In this section, we first recalled the following notions of cyclic Kannan contractions and Chatterjee contractions for the cyclic mapping $T : A \cup B \to A \cup B$.

Definition 2.1. *Let A and B be two nonempty subsets of a metric space (X, d), and let $T : A \cup B \to A \cup B$ be a cyclic mapping. Then*

(1) *T is said to be a cyclic Kannan contraction if*

$$d(Tx, Ty) \leq k(d(x, Tx) + d(y, Ty)) + (1 - 2k)d(A, B),$$

for all $x \in A$ and $y \in B$, where $k \in (0, \frac{1}{2})$.

(2) *T is said to be a cyclic Chatterjee contraction if*

$$d(Tx, Ty) \leq k(d(x, Ty) + d(y, Tx)) + (1 - 2k)d(A, B),$$

for all $x \in A$ and $y \in B$, where $k \in (0, \frac{1}{2})$.

By using the Meir-Keeler mapping, cyclic Kannan contraction and Chatterjee contraction, we establish the new notion of cyclic Meir-Keeler-Kannan-Chatterjea contraction, as follows:

Definition 2.2. *Let A and B be nonempty subsets of a metric space (X, d), and let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a Meir-Keeler mapping. Then the mapping $T : A \cup B \to A \cup B$ is said to be a cyclic Meir-Keeler-Kannan-Chatterjea contraction, if the following conditions hold:*

(1) *$T : A \cup B \to A \cup B$ is a cyclic mapping,*

(2) *for all $x \in A$ and $y \in B$,*

$$d(Tx, Ty) - d(A, B) \leq \phi(\frac{d(x, Tx) + d(y, Ty) + d(x, Ty) + d(y, Tx)}{4} - d(A, B)).$$

Lemma 2.3. *Let A and B be nonempty closed subsets of a metric space (X,d), and let $\phi:\mathbb{R}^+\to\mathbb{R}^+$ be an increasing Meir-Keeler mapping. Let $T:A\cup B\to A\cup B$ be a cyclic Meir-Keeler-Kannan-Chatterjea contraction. For $x_0\in A\cup B$, define $x_{n+1}=Tx_n$ for each $n\in\mathbb{N}\cup\{0\}$. Then*

$$d(x_n,x_{n+1})\to d(A,B), \quad as \;\; n\to\infty.$$

Proof. Since $T:A\cup B\to A\cup B$ is a cyclic Meir-Keeler-Kannan-Chatterjea contraction, we obtain that for each $n\in\mathbb{N}\cup\{0\}$,

$$\begin{aligned}
&d(x_{n+2},x_{n+1})-d(A,B)\\
=&d(Tx_{n+1},Tx_n)-d(A,B)\\
\leq&\phi(\frac{d(x_{n+1},Tx_{n+1})+d(x_n,Tx_n)+d(x_{n+1},Tx_n)+d(x_n,Tx_{n+1})}{4}-d(A,B))\\
=&\phi(\frac{d(x_{n+1},x_{n+2})+d(x_n,x_{n+1})+d(x_{n+1},x_{n+1})+d(x_n,x_{n+2})}{4}-d(A,B)).
\end{aligned}$$

Since ϕ is a Meir-Keeler mapping, we have that

$$\begin{aligned}
&d(x_{n+2},x_{n+1})-d(A,B)\\
\leq&\phi(\frac{d(x_{n+1},x_{n+2})+d(x_n,x_{n+1})+d(x_{n+1},x_{n+1})+d(x_n,x_{n+2})}{4}-d(A,B))\\
<&\frac{d(x_{n+1},x_{n+2})+d(x_n,x_{n+1})+0+d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})}{4}-d(A,B)\\
=&\frac{d(x_{n+1},x_{n+2})+d(x_n,x_{n+1})}{2}-d(A,B).
\end{aligned}$$

Thus, we can conclude that for each $n\in\mathbb{N}\cup\{0\}$,

$$d(x_{n+2},x_{n+1})-d(A,B)<d(x_{n+1},x_n)-d(A,B),$$

that is, $\{d(x_{n+1},x_n)-d(A,B)\}$ is decreasing and is bounded below, so there exists $\gamma\geq 0$ such that

$$d(x_n,x_{n+1})-d(A,B)\to\gamma, \;\; \text{as} \;\; n\to\infty.$$

Notice that

$$\gamma=inf\{d(x_n,x_{n+1})-d(A,B):\; n\in\mathbb{N}\cup\{0\}\}$$

We claim that $\gamma=0$. Suppose, on the the contrary, that $\gamma>0$. Since ϕ is a Meir-Keeler mapping, corresponding to γ , there exist a η and a natural number k_0 such that

$$\gamma\leq d(x_k,x_{k+1})-d(A,B)\leq\gamma+\eta, \;\; \text{for all } n\geq k_0.$$

Since $T:A\cup B\to A\cup B$ is a cyclic Meir-Keeler-Kannan-Chatterjea contraction and ϕ is an increasing Meir-Keeler mapping, we have that:

$$\begin{aligned}
&d(x_{k+2},x_{k+1})-d(A,B)\\
=&d(Tx_{k+1},Tx_k)-d(A,B)\\
\leq&\phi(\frac{d(x_{k+1},Tx_{k+1})+d(x_k,Tx_k)+d(x_{k+1},Tx_k)+d(x_k,Tx_{k+1})}{4}-d(A,B)\\
\leq&\phi(\frac{d(x_{k+1},x_{k+2})+d(x_k,x_{k+1})+d(x_{k+1},x_{k+1})+d(x_k,x_{k+2})}{4}-d(A,B))\\
\leq&\phi(d(x_k,x_{k+1})-d(A,B))<\gamma,
\end{aligned}$$

which implies a contradiction. Thus, we get $\gamma=0$, and we have

$$d(x_n, x_{n+1}) - d(A,B) \to 0, \text{ as } n \to \infty,$$

that is,

$$d(x_n, x_{n+1}) \to d(A,B), \text{ as } n \to \infty.$$

□

We now establish the following best proximity point theorem of the cyclic Meir-Keeler-Kannan-Chatterjea contraction $T : A \cup B \to A \cup B$.

Theorem 2.4. *Let A and B be nonempty closed subsets of a complete metric space (X,d), let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing Meir-Keeler mapping, and let $T : A \cup B \to A \cup B$ be a cyclic Meir-Keeler-Kannan-Chatterjea contraction. For $x_0 \in A \cup B$, define $x_{n+1} = Tx_n$ for each $n \in \mathbb{N} \cup \{0\}$. Then we have*

(1) *If $x_0 \in A$ and $\{x_{2n}\}$ has a subsequence $\{x_{2n_k}\}$ converges to $\mu \in A$, then μ is a best proximity point of T.*

(2) *If $x_0 \in B$ and $\{x_{2n-1}\}$ has a subsequence $\{x_{2n_k-1}\}$ converges to $\nu \in B$, then ν is a best proximity point of T.*

Proof. Assume that $x_0 \in A$. Since T is cyclic, $x_{2n} \in A$ and $x_{2n+1} \in B$ for all $n \in \mathbb{N} \cup \{0\}$. Now, if $\{x_{2n}\}$ has a subsequence $\{x_{2n_k}\}$ converges to $\mu \in A$ with $d(\mu,\mu) = 0$, then

$$\lim_{n\to\infty} d(x_{2n}, \mu) = d(\mu,\mu) = 0.$$

Since T is cyclic Meir-Keeler-Kannan-Chatterjea contraction and ϕ is an increasing Meir-Keeler mapping, we have

$$\begin{aligned}
&d(\mu, T\mu) - d(A,B)\\
&\leq d(\mu, x_{2n_k}) + d(x_{2n_k}, T\mu) - d(A,B)\\
&\leq d(\mu, x_{2n_k}) + d(Tx_{2n_k-1}, T\mu) - d(A,B)\\
&\leq d(\mu, x_{2n_k}) + \phi(\frac{d(x_{2n_k-1}, Tx_{2n_k-1}) + d(\mu, T\mu) + d(x_{2n_k-1}, T\mu) + d(\mu, Tx_{2n_k-1})}{4} - d(A,B))\\
&\leq d(\mu, x_{2n_k}) + \phi(\frac{d(x_{2n_k-1}, x_{2n_k}) + d(\mu, T\mu) + d(x_{2n_k-1}, T\mu) + d(\mu, x_{2n_k})}{4} - d(A,B))\\
&\leq d(\mu, x_{2n_k}) + \phi(\frac{2d(x_{2n_k-1}, x_{2n_k}) + 2d(\mu, T\mu) + 2d(\mu, x_{2n_k})}{4} - d(A,B))\\
&< d(\mu, x_{2n_k}) + \frac{d(x_{2n_k-1}, x_{2n_k}) + d(\mu, T\mu) + d(\mu, x_{2n_k})}{2} - d(A,B).
\end{aligned}$$

Letting $k \to \infty$, by Lemma 2.3, we obtain

$$d(\mu, T\mu) - d(A,B) < \frac{d(A,B) + d(\mu, T\mu)}{2} - d(A,B) = \frac{d(\mu, T\mu) - d(A,B)}{2}.$$

Thus, we can conclude that $d(\mu, T\mu) = d(A,B)$, that is, μ is a best proximity point of T.

The proof of (2) is similar to (1), we omit it. □ □

Apply Theorem 2.4, we are easy to obtain the following corollaries. We introduce the following notions of cyclic Meir-Keeler-Kannan contractions and cyclic Meir-Keeler-Chatterjea contractions.

Definition 2.5. *Let A and B be nonempty subsets of a metric space (X,d), and let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a Meir-Keeler mapping. Then the mapping $T : A \cup B \to A \cup B$ is said to be a cyclic Meir-Keeler-Kannan contraction, if the following conditions hold:*

(1) *$T : A \cup B \to A \cup B$ is a cyclic mapping.*

(2) *for all $x \in A$ and $y \in B$,*

$$d(Tx, Ty) - d(A, B) \leq \phi(\frac{d(x, Tx) + d(y, Ty)}{2} - d(A, B)).$$

Definition 2.6. *Let A and B be nonempty subsets of a metric space (X, d), and let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a Meir-Keeler mapping. Then the mapping $T : A \cup B \to A \cup B$ is said to be a cyclic Meir-Keeler-Chatterjea contraction, if the following conditions hold:*

(1) *$T : A \cup B \to A \cup B$ is a cyclic mapping.*

(2) *for all $x \in A$ and $y \in B$,*

$$d(Tx, Ty) - d(A, B) \leq \phi(\frac{d(x, Ty) + d(y, Tx)}{2} - d(A, B)).$$

Corollary 2.7. *Let A and B be nonempty closed subsets of a complete metric space (X, d), and let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing Meir-Keeler mapping. Let $T : A \cup B \to A \cup B$ be a cyclic Meir-Keeler-Kannan contraction. For $x_0 \in A \cup B$, define $x_{n+1} = Tx_n$ for each $n \in \mathbb{N} \cup \{0\}$. Then we have*

(1) *If $x_0 \in A$ and $\{x_{2n}\}$ has a subsequence $\{x_{2n_k}\}$ converges to $\mu \in A$, then μ is a best proximity point of T.*

(2) *If $x_0 \in B$ and $\{x_{2n-1}\}$ has a subsequence $\{x_{2n_k-1}\}$ converges to $\nu \in B$, then ν is a best proximity point of T.*

Corollary 2.8. *Let A and B be nonempty closed subsets of a complete metric space (X, d), and let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing Meir-Keeler mapping. Let $T : A \cup B \to A \cup B$ be a cyclic Meir-Keeler-Chatterjea contraction. For $x_0 \in A \cup B$, define $x_{n+1} = Tx_n$ for each $n \in \mathbb{N} \cup \{0\}$. Then we have*

(1) *If $x_0 \in A$ and $\{x_{2n}\}$ has a subsequence $\{x_{2n_k}\}$ converges to $\mu \in A$, then μ is a best proximity point of T.*

(2) *If $x_0 \in B$ and $\{x_{2n-1}\}$ has a subsequence $\{x_{2n_k-1}\}$ converges to $\nu \in B$, then ν is a best proximity point of T.*

3. Main Results (II)

On the other hand, the best proximity point theorems for various types of contractions have been obtained in [3, 5, 7, 8, 13]. Particularly, in [12] the authors prove some best proximity point theorems for the pair (T, S) of cyclic Kannan mappings and cyclic Chatterjea mappings in the frameworks of metric spaces.

Definition 3.1. *[12] Let A and B be nonempty subsets of a metric space (X, d). A pair (T, S) of mappings $T : A \to B$ and $S : B \to A$ is said to form a cyclic Kannan mapping between A and B if there exists a nonnegative real number $k < \frac{1}{2}$ such that*

$$d(TX, Sy) \leq k[d(x, Tx) + d(y, Sy)] + (1 - 2k)d(A, B),$$

for all $x \in A$ and $y \in B$.

Definition 3.2. *[12] Let A and B be nonempty subsets of a metric space (X, d). A pair (T, S) of mappings $T : A \to B$ and $S : B \to A$ is said to form a cyclic Chatterjea mapping between A and B if there exists a nonnegative real number $k < \frac{1}{2}$ such that*

$$d(TX, Sy) \leq k[d(y, Tx) + d(x, Sy)] + (1 - 2k)d(A, B),$$

for all $x \in A$ and $y \in B$.

By the Meir-Keeler mapping, Defintion 3.1 and Defintion 3.2, we introduce the new notion of cyclic Meir-Keeler-Kannan-Chatterjea contractive pair (T, S), as follows:

Definition 3.3. *Let A and B be nonempty subsets of a metric space (X, d), and let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a Meir-Keeler mapping. A pair (T, S) of mappings $T : A \to B$ and $S : B \to A$ is said to form a cyclic Meir-Keeler-Kannan-Chatterjea contractive pair between A and B if*

$$d(Tx, Sy) - d(A, B) \le \phi(\frac{d(x, Tx) + d(y, Sy) + d(y, Tx) + d(x, Sy)}{4} - d(A, B)),$$

for all $x \in A$ and $y \in B$.

Lemma 3.4. *Let A and B be nonempty subsets of a metric space (X, d), and let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be an incresing Meir-Keeler mapping. Suppose that the pair (T, S) of mappings $T : A \to B$ and $S : B \to A$ form a cyclic Meir-Keeler-Kannan-Chatterjea contractive pair between A and B. Then there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = d(A, B).$$

Proof. Let $x_0 \in A$ be given, and let $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. Since the pair (T, S) forms a cyclic Meir-Keeler-Kannan-Chatterjea contractive pair between A and B, we have that for $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} & d(x_{2n+1}, x_{2n+2}) - d(A, B) = d(Tx_{2n}, Sx_{2n+1}) - d(A, B) \\ \le\ & \phi(\frac{d(x_{2n}, Tx_{2n}) + d(x_{2n+1}, Sx_{2n+1}) + d(x_{2n+1}, Tx_{2n}) + d(x_{2n}, Sx_{2n+1})}{4} - d(A, B)) \\ \le\ & \phi(\frac{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1}) + d(x_{2n}, x_{2n+2})}{4} - d(A, B)). \end{aligned}$$

Since ϕ is a Meir-Keeler mapping, we obtain that for each $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} & d(x_{2n+1}, x_{2n+2}) - d(A, B) = d(Tx_{2n}, Sx_{2n+1}) - d(A, B) \\ <\ & \frac{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1}) + d(x_{2n}, x_{2n+2})}{4} - d(A, B) \\ =\ & \frac{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})}{2} - d(A, B). \end{aligned}$$

Thus, we conclude that $d(x_{2n+1}, x_{2n+2}) < d(x_{2n}, x_{2n+1})$ for all $n \in \mathbb{N} \cup \{0\}$, that is, for all $n \in \mathbb{N} \cup \{0\}$,

$$d(x_{2n+1}, x_{2n+2}) - d(A, B) < d(x_{2n}, x_{2n+1}) - d(A, B).$$

Similarly, we can conclude that $d(x_{2n}, x_{2n+1}) < d(x_{2n-1}, x_{2n})$ for all $n \in \mathbb{N}\cup\{0\}$, that is, for all $n \in \mathbb{N}\cup\{0\}$,

$$d(x_{2n}, x_{2n+1}) - d(A, B) < d(x_{2n-1}, x_{2n}) - d(A, B).$$

By the above argument, we conclude that $\{d(x_n, x_{n+1}) - d(A, B)\}_{n\in\mathbb{N}\cup\{0\}}$ is decreasing and bounded below, so there exists $\gamma \ge 0$ such that

$$d(x_n, x_{n+1}) - d(A, B) \to \gamma, \ \text{ as } \ n \to \infty.$$

Notice that

$$\gamma = inf\{d(x_n, x_{n+1}) - d(A, B) : \ n \in \mathbb{N} \cup \{0\}\}$$

We now claim that $\gamma = 0$. Suppose, on the the contrary, that $\gamma > 0$. Since ϕ is a Meir-Keeler mapping, corresponding to γ , there exist a η and a natural number k_0 such that

$$\gamma \le d(x_k, x_{k+1}) - d(A, B) \le \gamma + \eta, \ \text{ for all } n \ge k_0.$$

Since the pair (T, S) forms a cyclic Meir-Keeler-Kannan-Chatterjea contractive pair between A and B and ϕ is increasing, we have

$$
\begin{aligned}
& d(x_{k+1}, x_{k+2}) - d(A, B) \\
=& d(Tx_k, Tx_{k+1}) - d(A, B) \\
\leq& \phi(\frac{d(x_k, Tx_k) + d(x_{k+1}, Tx_{k+1}) + d(x_{k+1}, Tx_k) + d(x_k, Tx_{k+1})}{4} - d(A, B) \\
\leq& \phi(\frac{d(x_k, x_{k+1}) + d(x_{k+1}, x_{k+2}) + d(x_{k+1}, x_{k+1}) + d(x_k, x_{k+2})}{4} - d(A, B)) \\
\leq& \phi(d(x_k, x_{k+1}) - d(A, B)) < \gamma,
\end{aligned}
$$

which implies a contradiction. Thus, we get $\gamma = 0$, and we have

$$d(x_n, x_{n+1}) - d(A, B) \to 0, \text{ as } n \to \infty,$$

that is,

$$d(x_n, x_{n+1}) \to d(A, B), \text{ as } n \to \infty.$$

□

Lemma 3.5. *Let A and B be nonempty subsets of a metric space (X, d), and let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be an incresing Meir-Keeler mapping. Suppose that the pair (T, S) of mappings $T : A \to B$ and $S : B \to A$ form a cyclic Meir-Keeler-Kannan-Chatterjea contractive pair between A and B. For a fixed point $x_0 \in A$, let $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$. Then the sequence x_n is bounded.*

Proof. It follows from Lemma 3.4 that the sequence $\{d(x_{2n-1}, x_{2n})\}$ is convergent and hence it is bounded. Since the pair (T, S) forms a cyclic Meir-Keeler-Kannan-Chatterjea contractive pair between A and B such that for $x_0 \in A$ and $x_{2n-1} \in B$, we have

$$
\begin{aligned}
& d(x_{2n}, Tx_0) - d(A, B) \\
=& d(Sx_{2n-1}, Tx_0) - d(A, B) \\
=& d(Tx_0, Sx_{2n-1}) - d(A, B) \\
\leq& \phi(\frac{d(x_0, Tx_0) + d(x_{2n-1}, Sx_{2n-1}) + d(x_{2n-1}, Tx_0) + d(x_0, Sx_{2n-1})}{4} - d(A, B)) \\
\leq& \phi(\frac{d(x_0, Tx_0) + d(x_{2n-1}, x_{2n}) + d(x_{2n-1}, Tx_0) + d(x_0, x_{2n})}{4} - d(A, B)) \\
<& \frac{d(x_0, Tx_0) + d(x_{2n-1}, x_{2n}) + d(x_{2n-1}, Tx_0) + d(x_0, x_{2n})}{4} - d(A, B) \\
\leq& \frac{d(x_0, Tx_0) + d(x_{2n-1}, x_{2n}) + d(x_{2n}, Tx_0)}{2} - d(A, B).
\end{aligned}
$$

Thus, we conclude that

$$d(x_{2n}, Tx_0) < d(x_0, Tx_0) + d(x_{2n-1}, x_{2n}).$$

Therefore, the sequence $\{x_{2n}\}$ is bounded. Similarly, it can be shown that $\{x_{2n+1}\}$ is also bounded. So we complete the proof. □

Apply Lemma 3.4 and Lemma 3.5, we prove the best proximity points theorem of cyclic Meir-Keeler-Kannan-Chatterjea contractive pair (T, S).

Theorem 3.6. *Let A and B be nonempty subsets of a metric space (X, d), and let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be an incresing Meir-Keeler mapping. Suppose that the pair (T, S) of mappings $T : A \to B$ and $S : B \to A$ form a cyclic Meir-Keeler-Kannan-Chatterjea contractive pair between A and B. For a fixed point $x_0 \in A$, let $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$. Suppose that the sequence $\{x_{2n}\}$ has a subsequence converging to some element x in A. Then, x is a best proximity point of T.*

Proof. Suppose that a subsequence $\{x_{2n_k}\}$ converges to x in A. It follows from Lemma 3.4 that $\{d(x_{2n_k-1}, x_{2n_k})\}$ converges to $d(A, B)$. Since the pair (T, S) forms a cyclic Meir-Keeler-Kannan-Chatterjea contractive pair between A and B, we have that for each $2n_k \in \mathbb{N}$,

$$\begin{aligned}
& d(x_{2n_k}, Tx) - d(A, B) \\
= & d(Tx, x_{2n_k}) - d(A, B) \\
= & d(Tx, Sx_{2n_k-1}) - d(A, B) \\
\leq & \phi(\frac{d(x, Tx) + d(x_{2n_k-1}, Sx_{2n_k-1}) + d(x_{2n_k-1}, Tx) + d(x, Sx_{2n_k-1})}{4} - d(A, B)) \\
\leq & \phi(\frac{d(x, Tx) + d(x_{2n_k-1}, x_{2n_k}) + d(x_{2n_k-1}, Tx) + d(x, x_{2n_k})}{4} - d(A, B)) \\
< & \frac{d(x, Tx) + d(x_{2n_k-1}, x_{2n_k}) + d(x_{2n_k-1}, Tx) + d(x, x_{2n_k})}{4} - d(A, B) \\
\leq & \frac{d(x, Tx) + 2d(x_{2n_k-1}, x_{2n_k}) + d(x_{2n_k}, Tx) + d(x, x_{2n_k})}{4} - d(A, B).
\end{aligned}$$

Letting $k \to \infty$. Then we conclude that

$$d(x, Tx) \leq \frac{d(x, Tx) + d(A, B)}{2}, \text{ that is, } d(x, Tx) \leq d(A, B).$$

So we can conclude that $d(x, Tx) = d(A, B)$, so x is a best proximity point of T. □

We next introduce the notions of cyclic Meir-Keeler-Kannan contractive pair and cyclic Meir-Keeler-Chatterjea contractive pair.

Definition 3.7. *Let A and B be nonempty subsets of a metric space (X, d), and let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a Meir-Keeler mapping. A pair (T, S) of mappings $T : A \to B$ and $S : B \to A$ is said to form a cyclic Meir-Keeler-Kannan contractive pair between A and B if*

$$d(Tx, Sy) - d(A, B) \leq \phi(\frac{d(x, Tx) + d(y, Sy)}{2} - d(A, B)),$$

for all $x \in A$ and $y \in B$.

Definition 3.8. *Let A and B be nonempty subsets of a metric space (X, d), and let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a Meir-Keeler mapping. A pair (T, S) of mappings $T : A \to B$ and $S : B \to A$ is said to form a cyclic Meir-Keeler-Chatterjea contractive pair between A and B if*

$$d(Tx, Sy) - d(A, B) \leq \phi(\frac{d(y, Tx) + d(x, Sy)}{2} - d(A, B)),$$

for all $x \in A$ and $y \in B$.

Apply Theorem 3.6, we are easy to get the following corollaries.

Corollary 3.9. *Let A and B be nonempty subsets of a metric space (X, d), and let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be an incresing Meir-Keeler mapping. Suppose that the pair (T, S) of mappings $T : A \to B$ and $S : B \to A$ form a cyclic Meir-Keeler-Kannan contractive pair between A and B. For a fixed point $x_0 \in A$, let $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$. Suppose that the sequence $\{x_{2n}\}$ has a subsequence converging to some element x in A. Then, x is a best proximity point of T.*

Corollary 3.10. *Let A and B be nonempty subsets of a metric space (X, d), and let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be an incresing Meir-Keeler mapping. Suppose that the pair (T, S) of mappings $T : A \to B$ and $S : B \to A$ form a cyclic Meir-Keeler-Chatterjea contractive pair between A and B. For a fixed point $x_0 \in A$, let $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$. Suppose that the sequence $\{x_{2n}\}$ has a subsequence converging to some element x in A. Then, x is a best proximity point of T.*

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