Konuralp Journal of Mathematics, 9 (2) (2021) 337-341

Konuralp Journal of Mathematics

Research Paper

https://dergipark.org.tr/en/pub/konuralpjournalmath

e-ISSN: 2147-625X

Relationships between Mahler Expansion and Higher Order q-Daehee polynomials

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Abstract

In this paper, multifarious formulas for p-adic gamma function by means of their Mahler expansion and higher order q-Volkenborn integral on $\mathbb{Z}_p$ are investigated. Then, some higher order q-Volkenborn integrals of p-adic gamma function in terms of both the higher order q-Daehee polynomials and higher order q-Daehee polynomials of the second kind are derived. Moreover, diverse higher order q-Volkenborn integrals of the derivative of p-adic gamma function associated with the Stirling numbers of the both kinds and the q-Bernoulli polynomials of order k are acquired.

Keywords: *q-numbers; p-adic numbers; p-adic gamma function; Mahler expansion; Bernoulli polynomials; Daehee polynomials; Stirling numbers of the first kind; Stirling numbers of the second kind.*

2010 Mathematics Subject Classification: *05A10, 05A30, 11B65, 11S80, 33B15*

1. Introduction

The usual Daehee polynomials $D_m(y)$ are given by (*cf.* [13])

$$\frac{\log(1+t)}{t}(1+t)^y = \sum_{m=0}^{\infty} D_m(y)\frac{t^m}{m!}. \tag{1.1}$$

When $y=0$ in (1.1), we attain $D_m(0) := D_m$ called m-th Daehee number, see [2-4, 6, 10, 12, 13, 18, 20] for more details.
Let $\mathbb{Z}$ be the set of all integers, $\mathbb{Q}$ be the field of rational numbers, $\mathbb{Z}_p$ be the ring of the p-adic integers, $\mathbb{Q}_p$ be the field of the p-adic numbers, and $\mathbb{C}_p$ be the p-adic completion of an algebraic closure of $\mathbb{Q}_p$, where p be a fixed prime number (*cf.* [1-20]). Let $\mathbb{N}^* = \mathbb{N} \cup \{0\}$ and $\mathbb{N} = \{1,2,3,\cdots\}$.
The familiar p-adic Haar distribution μ_0 and the Volkenborn integral $I_0(f)$ of a function

$$f \in UD(\mathbb{Z}_p, \mathbb{C}_p) = \left\{ f \,\middle|\, f : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function} \right\},$$

respectively, are given by (*cf.* [2-4, 6, 8-10, 12-17, 18-20])

$$\mu_0\left(z + p^m\mathbb{Z}_p\right) = 1/p^m \tag{1.2}$$

and

$$I_0(f) = \int_{\mathbb{Z}_p} f(y)\, d\mu_0(y) = \lim_{m\to\infty} \frac{1}{p^m} \sum_{y=0}^{p^m-1} f(y) \tag{1.3}$$

which yields the Daehee polynomials $D_m(y)$ and Daehee numbers D_m, for $m \in \mathbb{N}^*$, as follows

$$D_m(y) = \int_{\mathbb{Z}_p} (y+z)_m\, d\mu_0(z) \text{ and } D_m = \int_{\mathbb{Z}_p} (z)_m\, d\mu_0(z),$$

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where $(y)_m = y(y-1)(y-2)\cdots(y-m+1)$ with $(y)_0 = 1$ and (*cf.* [2-4, 6, 10, 12, 13, 18, 20]). The following relation is valid:

$$(y)_m = \sum_{k=0}^{m} S_1(m,k)\, y^k, \tag{1.4}$$

where $S_1(m,k)$ are called the Stirling numbers of the first kind (see [1-4, 6, 7, 10, 12, 13, 18, 20]).
The notation q may be varyingly considered as indeterminate, complex number $q \in \mathbb{C}$ with $0 < |q| < 1$, or p-adic number $q \in \mathbb{C}_p$ with $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^y = \exp(y \log q)$ for $|y|_p \le 1$, where $|.|_p$ indicates the p-adic norm on $\mathbb{C}_p$ normalized by $|p|_p = 1/p$.
The q-extension of the Volkenborn integral of a function $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$ is defined by (*cf.* [2-4, 6, 8, 12, 18])

$$I_q(f) = \lim_{m\to\infty} \frac{1}{[p^m]_q} \sum_{y=0}^{p^m-1} f(y)\, q^y = \int_{\mathbb{Z}_p} f(y)\, d\mu_q(y). \tag{1.5}$$

Suppose that $f_1(y) = f(y+1)$. Then, we see that

$$qI_q(f_1) = (q-1)\, f(0) + \frac{q-1}{\log q} f'(0) + I_q(f). \tag{1.6}$$

For $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$, the higher order q-Daehee numbers $D_{m,q}^{(k)}$ and polynomials $D_{m,q}^{(k)}(y)$ are defined by (*cf.* [4])

$$D_{m,q}^{(k)} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (y_1 + \cdots + y_k)_m\, d\mu_q(y_1) \cdots d\mu_q(y_k), \tag{1.7}$$

$$D_{m,q}^{(k)}(y) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (y_1 + \cdots + y_k + y)_m\, d\mu_q(y_1) \cdots d\mu_q(y_k). \tag{1.8}$$

When $k = 1$, it is obvious that $\lim_{q\to 1} D_{m,q}^{(1)} := D_m$ and $\lim_{q\to 1} D_{m,q}^{(1)}(y) := D_m(y)$.
For $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$, the higher order q-Daehee numbers $\widehat{D}_{m,q}^{(k)}$ and polynomials $\widehat{D}_{m,q}^{(k)}(y)$ of the second kind are introduced by (*cf.* [4])

$$\widehat{D}_{m,q}^{(k)} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-y_1 - \cdots - y_k)_m\, d\mu_q(y_1) \cdots d\mu_q(y_k), \tag{1.9}$$

$$\widehat{D}_{m,q}^{(k)}(y) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-y_1 - \cdots - y_k + y)_m\, d\mu_q(y_1) \cdots d\mu_q(y_k). \tag{1.10}$$

The Daehee numbers and polynomials in conjuction with their diverse generalizations have been recently studied by many mathematicians, *cf.* [2-4, 6, 10, 12, 13, 18, 20]. For example, Araci [1] considered degenerate q-Daehee polynomials with weight α and the degenerate q-Daehee polynomials of higher order with weight α by using q-Volkenborn integrals on $\mathbb{Z}_p$ and then he derived some summation formulae and properties. Jang et al. [10] defined the degenerate Daehee polynomials of the third kind and developed several novel identities and relations between the degenerate Daehee polynomials of the third kind and the Korobov polynomials. Cho et al. introduced the Daehee numbers and polynomials of order k and provided their some identities and relationships. Simsek and Yardimci [20], by utilizing generating functions and p-adic Volkenborn integral, derived diverse properties of the Bernstein basis functions, the Apostol-Daehee numbers and polynomials, Apostol-Bernoulli polynomials, some special numbers including the Stirling numbers, the Euler numbers, the Daehee numbers, and the Changhee numbers. By using functional equations and an integral equation of their partial differential equations and the generating functions, they gave a recurrence relation for the Apostol-Daehee polynomials in [20].
In this paper, we investigate multifarious formulas for p-adic gamma function by means of their Mahler expansion and higher order q-Volkenborn integral on $\mathbb{Z}_p$. Then, we derive some higher order q-Volkenborn integrals of p-adic gamma function in terms of both the higher order q-Daehee polynomials and higher order q-Daehee polynomials of the second kind. Moreover, we acquire diverse higher order q-Volkenborn integrals of the derivative of p-adic gamma function associated with the Stirling numbers of the both kinds and the q-Bernoulli polynomials of order k.

2. Higher Order q-Daehee Polynomials Associated with Mahler Theorem

The p-adic gamma function is defined as follows

$$\Gamma_p(y) = \lim_{m\to y} (-1)^m \prod_{\substack{j<m \\ (p,j)=1}} j \qquad (y \in \mathbb{Z}_p), \tag{2.1}$$

where m approaches y through positive integers. The p-adic gamma function in conjuction with its diverse extensions have been studied and progressed broadly by many mathematicians, *cf.* [5, 6, 8, 9, 11, 14, 16, 17, 19].
For $y \in \mathbb{Z}_p$. let $\binom{y}{m} = \frac{y(y-1)\cdots(y-m+1)}{m!}$ $(m \in \mathbb{N})$ with $\binom{y}{0} = 1$.
For $y \in \mathbb{Z}_p$ and $m \in \mathbb{N}$, the functions $y \to \binom{y}{m}$ form an orthonormal base of the space $C(\mathbb{Z}_p \to \mathbb{C}_p)$ with respect to the Euclidean norm $\|\cdot\|_\infty$, which satisfies the following equality (see [6, 8, 9, 16, 19])

$$\binom{y}{m}' = \sum_{j=0}^{m-1} \frac{(-1)^{m-j-1}}{m-j} \binom{y}{j}. \tag{2.2}$$

An extension for continuous maps of a p-adic variable using the special functions as binomial coefficient polynomial is investigated by Mahler [15], which implies that for any $f \in C(\mathbb{Z}_p \to \mathbb{C}_p)$, there exist unique elements $a_0, a_1, a_2, \ldots$ of $\mathbb{C}_p$ such that

$$f(y) = \sum_{m=0}^{\infty} a_m \binom{y}{m}.$$

The base $\left\{\binom{*}{m} : m \in \mathbb{N}\right\}$ is termed as Mahler base of the space $C\left(\mathbb{Z}_p \to \mathbb{C}_p\right)$, and the components $\{a_m : m \in \mathbb{N}\}$ in $f(y) = \sum_{m=0}^{\infty} a_m \binom{y}{m}$ are termed Mahler coefficients of $f \in C\left(\mathbb{Z}_p \to \mathbb{C}_p\right)$.
The Mahler expansion with coefficients of the p-adic gamma function Γ_p is provided (*cf.* [19]) as follows.

Theorem 2.1. *For $y \in \mathbb{Z}_p$, let $\Gamma_p(y+1) = \sum_{m=0}^{\infty} a_m \binom{y}{m}$ be Mahler series of Γ_p. Then its coefficients satisfy the following identity:*

$$\sum_{m \geqq 0} (-1)^{m+1} a_m \frac{y^m}{m!} = \frac{1-y^p}{1-y} \exp\left(y + \frac{y^p}{p}\right). \tag{2.3}$$

We give the following theorem.

Theorem 2.2. *For $y, y_i \in \mathbb{Z}_p$ where $i \in \{1,2,\ldots k\}$, we have*

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \Gamma_p(y_1 + \cdots + y_k + y + 1)\, d\mu_q(y_1) \cdots d\mu_q(y_k) = \sum_{m=0}^{\infty} a_m \frac{D_{m,q}^{(k)}(y)}{m!}, \tag{2.4}$$

where a_m is provided by Theorem 2.1.

Proof. Let $y, y_i \in \mathbb{Z}_p$ where $i \in \{1,2,\ldots k\}$. By using the relation $\binom{y_1+y_2}{m} = \frac{(y_1+y_2)_m}{m!}$ and Theorem 2.1, we get

$$\begin{aligned} & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \Gamma_p(y_1 + \cdots + y_k + y + 1)\, d\mu_q(y_1) \cdots d\mu_q(y_k) \\ = & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{m=0}^{\infty} a_m \frac{(y_1 + \cdots + y_k + y)_m}{m!} d\mu_q(y_1) \cdots d\mu_q(y_k) \\ = & \sum_{m=0}^{\infty} a_m \frac{1}{m!} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (y_1 + \cdots + y_k + y)_m\, d\mu_q(y_1) \cdots d\mu_q(y_k), \end{aligned}$$

which is the desired result (2.4) via the formula (1.8). □

We here analyze an outcome of the Theorem 2.2 as follows.

Remark 2.3. *Taking $y = 0$ in Theorem 2.2 means the following relation*

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \Gamma_p(y_1 + \cdots + y_k + 1)\, d\mu_q(y_1) \cdots d\mu_q(y_k) = \sum_{m=0}^{\infty} a_m \frac{D_{m,q}^{(k)}}{m!}, \tag{2.5}$$

where a_m is provided by Theorem 2.1.

Let $m \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Cho *et al.* [4] gave the following correlation:

$$D_{m,q}^{(k)}(y) = \sum_{u=0}^{m} S_1(m,u) B_{u,q}^{(k)}(y), \tag{2.6}$$

where $B_{u,q}^{(k)}(y)$ denotes the u-th Bernoulli polynomials of order k defined by

$$B_{u,q}^{(k)}(y) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (y_1 + \cdots + y_k + y)^u\, d\mu_q(y_1) \cdots d\mu_q(y_k) \quad (u \in \mathbb{N}_0). \tag{2.7}$$

As a result of Theorem 2.2 and relation (2.6), one other higher order q-Volkenborn integrals of the p-adic gamma function by means of the q-Bernoulli polynomials of order k is given below.

Remark 2.4. *Let $y, y_i \in \mathbb{Z}_p$ where $i \in \{1,2,\ldots k\}$. We acquire*

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \Gamma_p(y_1 + \cdots + y_k + y + 1)\, d\mu_q(y_1) \cdots d\mu_q(y_k) = \sum_{m=0}^{\infty} \sum_{u=0}^{m} a_m \frac{S_1(m,u)}{m!} B_{u,q}^{(k)}(y).$$

We give the following theorem.

Theorem 2.5. *For $y, y_i \in \mathbb{Z}_p$ where $i \in \{1,2,\ldots k\}$, we have*

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \Gamma_p'(y_1 + \cdots + y_k + y + 1)\, d\mu_q(y_1) \cdots d\mu_q(y_k) = \sum_{m=0}^{\infty} \sum_{j=0}^{m-1} a_m \frac{(-1)^{m-j-1} D_{j,q}^{(k)}(y)}{(m-j)\, j!}.$$

Proof. By Theorem 2.1, we acquire

$$\begin{aligned} & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \Gamma_p'(y_1 + \cdots + y_k + y + 1)\, d\mu_q(y_1) \cdots d\mu_q(y_k) \\ = & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{m=0}^{\infty} a_m \binom{y_1 + \cdots + y_k + y}{m}' d\mu_q(y_1) \cdots d\mu_q(y_k) \\ = & \sum_{m=0}^{\infty} a_m \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{y_1 + \cdots + y_k + y}{m}' d\mu_q(y_1) \cdots d\mu_q(y_k) \end{aligned}$$

and using (2.2), we derive

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\Gamma_p'(y_1+\cdots+y_k+y+1)\,d\mu_q(y_1)\cdots d\mu_q(y_k)\\ =&\sum_{m=0}^{\infty}\sum_{j=0}^{m-1}a_m\frac{(-1)^{m-j-1}}{m-j}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{y_1+\cdots+y_k+y}{j}d\mu_q(y_1)\cdots d\mu_q(y_k)\\ =&\sum_{m=0}^{\infty}\sum_{j=0}^{m-1}a_m\frac{(-1)^{m-j-1}}{m-j}\frac{D_{j,q}^{(k)}(y)}{j!}.\end{aligned}$$

□

A special consequence of Theorem 2.5 is stated below.

Remark 2.6. *Let $y_i\in\mathbb{Z}_p$ where $i\in\{1,2,\ldots k\}$. We have*

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\Gamma_p'(y_1+\cdots+y_k+1)\,d\mu_q(y_1)\cdots d\mu_q(y_k)=\sum_{m=0}^{\infty}\sum_{j=0}^{m-1}a_m\frac{(-1)^{m-j-1}D_{j,q}^{(k)}}{(m-j)\,j!}.$$

A relation between $\Gamma_p(y)$ and $\widehat{D}_{m,q}^{(k)}(y)$ is provided by the following theorem.

Theorem 2.7. *For $y,y_i\in\mathbb{Z}_p$ where $i\in\{1,2,\ldots k\}$, we have*

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\Gamma_p(-y_1-\cdots-y_k-y+1)\,d\mu_q(y_1)\cdots d\mu_q(y_k)=\sum_{m=0}^{\infty}a_m\frac{\widehat{D}_{m,q}^{(k)}(y)}{m!},$$

where a_m is provided by Theorem 2.1.

Proof. For $y,y_i\in\mathbb{Z}_p$ where i lies in$\{1,2,\ldots k\}$, by utilizing the relation $\binom{-y_1-y_2}{m}=\frac{(-y_1-y_2)_m}{m!}$ and Theorem 2.1, we get

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\Gamma_p(-y_1-\cdots-y_k+y+1)\,d\mu_q(y_1)\cdots d\mu_q(y_k)\\ =&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{m=0}^{\infty}a_m\frac{(-y_1-\cdots-y_k+y)_m}{m!}d\mu_q(y_1)\cdots d\mu_q(y_k)\\ =&\sum_{m=0}^{\infty}a_m\frac{1}{m!}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-y_1-\cdots-y_k+y)_m\,d\mu_q(y_1)\cdots d\mu_q(y_k),\end{aligned}$$

which means the asserted result (1.10). □

An outcome of Theorem 2.7 is stated below.

Remark 2.8. *Letting $y=0$ in Theorem 2.7 reduces the following relation*

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\Gamma_p(-y_1-\cdots-y_k+1)\,d\mu_q(y_1)\cdots d\mu_q(y_k)=\sum_{m=0}^{\infty}a_m\frac{\widehat{D}_{m,q}^{(k)}}{m!},$$

where a_m is given by Theorem 2.1.

Let $m\in\mathbb{N}_0$ and $k\in\mathbb{N}$. Cho et al. [4] gave the following correlation:

$$\widehat{D}_{m,q}^{(k)}(y)=\sum_{u=0}^{m}(-1)^{m-u}S_1(m,u)\,B_{u,q}^{(k)}(-y),$$

which yields the following result with the help of Theorem 2.7.

Remark 2.9. *For $y,y_i\in\mathbb{Z}_p$ where $i\in\{1,2,\ldots k\}$, we acquire*

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\Gamma_p(-y_1-\cdots-y_k+y+1)\,d\mu_q(y_1)\cdots d\mu_q(y_k)=\sum_{m=0}^{\infty}\sum_{u=0}^{m}(-1)^{m-u}a_m\frac{S_1(m,u)}{m!}B_{u,q}^{(k)}(-y).$$

We observe that

$$\begin{aligned}(-z)_m&=(-z)(-z+1)\cdots(-z-m+1)\\&=(-1)^m z(z-1)\cdots(z+m-1)\\&=(-1)^m\sum_{u=0}^{m}S_2(m,u)\,z^u,\end{aligned}$$

where $S_2(m,u)$ called the Stirling numbers of the second kind (*cf.* [7,20]) is given by

$$\frac{(e^t-1)^u}{u!}=\sum_{m\geq 0}S_2(m,u)\frac{t^m}{m!}.$$

We lastly state one other higher q-Volkenborn integrals of the derivative of the p-adic gamma function by means of the Stirling numbers of the second kind and the q-Bernoulli polynomials of order k.

Theorem 2.10. *For $y, y_i \in \mathbb{Z}_p$ where $i \in \{1,2,\ldots k\}$, we have*

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\Gamma'_p\left(-y_1-\cdot-y_k-y+1\right)d\mu_q\left(y_1\right)\cdots d\mu_q\left(y_k\right)=\sum_{m=0}^{\infty}\sum_{j=0}^{m-1}\sum_{u=0}^{j}S_2\left(j,u\right)a_m\frac{(-1)^{m-1}}{m-j}\frac{B_{j,q}^{(k)}\left(-y\right)}{j!}.$$

Proof. Via Theorem 2.1, we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\Gamma'_p\left(-y_1-\cdots-y_k+y+1\right)d\mu_q\left(y_1\right)\cdots d\mu_q\left(y_k\right)\\
=\;&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{m=0}^{\infty}a_m\binom{-y_1-\cdots-y_k+y}{m}'d\mu_q\left(y_1\right)\cdots d\mu_q\left(y_k\right)\\
=\;&\sum_{m=0}^{\infty}a_m\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{-y_1-\cdots-y_k+y}{m}'d\mu_q\left(y_1\right)\cdots d\mu_q\left(y_k\right)
\end{aligned}$$

and utilizing (2.2) and (2.7), we derive

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\Gamma'_p\left(-y_1-\cdots-y_k+y+1\right)d\mu_q\left(y_1\right)\cdots d\mu_q\left(y_k\right)\\
=\;&\sum_{m=0}^{\infty}\sum_{j=0}^{m-1}a_m\frac{(-1)^{m-j-1}}{m-j}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{-y_1-\cdots-y_k+y}{j}d\mu_q\left(y_1\right)\cdots d\mu_q\left(y_k\right)\\
=\;&\sum_{m=0}^{\infty}\sum_{j=0}^{m-1}a_m\frac{(-1)^{m-j-1}}{(m-j)\,j!}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(-y_1-\cdots-y_k+y\right)_j d\mu_q\left(y_1\right)\cdots d\mu_q\left(y_k\right)\\
=\;&\sum_{m=0}^{\infty}\sum_{j=0}^{m-1}a_m\frac{(-1)^{m-j-1}}{(m-j)\,j!}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-1)^j\sum_{u=0}^{j}S_2\left(j,u\right)\left(y_1+\cdots+y_k-y\right)^u d\mu_q\left(y_1\right)\cdots d\mu_q\left(y_k\right)\\
=\;&\sum_{m=0}^{\infty}\sum_{j=0}^{m-1}\sum_{u=0}^{j}S_2\left(j,u\right)a_m\frac{(-1)^{m-1}}{m-j}\frac{B_{j,q}^{(k)}\left(-y\right)}{j!}.
\end{aligned}$$

□

3. Conclusion and Observations

In the present paper, several relations for p-adic gamma function by means of their Mahler expansion and higher order q-Volkenborn integral on $\mathbb{Z}_p$ have been derived. Then, some higher order q-Volkenborn integrals of p-adic gamma function in terms of both the higher order q-Daehee polynomials and higher order q-Daehee polynomials of the second kind have been acquired. Moreover, multifarious higher order q-Volkenborn integrals of the derivative of p-adic gamma function associated with the Stirling numbers of the both kinds and the q-Bernoulli polynomials of order k have been investigated. Some results attained in this paper reduce to the results in the paper [6].

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