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Generalizations of Hyperconvex Metric Spaces

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Abstract

Since Khamsi found a KKM theorem for hyperconvex metric spaces in 1996, there have appeared a large number of works on them related to the KKM theory. In our previous review [34], we followed the various stages of developments of the KKM theory of hyperconvex metric spaces. In fact, we introduced abstracts of articles on such theory and gave comments or generalizations of the results there if necessary. We noted that many results in those articles are consequences of (partial) KKM space theory developed by ourselves from 2006. The present survey is a continuation of [34] and aims to collect further generalizations of hyperconvex metric spaces related to the KKM theory.

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1. Introduction and Preliminaries

For a long period, the study of hyperconvex metric spaces introduced by Aronszajn and Panitchpackdi [2] in 1956 was concentrated to the fixed point property of nonexpansive maps. Thirty-seven years later Horvath [8] in 1993 found that the space is one of his c-spaces, an important object of the KKM theoretic study. In fact, Khamsi [11] in 1996 stated a KKM theorem for hyperconvex metric spaces, and then there have appeared a large number of works on them related to the KKM theory; see [34] and the references therein. Later, it is known that most of the results in the KKM theory of hyperconvex metric spaces are simple consequences of much more general results on G-convex spaces due to ourselves.

In 2005, Amini-Fakhar-Zafarani [1] first attempted to generalize certain results on the KKM theory of hyperconvex metric spaces to the corresponding ones of mere metric spaces. In fact, they introduced the class of KKM-type maps on metric spaces and established some fixed point theorems and others. Their works

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are extended by Turkoglu-Abuloha-Abdeljawad [43], Khamsi-Hussain [12], Hussain-Shah [10], and others for various metric type spaces.

Since 2006, we generalized G-convex spaces to abstract convex spaces and introduced the multimap classes $\mathfrak{KC}$ and $\mathfrak{KO}$ related to generalizations of KKM maps; see [21]-[28]. As applications of our new KKM theory of abstract convex spaces, in our previous work [35], we deduced generalizations of recent results on KKM maps on metric spaces in Amini et al. [1] and known generalized KKM theorems on hyperconvex metric spaces.

More recently, Horvath [9] in 2009 introduced continuous midpoint spaces as a generalization of various types of metric spaces. Niculescu and Rovenţa [17] in 2009 extended Fan's minimax inequality to the context of metric spaces with global nonpositive curvature (NPC). As a consequence, a general result on the existence of a Nash equilibrium is obtained by them. Note that such spaces in [9] and [17] contain hyperconvex metric spaces as one of typical examples.

Our aim in this article is to introduce such various generalizations of hyperconvex metric spaces and to deepen the understanding of their nature.

This article is organized as follows: Section 2 recalls the several stages of extensions of hyperconvex metric spaces to various subclasses of abstract convex spaces. In Section 3, some contents of our KKM theory and one of the most general KKM type theorems are introduced. Section 4 devotes to metric abstract convex spaces and their basic properties. In Section 5, some metric type spaces extending hyperconvex metric spaces in [1, 10, 12, 43] are introduced, and note that the spaces in [10, 12] are not known to be partial KKM spaces. Section 6 devotes to introduce continuous midpoint spaces due to Horvath [9] generalizing hyperconvex metric spaces and many KKM theorems. In Section 7, global nonpositive curvature (NPC) spaces due to Niculescu and Rovenţa [17] are introduced. This kind of spaces are rich sources of partial KKM spaces extending hyperconvex metric spaces. Section 8 devotes to $\aleph_0$-hyperconvexity of Espinola and Lopez [6] and $\aleph_0$-spaces due to El Bansami and Riahi [5]. Finally, in Section 9, we introduce some recent works related to the subject of this article which appeared after our previous historical article [34] in 2015.

2. From hyperconvex spaces to partial KKM spaces

In this section, we follow our previous works and the references therein.

The following is due to Aronszajn and Panitchpakdi [2] in 1956:

Definition. A metric space (H, d) is said to be *hyperconvex* if $\bigcap_\alpha \mathrm{B}(x_\alpha, r_\alpha) \neq \emptyset$ for any collection $\{\mathrm{B}(x_\alpha, r_\alpha)\}$ of closed balls in H for which $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$.

It is known that a normed vector space X is not hyperconvex in general, and the spaces $(\mathbb{R}^n, ||\cdot||_\infty)$, l^∞, L^∞ and $\mathbb{R}$-trees are hyperconvex.

A nonempty topological space X is *homotopically trivial* if for any natural number n and any continuous function $f : \vartheta\Delta_n \to X$, defined on the boundary of the standard n-dimensional simplex Δ_n, there exists its continuous extension $g : \Delta_n \to X$.

Let $\langle D\rangle$ be the class of all nonempty finite subsets of a nonempty set D. We derived the following in 2019 [40]:

Definition. A triple $(X \supset D; \Gamma)$ is called a *Horvath space* if X is a topological space and $\Gamma = \{\Gamma_A\}$ a family of homotopically trivial subsets of X indexed by $A \in \langle D\rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D\rangle$.

When $D = X$ and $\Gamma = \{\Gamma_A\}$ is a family of contractible subsets of X, $(X; \Gamma) := (X, X; \Gamma)$ is called a c-space by Horvath [6, 7] or an H-space by Bardaro and Ceppitelli [4].

Horvath noted that a torus, the Möbius band, or the Klein bottle can be regarded as c-spaces, and are examples of compact c-spaces without having the fixed point property.

For any hyperconvex metric space (H, d), an *admissible subset* is defined and denoted by

$$\mathrm{BI}(A) = \mathrm{ad}(A) := \bigcap\{B \subset M \mid B \text{ is a closed ball in } H \text{ such that } A \subset B\}$$

for each $A \in \langle H \rangle$.

Since each admissible subset of a hyperconvex metric space is hyperconvex and hence contractible, the following is due to Horvath [9]:

Lemma 2.1. *Any hyperconvex metric space H is a c-space $(H; \Gamma)$, where $\Gamma_A = \mathrm{BI}(A)$ for each $A \in \langle H \rangle$.*

Note that a Horvath space becomes clearly the following well-known type of spaces due to ourselves.

Definition. A *generalized convex space* or a G-convex space $(X, D; \Gamma)$ is an abstract convex space such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n+1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$. Here, Δ_J is the face of Δ_n corresponding to $J \in \langle A \rangle$.

Definition. A *space having a family* $\{\phi_A\}_{A \in \langle D \rangle}$ or simply a ϕ_A-*space*

$$(X, D; \{\phi_A\}_{A \in \langle D \rangle}) \text{ or simply } (X, D; \phi_A)$$

consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplices) for $A \in \langle D \rangle$ with the cardinality $|A| = n+1$.

Finally we introduced the following in 2006; see [21]-[26]:

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$, where $\langle D \rangle$ is the set of all nonempty finite subsets of D, such that, for any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$.

In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Definition. Let $(E, D; \Gamma)$ be an abstract convex space and Z be a set. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_N) \subset G(N) := \bigcup_{y \in N} G(y) \qquad \text{for all } N \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

Definition. A multimap $F : E \multimap Z$ to a set Z is called *a $\mathfrak{K}$-map* if, for a KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(E, Z) := \{F : E \multimap Z \mid F \text{ is a } \mathfrak{K}\text{-map}\}.$$

Similarly, when Z is a topological space, a $\mathfrak{KC}$-*map* is defined for closed-valued maps G, and a $\mathfrak{KO}$-*map* for open-valued maps G. In this case, we denote $F \in \mathfrak{KC}(E, Z)$ [resp. $F \in \mathfrak{KO}(E, Z)$].

Definition. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement $1_E \in \mathfrak{KC}(E, E)$; that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, E) \cap \mathfrak{KO}(E, E)$; that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, respectively.

Our KKM theory concerns with the study of partial KKM spaces and their applications.

Now we have the following diagram of subclasses for abstact convex spaces $(E, D; \Gamma)$:

$$\text{Simplex} \Longrightarrow \text{Convex subset of a t.v.s.} \Longrightarrow \text{Lassonde's convex space}$$
$$\Longrightarrow \text{Horvath space} \Longrightarrow \text{G-convex space} \Longleftrightarrow \phi_A\text{-space}$$
$$\Longrightarrow \text{KKM space} \Longrightarrow \text{Partial KKM space}$$
$$\Longrightarrow \text{Abstract convex space.}$$

For typical examples of KKM spaces, see [31, 37, 38] and the references therein.

3. Contents of the KKM theory

Recall that KKM spaces satisfy a large number of results of our [30, 31].

For example, in [30], we derived generalized forms of the Ky Fan minimax inequality, the von Neumann-Sion minimax theorem, the von Neumann–Fan intersection theorem, the Fan type analytic alternative, and the Nash equilibrium theorem for partial KKM spaces. Consequently, our results in [30] unify and generalize most of previously known particular cases of the same nature.

Moreover, in [31], we clearly derived a sequence of a dozen statements which characterize the KKM spaces and equivalent formulations of the partial KKM principle. As their applications, we add more than a dozen statements including generalized formulations of von Neumann minimax theorem, von Neumann intersection lemma, the Nash equilibrium theorem, the Fan type minimax inequalities, and several variational inequality results for any KKM spaces. Consequently, [31] unifies and enlarges previously known many proper examples of such statements for particular types of partial KKM spaces. For some corrections, see [33].

From Lemma 2.1, we have

Lemma 3.1. *Every hyperconvex metric space is a KKM metric space, that is, a metric space satisfying the KKM principle.*

From this, hyperconvex metric spaces satisfy all of the results in the KKM theory appeared in [30, 31] and many other works of ourselves. Moreover, the following useful result is due to Horvath [8]:

Lemma 3.2. *Any hyperconvex metric space is a a complete metric l.c. space.*

Now we prepare to introduce one of the most general forms of the KKM theorem.

Consider the following related four conditions for a map $G : D \multimap Z$ with a topological space Z:

(a) $\bigcap_{y\in D} \overline{G(y)} \neq \emptyset$ implies $\bigcap_{y\in D} G(y) \neq \emptyset$.

(b) $\bigcap_{y\in D} \overline{G(y)} = \overline{\bigcap_{y\in D} G(y)}$ (G is *intersectionally closed-valued*).

(c) $\bigcap_{y\in D} \overline{G(y)} = \bigcap_{y\in D} G(y)$ (G is *transfer closed-valued*).

(d) G is closed-valued.

Note that Luc et al. [16] showed that (a) $\Longleftarrow$ (b) $\Longleftarrow$ (c) $\Longleftarrow$ (d), and not conversely in each step.

The following is one of the most general KKM type theorems in [32] for abstract convex spaces:

Theorem 3.3. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a topological space, $F \in \mathfrak{KC}(E, Z)$, and $G : D \multimap Z$ a map such that*

(1) *$\overline{G}$ is a KKM map w.r.t. F; and*

(2) *there exists a nonempty compact subset K of Z such that one of the following coercivity conditions holds:*

(i) *$K = Z$;*

(ii) *$\bigcap\{\overline{G(y)} \mid y \in M\} \subset K$ for some $M \in \langle D\rangle$; or*

(iii) *for each $N \in \langle D\rangle$, there exists a Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$, $\overline{F(L_N)}$ is compact, and*

$$\overline{F(L_N)} \cap \bigcap_{y \in D'} \overline{G(y)} \subset K.$$

Then we have

$$\overline{F(E)} \cap K \cap \bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

Furthermore,
(α) *if* G *is transfer closed-valued, then* $\overline{F(E)} \cap K \cap \bigcap\{G(y) \mid y \in D\} \neq \emptyset$*; and*
(β) *if* G *is intersectionally closed-valued, then* $\bigcap\{G(y) \mid y \in D\} \neq \emptyset$.

Of course, this applies to hyperconvex metric spaces and their generalizations.

From the partial KKM principle [31] or Theorem 3.3, we have a whole intersection property of the Fan type as follows:

Corollary 3.4. *Let* $(E, D; \Gamma)$ *be a partial KKM space and* $G : D \multimap E$ *a map such that*
(1) G *is a closed-valued KKM map; and*
(2) $\bigcap_{z \in M} G(z)$ *is compact for some* $M \in \langle D \rangle$
Then we have $\bigcap_{z \in D} G(z) \neq \emptyset$.

For hyperconvex metric space, the following was originally given by Khamsi [11]:

Corollary 3.5. [KKM-Map Principle] *Let* H *be a hyperconvex metric space,* X *an arbitrary subset of* H*, and* $G : X \multimap H$ *a KKM map such that each* $G(x)$ *is finitely (metrically) closed. Then the family* $\{G(x) : x \in X\}$ *has the finite intersection property.*

We need the following:

Definition. Let $(E \supset D; \Gamma)$ be an abstract convex space. Then $E_0 \subset E$ is called a *subspace* of the given abstract convex space whenever $(E_0 \supset E_0 \cap D; \Gamma')$ is an abstract convex space, where $\Gamma'_A = \Gamma_A$ for each $A \in \langle E_0 \cap D \rangle$.

From now on, we are going to find generalizations of hyperconvex metric spaces within the category of partial KKM spaces or else.

4. Metric spaces

Let A be a nonempty bounded subset of a metric space (M, d). Then we define the following as in Khamsi [11]:

(i) $\mathrm{BI}(A) = \mathrm{ad}(A) := \bigcap\{B \subset M \mid B$ is a closed ball in M such that $A \subset B\}$.
(ii) $\mathcal{A}(M) := \{A \subset M \mid A = \mathrm{ad}(A)\}$, i.e., $A \in \mathcal{A}(M)$ iff A is an intersection of closed balls. In this case we will say A is an *admissible* subset of M.
(iii) A is called *subadmissible*, if for each $N \in \langle A \rangle$, $\mathrm{ad}(N) \subset A$. Obviously, if A is an admissible subset of M, then A must be subadmissible.

For a point $x \in M$ and $t > 0$, let

$$\mathrm{B}(x, t) := \{y \in M \mid d(x, y) \leq t\} \text{ and } \mathrm{N}(x, t) := \{y \in M \mid d(x, y) < t\}.$$

We introduce new definitions:

Definition. An abstract convex space $(M \supset D; \Gamma)$ is called simply a *metric space* if (M, d) is a metric space, D is a nonempty set, and $\Gamma : \langle D \rangle \to \mathcal{A}(M)$ is a map having admissible values. A map $G : D \multimap M$ is a KKM map if $\Gamma_A \subset G(A)$ for each $A \in \langle D \rangle$.

A Γ-convex subset of $(M \supset D; \Gamma)$ is said to be *subadmissible* by some authors.

Example. For the following examples, we can make metric spaces $(M, D; \Gamma)$.

(1) Any normed vector space.
(2) Any metrizable topological vector space.
(3) Any hyperconvex metric spaces.

The following are from Park [29]:

Definition. Let X be a nonempty subset of a metric space (M, d). A map $F : X \multimap M$ is said to have the *almost fixed point property* (simply, a.f.p.p.) if for any $\varepsilon > 0$, there exists an $x_\varepsilon \in X$ such that $F(x_\varepsilon) \cap \mathrm{B}(x_\varepsilon, \varepsilon) \neq \emptyset$.

Theorem 4.1. *Let $(M \supset D; \Gamma)$ be a metric space and X a Γ-convex subset of M such that $X \cap D$ is dense in X. Suppose that $F \in \mathfrak{KC}(X, X)$ [resp. $F \in \mathfrak{KO}(X, X)$] such that $F(X)$ is totally bounded. Then F has the a.f.p.p.*

Theorem 4.2. *Let $(M \supset D; \Gamma)$ be a metric space and X a Γ-convex subset of M such that $X \cap D$ is dense in X. Then any compact closed map $F \in \mathfrak{KC}(X, X)$ [resp. $\mathfrak{KO}(X, X)$] has a fixed point.*

In [26], we had the following:

Lemma 4.3. *Let $(E, D; \Gamma)$ be an abstract convex space, Z, W two topological spaces, $F \in \mathfrak{KC}(E, Z)$ and $f : Z \to W$ a continuous function. Then $fF \in \mathfrak{KC}(E, W)$. This also holds for $\mathfrak{KO}$ instead of $\mathfrak{KC}$.*

As a consequence of Theorem 4.2 and Lemma 4.3, we obtain a Schauder type fixed point theorem for metric spaces:

Corollary 4.4. *Let $(M \supset D; \Gamma)$ be a metric space and X a Γ-convex subset of M such that $X \cap D$ is dense in X. If the identity map $1_X \in \mathfrak{KC}(X, X)$ [resp. $1_X \in \mathfrak{KO}(X, X)$], then any compact continuous function $f : X \multimap X$ has a fixed point.*

Note that Theorems 4.1, 4.2, Lemma 4.3 and Corollary 4.4 properly generalize the corresponding ones in [1]. Moreover, the authors of [1] claimed that acyclic maps defined on G-convex spaces have the KKM property; but it was already shown by Park and Kim in 1997.

Corollary 4.5. *Let H be a hyperconvex metric space and $f : H \to H$. If f is continuous and compact, then f has a fixed point.*

5. Metric type spaces

In 2005, motivated by Khamsi's earlier work [10] on hyperconvex metric spaces, Amini-Fakhar-Zafaran [1] adopted the terminology as in the previous section for metric spaces. Then they obtained a Schauder type fixed point theorem for metric spaces as follows ([1], Corollary 2.4):

Proposition 5.1. [1] *Let (M, d) be a metric space and X a nonempty subadmissible subset of M. Suppose that the identity map $1_X : X \to X$ belongs to $\mathfrak{KC}(X, X)$, then any continuous map $f : X \to X$ such that $\mathrm{cl} f(X)$ is compact, has a fixed point.*

In our terminology, *if X is a partial KKM space, every continuous compact selfmap of X has a fixed point.* Note that Proposition 5.1 follows from Corollary 4.4.

It is stated in [1] that Fakhar - Zafarani have shown that those multifunctions defined on G-convex spaces which are closed, compact and acyclic valued have the KKM property. However we noted earlier that $\mathfrak{A}_c^\kappa$ has the KKM property.

In the following, we give some facts on the metric spaces X for which the identity mapping $1_X : X \to X$ belongs to $\mathfrak{KC}(X, X)$; that is, X is a partial KKM space.

Definition. [1] We say that (M, d) is an $\mathcal{NR}$-*metric space*, if there exists a closed convex subset (E, ρ) of a completely metrizable Hausdorff topological vector space (V, ρ) in which

$$\rho(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2) \leq \max(\rho(x_1, y_1), \rho(x_2, y_2))$$

$$\text{for each } x_1, x_2, y_1, y_2 \in E,\ \alpha+\beta=1,\ \alpha,\beta>0$$

such that (M, d) can be isometrically embedded into (E, ρ) and there exists a nonexpansive retraction $r : E \to M$.

Every hyperconvex metric space is an $\mathcal{NR}$-metric space, and every $\mathcal{NR}$-metric space is a generalized convex space in the sense of Park; see [1].

Later in 2010, Turkoglu-Abuloha-Abdeljawad [43] defined KKM maps in cone metric spaces and obtained some fixed point theorems. Also they defined $\mathcal{NR}$-cone metric spaces and hyperconvex cone metric spaces as generalizations to the same concepts for metric spaces, and hence generalized the results of Amini-Fakhar-Zafarani [1] in 2005.

The following is an example ([43], Corollary 17):

Proposition 5.2. [43] *Let (M, d) be a cone metric space and X a nonempty subadmissible subset of M. Suppose that X is a partial KKM space (that is, the identity map $1_X \in \mathfrak{KC}(X, X)$), then any continuous compact map $f : X \to X$ has a fixed point.*

It is shown that in every $\mathcal{NR}$-cone metric space $(M; d)$, any subadmissible subset X of M is a partial KKM space; see ([43], Lemma 25).

Moreover, in 2010, Khamsi and Hussain [12] discussed some recent results about KKM maps in cone metric spaces, and also discussed the fixed point existence results of multimaps defined on such metric spaces. In particular they showed that most of the new results are merely copies of the classical ones and do not necessitate the underlying Banach space nor the associated cone.

They defined metric type spaces and obtained the following ([12], Theorem 4.3):

Theorem 5.3. [12] *Let (M, d) be a metric type space and X a nonempty subadmissible subset of M. Suppose that X is a partial KKM space, then any continuous compact map $f : X \to X$ has a fixed point.*

Furthermore, in 2011, Hussain and Shah [10] introduced cone b-metric spaces and defined KKM maps on them. The following is a sample result ([10], Theorem 4.4):

Theorem 5.4. [10] *Let (M, D) be a cone b-metric space and X a nonempty subadmissible subset of M. Suppose that X is a partial KKM space. Then any continuous compact map $f : X \to X$ has a fixed point.*

Note that, in [10] and [12], their authors assumed the existence of partial KKM spaces, but did not give any concrete example of them.

6. Metric spaces with continuous midpoints

In 2009, Horvath [9] introduced continuous midpoint spaces as a generalization of various types of metric spaces as follows:

Definition. [9] A *continuous midpoint map* on a metric space (X, d) is a continuous map $\mu : X \times X \to X$ such that, for all $(a, b) \in X \times X$, $d(a, \mu(a, b)) = (1/2)d(a, b) = d(b, \mu(a, b))$. If μ is a continuous midpoint map then $\check{\mu}(a, b) = \mu(b, a)$ is also a continuous midpoint map. The triple (X, d, μ) is called *a continuous midpoint space.* Given a continuous midpoint space (X, d, μ) it is natural to say that a closed subset C of X is convex if, for all $(a, b) \in C \times C$, $\mu(a, b) \in C$.

Example. Horvath [9] gave a large number of examples of continuous midpoint spaces. We give some of them which are complete as follows:

(1) Closed convex subsets of Banach spaces.
(2) Hyperconvex metric spaces due to Aronszajn and Panitchpackdi.
(3) Hilbert spaces.
(4) Completion of Bruhat-Tits spaces [= Hadamard spaces, that is, complete and simply connected metric spaces of nonpositive curvature (= complete CAT(0) spaces)].

(5) Complete $\mathbb{R}$-trees [= hyperconvex metric spaces with unique metric segments].

(6) Buseman midpoint spaces [includes hyperbolic metric spaces in the sense of Kirk and Reich-Shafrir; see [38].].

Lemma 6.1. [9] *Nonempty geodesically convex subsets of a complete continuous midpoint space* (X, d, μ) *are contractible.*

Here we stop to quote Horvath [9], and give a new definition.

Definition. An abstract convex space $(X, D; \Gamma)$ is called a *Horvath midpoint space* whenever X is a complete continuous midpoint metric space, D is a nonempty subset of X, and $\Gamma : \langle D \rangle \multimap X$ is a multimap such that $\Gamma(A) = \Gamma_A$ is a geodesically convex subset containing A and $\Gamma_A \subset \Gamma_B$ if $A \subset B \in \langle D \rangle$.

Proposition 6.2. *Any Horvath midpoint space is a Horvath space and hence a KKM space.*

Therefore, any Horvath midpoint space satisfies all results in [30, 31]. The above Examples (1)–(6) are Horvath midpoint spaces. However, it seems to be hard to check whether their subsets is geodesically convex in the sense of Horvath, in general.

Also, note that $\Gamma(A) = \Gamma_A$ in Definition and Proposition 6.2 can be replaced by any homotopically trivial values.

7. Global NPC spaces with CHFP

In 2009, Niculescu and Rovenţa [17] extended Fan's minimax inequality to the context of metric spaces with global nonpositive curvature (NPC). As a consequence, a general result on the existence of a Nash equilibrium is obtained by them:

Definition. [17] A *global NPC space* is a complete metric space $E = (E, d)$ such that, for each pair of points $x_0,\ x_1 \in E$, there exists a point $y \in E$ such that

$$d(x_0, y) = d(y, x_1) = \frac{1}{2} d(x_0, x_1).$$

In a global NPC space E each pair of points $x_0,\ x_1 \in E$ can be connected by a *geodesic* (that is, by a rectifiable curve $\gamma : [0, 1] \to E$ such that the length of $\gamma_{[s,t]}$ is $d(\gamma(s), \gamma(t))$ for all $0 \le s \le t \le 1$). Moreover, this geodesic is unique.

Definition. [17] A set $C \subset E$ is called *convex* if $\gamma([0, 1]) \subset C$ for each geodesic $\gamma : [0, 1] \to C$ joining $\gamma(0),\ \gamma(1) \in C$.

Examples of global NPC spaces E are given in [17]:

(1) Every Hilbert space.

(2) A complete, simply connected Riemannian manifold (M, g) with a nonpositive sectional curvature.

(3) The Bruhat-Tits building (in particular, trees).

(4) All closed convex subset of a global NPC space. etc.

We note that the concept of *a global NPC space is slightly general than a complete continuous midpoint metric space* of Horvath [9] since the former does not required the continuity of midpoints.

Therefore, the following popular types of Horvath spaces are global NPC spaces:

Hadamard manifolds, hyperconvex metric spaces, hyperbolic metric spaces, complete CAT(0) spaces (Hadamard spaces), complete $\mathbb{R}$-trees, etc.

Niculescu and Rovenţa [17] begins with the 1961 KKM lemma of Ky Fan, where Hausdorffness is redundant. They recalled that the notion of a convex hull of a finite set of E is introduced via the formula

$$\mathrm{co}F = \bigcup_{n=1}^{\infty} F_n,$$

where $F_0 = F$ and for $n \geq 1$ the set F_n consists of all points in E which lie on geodesics which start and end in F_{n-1}.

Lemma 7.1. [17] *The KKM Lemma extends to any global NPC space E, provided that the closed convex hull of every nonempty finite subsets of E has the fixed point property.*

From this, they obtain a Ky Fan type minimax inequality (Theorem 1), a much more general result (Theorem 2), its non-symmetric version (Theorem 3), and existence of a g-equilibrium, a fact that generalizes the well-known result on the Nash equilibrium (Theorem 4).

Let the abstract convex space $(E \supset D; \Gamma)$ consists of a global NPC space E, provided that the closed convex hull $\Gamma(A)$ of every nonempty finite subsets A of D has the fixed point property. Then Lemma 7.1 implies

Theorem 7.2. $(E \supset D; \Gamma)$ *is a partial KKM space.*

Consequently, a large number of results in Park [31] holds; that is, the Ky Fan type minimax inequality, the von Neumann-Sion minimax theorem, the Nash-Fan type equilibrium theorem, and many others.

Example. (1) Horvath [8] and Khamsi [11] showed that hyperconvex metric spaces are KKM spaces. This seems to be the first example of the above Theorem 7.2 and the origin of the method of this section.

In his proof Khamsi applied that $C = \mathrm{conv}(A)$ for $A \in \langle H_\infty \rangle$ has the fixed point property. This seems to be the origin of the CHFP of Niculescu-Rovenţa [17].

(2) Kirk-Panyanak [15] proved a KKM theorem for geodesically bounded $\mathbb{R}$-trees. The proof is almost same to that of Khamsi [10]. Note that an $\mathbb{R}$-tree is a CAT(0) space and a complete $\mathbb{R}$-tree is a hyperconvex metric space.

(3) Shabanian-Vaezpour [42]: A CAT(0) space X has *convex hull finite property* if the closed convex hull of every nonempty finite subset of X has the fixed point property. Suppose that E is a complete CAT(0) space with the convex hull finite property and X is a nonempty subset of E. Then X is a partial KKM space. This KKM principle is applied to some fixed point theorems and best approximation theorems in CAT(0) spaces.

(4) Kimura-Kishi [14]: An Hadamard space X has *Convex Hull Finite Property* (CHFP for short) if every continuous mapping $f : \mathrm{clco}\, A \to \mathrm{clco}\, A$ has a fixed point for every finite subset A of X, where $\mathrm{clco}\, E$ is the closure of $\mathrm{co}\, A$. Adopted the KKM lemma of Niculescu-Rovenţa [17] that any nonempty subset C of a Hadamard space X with CHFP is a KKM space. On such spaces with the CHFP, they defined and studied resolvent operators for an equilibrium problem.

Note that global NPC spaces with CHFP seem to be not comparable to Horvath midpoint spaces.

8. $\aleph_0$-spaces

In 2012, Bansami and Riahi [5] introduced the topological structure of $\aleph_0$-spaces which is a generalization of hyperconvex metric spaces. Then they established an associated KKM finite intersection lemmas. As applications they gave an $\aleph_0$-space version of Fan's best approximation theorem for set-valued mappings and some fixed point theorems.

The following is due to Espinola and Lopez [6]:

Definition. [6] Let $m \in \mathbb{N}$. A metric space (X, d) is said to be *m-hyperconvex* if for any class $\{x_\alpha : \alpha \in \mathcal{A}\}$ in X and $\{r_\alpha : \alpha \in \mathcal{A}\}$ in $\mathbb{R}^+$ with $card(\mathcal{A}) < m$, one has

$$d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta, \forall \alpha, \beta \in \mathcal{A} \Longrightarrow \bigcap_{\alpha \in \mathcal{A}} \mathrm{B}(x_\alpha, r_\alpha) \neq \emptyset.$$

If X is m-hyperconvex for every $m \in \mathbb{N}$, we say that X is $\aleph_0$-*hyperconvex*.

Remark that if cardinality of $\mathcal{A}$ is not fixed, we only say X to be hyperconvex. It is clear that hyperconvexity is stronger than m-hyperconvexity, for each m. The notion of m-hyperconvexity is also stronger than m_0-hyperconvexity if $m_0 < m$, and the inclusion is strict.

Espinola and Lopez [6] gave characterization of a complete $\aleph_0$-hyperconvex metric space.

Moreover, Bansami and Riahi [5] introduced the following and some examples:

Definition. [5] Let H be a nonempty set, M be a topological space and $R : \langle H \rangle \multimap M$. The triplet (H, M, R) is said to be an $\aleph_0$-*space* if

(1) $\forall A \in \langle H \rangle,\ R(A)$ is a nonempty complete $\aleph_0$-hyperconvex metric space;

(2) $\forall A, B \in \langle H \rangle,\ A \subset B$ implies $R(A) \subset R(B)$ metrically.

Note that (H, M, R) is an abstract convex space $(M, H; R)$ in our sense; see Section 2. Actually, the authors of [5] showed several KKM theoretic results on such spaces. It might interesting to check that $(M, H; R)$ is a partial KKM space.

9. Literature on History

In our previous work [34], we collected a large number of references on hyperconvex metric spaces related to the KKM theory and later, we found that there are too many articles on that spaces. In this section, we introduce some recent works related to the subject of this article which appeared after the publication of our previous article [34].

Park [34] — JNAS54(2)

Abstract. For a long period, the study of hyperconvex metric spaces introduced by Aronszajn and Panitchpackdi in 1956 was concentrated to the fixed point property of nonexpansive maps. However, since Khamsi in 1996 found a KKM theorem for such spaces, there have appeared a large number of works on them related to the KKM theory. In the present review, we follow the various stages of developments of the KKM theory of hyperconvex metric spaces. In fact, we introduce abstracts of articles on such theory and give comments or generalizations of the results there if necessary. We show that many results in those articles are consequences of (partial) KKM space theory developed by ourselves from 2006.

Comments: Horvath first noted that hyperconvex metric spaces are c-spaces (or Horvath spaces) [8] and later Horvath midpoint spaces [9]. In Park [34], there is a list of 53 related articles to the KKM theory of hyperconvex metric spaces beginning with Khamsi's paper in 1996 and ending a paper of Markin and Shahzad in 2015. So we will not mention most of them in the present article. However, some articles not appeared in Park [34] will mention here.

Park [35] — LNA 2(1)

Abstract. Recent results in the KKM theory of abstract convex spaces and the related multimap classes $\mathfrak{KC}$ and $\mathfrak{KO}$ are applied to deduce generalizations of results on KKM maps in metric type spaces of Khamsi - Hussain [12].

Park [36] — NAF 22(2)

Abstract. In 1996, Khamsi established the KKM theorem for hyperconvex metric spaces and applied it to obtain a Schauder type fixed point theorem. This line of study has been followed by a large number of authors. In this article, we show that the KKM theorem, best approximation theorem, and the Schauder type fixed point theorem for hyperconvex metric spaces due to Khamsi can be extended to partial KKM metric spaces.

Park [37] — JNAS 56(2)

Abstract. We review briefly the history of the KKM theory from the original Knaster-Kuratowski-Mazurkiewicz (KKM) theorem on simplices in 1929 to the birth of the new (partial) KKM spaces by the following stages:

(1) We recall some early equivalent formulations of the Brouwer fixed point theorem and some statements implying the KKM theorem.
(2) We summarize Fan's foundational works on the KKM theory from 1960s to 1980s.
(3) Recall the early leading works of Granas since 1978.
(4) In 1983-2005, basic results in the theory were extended to convex spaces by Lassonde, to H-spaces by Horvath, and to G-convex spaces due to Park.
(5) In 2006, we introduced the concept of abstract convex spaces $(E, D; \Gamma)$ on which we can construct the KKM theory. Moreover, abstract convex spaces satisfying an abstract form of the KKM theorem and its 'open' version are called *KKM spaces*. Now the KKM theory becomes mainly the study of KKM spaces.
(6) Finally, we introduce the KKM space versions of the von Neumann minimax theorem, the von Neumann intersection lemma, the Nash equilibrium theorem, and the Himmelberg fixed point theorem.

Balaj - Jorquera - Khamsi [3] — JFPTA 20

Abstract. We establish several common fixed point theorems for families of set-valued mappings defined in hyperconvex metric spaces. Then we give several applications of our results.

Comments. Some results may be stated in more general forms.

Park [40] — JNAS 58(1)

Abstract. In the last decade the KKM theoretic results on the Hadamard manifolds have been extended to various types of H-spaces or Horvath spaces or partial KKM spaces. Such types include the *complete continuous midpoint metric spaces* due to Horvath in 2009 and the *metric spaces with global nonpositive curvature* (NPC) *and convex hull finite property* (CHFP) due to Niculescu–Rovenţa in the same year. Based on our KKM theory on abstract convex spaces, in the present article, we show that our method can be applied to various types of new spaces. Such results are the KKM theorem, the Fan type minimax inequality, the Fan-Browder fixed point theorem, variational inequalities, von Neumann minimax theorem, Nash equilibrium theorem, etc. Historical remarks are added to the literature on the KKM type results or others on such new spaces.

Park [41] — JNCA

Abstract. We introduce a typical subclass of KKM spaces called the Horvath spaces including c-spaces due to him. We show that hyperbolic metric spaces, metric spaces with continuous midpoints, metric spaces with global nonpositive curvature (NPC) and convex hull finite property (CHFP), certain Riemannian manifolds, $\mathbb{B}$-spaces as relatively new examples of Horvath spaces. Many of their properties are introduced by following our previous works [30, 31].

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