Oscillation theorems for second-order nonlinear delay differential equations of neutral type

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Abstract

In this paper, new sufficient conditions are obtained for oscillation of second-order neutral delay differential equations of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[r(t)\frac{\mathrm{d}}{\mathrm{d}t}[x(t) + p(t)x(\tau(t))]\right] + q(t)G(x(\sigma(t))) = 0 \text{ for } t \geq t_0,$$

under the assumptions $\int^\infty \frac{1}{r(\eta)}\mathrm{d}\eta = \infty$ and $\int^\infty \frac{1}{r(\eta)}\mathrm{d}\eta < \infty$ for various ranges of the bounded neutral coefficient p. Unlike most of the previous results, τ' is allowed to be oscillatory. Further, some illustrative examples showing applicability of the new results are included.

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1. Introduction

This article is concerned with sufficient conditions for oscillation of a nonlinear neutral second-order delay differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[r(t)\frac{\mathrm{d}}{\mathrm{d}t}[x(t) + p(t)x(\tau(t))]\right] + q(t)G(x(\sigma(t))) = 0 \text{ for } t \geq t_0. \qquad (1.1)$$

We also suppose that the following assumptions hold:

(A1) $r \in \mathrm{C}([t_0,\infty),(0,\infty))$, $p \in \mathrm{C}([t_0,\infty),\mathbb{R})$ and $q \in \mathrm{C}([t_0,\infty),[0,\infty))$, where q is not identically zero eventually;

(A2) $G \in \mathrm{C}(\mathbb{R},\mathbb{R})$ satisfies $uG(u) > 0$ for $u \neq 0$;

(A3) $\tau, \sigma \in \mathrm{C}([t_0,\infty),\mathbb{R})$ such that $\tau(t), \sigma(t) \leq t$ for $t \geq t_0$, $\tau(t), \sigma(t) \to \infty$ as $t \to \infty$ with differentiable or invertible τ when necessary.

Our investigation on the asymptotic behavior of solutions depend on the following two possible conditions:

(C1) $\int^\infty \frac{1}{r(\eta)}\mathrm{d}\eta = \infty$;

(C2) $\int^\infty \frac{1}{r(\eta)}\mathrm{d}\eta < \infty$.

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Baculikova *et al.* [1] have studied the linear counterpart of (1.1),

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[r(t)\frac{\mathrm{d}}{\mathrm{d}t}[x(t) + p(t)x(\tau(t))]\right] + q(t)x(\sigma(t)) = 0 \tag{1.2}$$

when $0 \leq p(t) \leq p_0 < \infty$ and (C1) hold. The authors have obtained sufficient conditions for oscillation of solutions of (1.2) through some comparison results, where the comparison results are unpredictable. In [5], Džurina have studied (1.2) when $0 \leq p(t) \leq p_0 < \infty$ and (C1) holds true. He has established sufficient condition for oscillation of solutions of (1.2) by comparison techniques. In another paper [10], Karpuz *et al.* have considered the linear equation

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}[x(t) + p(t)x(\tau(t))] + q(t)x(\sigma(t)) = 0,$$

and established some new oscillation results based on comparison theorems when $-1 < p(t) < 0$ and $0 < p(t) < 1$ for all large t. In [15], under various ranges of p, Santra studied oscillatory behaviour of the solutions of the following neutral differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) + p(t)x(t - \tau)] + q(t)G(x(t - \sigma)) = 0$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) + p(t)x(t - \tau)] + q(t)G(x(t - \sigma)) = f(t). \tag{1.3}$$

Also, sufficient conditions are obtained for existence of bounded positive solutions of (1.3). The motivation of the present work come from the above studies. Hence, in this work, an attempt is made to study the more general form of (1.2) without making any comparison. It seems that our method is the next alternative to the works [1,5] and [10], when p is bounded. Here, we are connected to both (C1) and (C2).

The neutral differential equations find numerous applications in natural sciences and technology. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines (see for e.g [7]). In this paper, we restrict our attention to study (1.1), which includes a class of nonlinear functional differential equations of neutral type. In this direction we refer the reader to some of the works ([2–4,6,8,11–14,17]) and the references cited therein.

By a solution to equation (1.1), we mean a function $x \in \mathrm{C}([T_x,\infty),\mathbb{R})$, where $T_x \geq t_0$, such that $rz' \in \mathrm{C}^1([T_x,\infty),\mathbb{R})$, where

$$z(t) := x(t) + p(t)x(\tau(t)) \text{ for } t \geq T_x, \tag{1.4}$$

and satisfies (1.1) on the interval $[T_x,\infty)$. A solution x of (1.1) is said to be proper if x is not identically zero eventually, i.e., $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. We assume that (1.1) possesses such solutions. A solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[T_x,\infty)$; otherwise, it is said to be nonoscillatory. (1.1) itself is said to be oscillatory if all of its solutions are oscillatory.

Remark 1.1. When the domain is not specified explicitly, all functional inequalities considered in this paper are assumed to hold eventually, i.e., they are satisfied for all t large enough.

2. Sufficient conditions for oscillation

In this section, sufficient conditions are obtained for oscillatory and asymptotic behaviour of second order nonlinear neutral differential equations of the form (1.1).

2.1. Oscillation under the condition (C1)

Lemma 2.1. *Assume that (C1) and (A1)–(A3) hold. If x is an eventually positive solution of* (1.1) *such that the companion function z defined by* (1.4) *is also eventually positive, then z satisfies*

$$z'(t) > 0 \text{ and } (rz')'(t) < 0 \text{ for all large } t. \tag{2.1}$$

Proof. Suppose that $x(t) > 0$ and $z(t) > 0$ for $t \geq t_1$, where $t \geq t_0$. By (A3), we may assume without loss of generality that $x(\sigma(t)) > 0$ for $t \geq t_1$. From (1.1) and (A2), it follows that

$$(rz')'(t) = -q(t)G(x(\sigma(t))) < 0 \text{ for } t \geq t_1. \tag{2.2}$$

Consequently, rz' is nonincreasing on $[t_1, \infty)$ and thus either $z'(t) < 0$ or $z'(t) > 0$ for $t \geq t_2$, where $t_2 \geq t_1$. If $z'(t) < 0$, then there exists $\varepsilon > 0$ such that $r(t)z'(t) \leq -\varepsilon$ for $t \geq t_2$, which yields upon integration over $[t_2, t) \subset [t_2, \infty)$ after dividing through by r that

$$z(t) \leq z(t_2) - \varepsilon \int_{t_2}^{t} \frac{1}{r(\eta)} d\eta \text{ for } t \geq t_2. \tag{2.3}$$

In view of (C1), letting $t \to \infty$ in (2.3) yields $z(t) \to -\infty$, which is a contradiction. Therefore, $z'(t) > 0$ for $t \geq t_2$. This completes the proof. $\qquad\square$

Remark 2.2. It follows from Lemma 2.1 that $\lim_{t\to\infty} z(t) > 0$, i.e., there exists $\varepsilon > 0$ such that $z(t) \geq \varepsilon$ for all large t.

Lemma 2.3. *Assume that (C1) and (A1)–(A3) hold. If x is an eventually positive solution of* (1.1) *such that the companion function z defined by* (1.4) *is bounded, then z satisfies* (2.1) *for all large t.*

Proof. The proof can be obtained from the proof of Lemma 2.1. $\qquad\square$

Theorem 2.4. *Let $0 \leq p(t) \leq p < 1$ for $t \geq t_0$, where p is a constant. Assume that (C1) and (A1)–(A3) hold. Furthermore, assume that the followings hold:*
(A4) G is a nondecreasing function;
(A5) $\int^{\infty} q(\eta)d\eta = \infty$.
Then, every solution of (1.1) *is oscillatory.*

Proof. Suppose the contrary that x is a nonoscillatory solution of (1.1). Then, there exists $t_1 \geq t_0$ such that either $x(t) > 0$ or $x(t) < 0$ for $t \geq t_1$. Assume that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. Clearly, z defined by (1.4) is positive on $[t_1, \infty)$. By Lemma 2.1 and Remark 2.2, there exists $\varepsilon > 0$ such that $z(t) \geq \varepsilon$ for $t \geq t_2$, where $t_2 \geq t_1$. On the other hand, z being increasing implies that

$$(1-p)z(t) \leq (1 - p(t))z(t) \leq z(t) - p(t)z(\tau(t))$$
$$= x(t) - p(t)p(\tau(t))x(\tau(\tau(t))) \leq x(t)$$

for $t \geq t_3$, where $t_3 \geq t_2$. Consequently, $x(t) \geq (1-p)\varepsilon > 0$ for $t \geq t_3$. From (2.2), we have

$$(rz')'(t) + G((1-p)\varepsilon)q(t) \leq 0 \text{ for } t \geq t_3.$$

Integrating the last inequality over the interval $[t_3, t) \subset [t_3, \infty)$, we get

$$G((1-p)\varepsilon) \int_{t_3}^{t} q(\eta)d\eta \leq r(t_3)z'(t_3) \text{ for all } t \geq t_3.$$

This contradicts (A5).

If $x(t) < 0$ for $t \geq t_1$, then we set $y(t) := -x(t)$ for $t \geq t_1$ in (1.1). Using (A4), we find

$$\frac{d}{dt}\left[r(t)\frac{d}{dt}[y(t) + p(t)y(\tau(t))]\right] + q(t)H(y(\sigma(t))) = 0 \text{ for } t \geq t_1,$$

where $H(u) := -G(-u)$ for $u \in \mathbb{R}$. Clearly, H also satisfies (A2) and (A4). Then, proceeding as above, we find the same contradiction. This completes the proof. $\square$

Theorem 2.5. *Let $1 \leq p(t) \leq p$ for $t \geq t_0$, where p is a constant. Assume that (C1) and (A1)–(A4) hold. Furthermore, assume that the following conditions hold:*
(A6) there exists $\lambda > 0$ such that

$$G(u) + G(v) \geq \lambda G(u+v) \text{ for } u, v \geq 0$$

and

$$G(u) + G(v) \leq \lambda G(u+v) \text{ for } u, v \leq 0;$$

(A7)

$$G(uv) \leq G(u)G(v) \text{ for } u, v \geq 0$$

and

$$G(uv) \geq G(u)G(v) \text{ for } u, v \leq 0;$$

(A8) $\tau(\sigma(t)) = \sigma(\tau(t))$ for $t \geq t_0$;
(A9) $\int^{\infty} Q(\eta)\mathrm{d}\eta = \infty$, where

$$Q(t) := \min\{q(t), q(\tau(t))\tau'(t)\} \text{ for } t \geq t_0.$$

Then, every solution of (1.1) is oscillatory.

Proof. Without loss of generality, suppose the contrary that x is an eventually positive solution of (1.1). Then, there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. Clearly, z defined by (1.4) is positive on $[t_1, \infty)$. By Lemma 2.1 and Remark 2.2, there exists $\varepsilon > 0$ such that $z(t) \geq \varepsilon$ for $t \geq t_2$, where $t_2 \geq t_1$. Let us define

$$w(t) := r(t)z'(t) + G(p)r(\tau(t))z'(\tau(t)) > 0 \text{ for } t \geq t_3,$$

where $t_3 \geq t_2$. From (1.1), we compute that

$$
\begin{aligned}
0 =& (rz')'(t) + q(t)G(x(\sigma(t))) \\
&+ G(p)\tau'(t)\big[(rz')'(\tau(t)) + q(\tau(t))G(x(\sigma(\tau(t))))\big] \\
=& w'(t) + q(t)G(x(\sigma(t))) + q(\tau(t))\tau'(t)G(p)G(x(\tau(\sigma(t))))
\end{aligned}
$$

for $t \geq t_4$, where $t_4 \geq t_3$. Using (A6) and (A7), we obtain

$$
\begin{aligned}
0 \geq& w'(t) + Q(t)G(x(\sigma(t))) + G(px(\tau(\sigma(t)))) \\
\geq& w'(t) + \lambda Q(t)G(x(\sigma(t)) + px(\tau(\sigma(t)))) \\
\geq& w'(t) + \lambda Q(t)G(x(\sigma(t)) + p(\sigma(t))x(\tau(\sigma(t)))) \\
=& w'(t) + \lambda Q(t)G(z(\sigma(t)))
\end{aligned}
\tag{2.4}
$$

for $t \geq t_4$. Consequently,

$$w'(t) + \lambda Q(t)G(\varepsilon) \leq 0 \text{ for } t \geq t_4,$$

which upon integration over the interval $[t_4, t) \subset [t_4, \infty)$ yields that

$$\lambda G(\varepsilon) \int_{t_4}^{t} Q(\eta)\mathrm{d}\eta \leq w(t_4) \text{ for all } t \geq t_4.$$

This contradicts (A9). Thus, $x(t) > 0$ for $t \geq t_1$ cannot hold.

The case where x is eventually negative is very similar and we omit it here. Thus, the theorem is proved. $\square$

Let us give an important example for Theorem 2.5, where the results in [1–5, 12, 16] cannot be applied because of the limitation $\inf\{\tau'(t) : t \geq t_0\} > 0$. Further, the results in [8, 13] cannot be applied either because of the nonlinear form of the delay τ.

Example 2.6. Consider the differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}[x(t) + 2x(t - \sin(\tfrac{\pi}{2}t) - 1)] + x(t - 4) = 0 \text{ for } t \geq 0, \tag{2.5}$$

where $r(t) :\equiv 1$, $p(t) :\equiv 2$, $\tau(t) := t - \sin(\tfrac{\pi}{2}t) - 1$, $q(t) :\equiv 1$, $\sigma(t) := t - 4$ and $G(u) := u$ for $t \geq 0$ and $u \in \mathbb{R}$. We have

$$\tau(\sigma(t)) = t - \sin(\tfrac{\pi}{2}t) - 5 = \sigma(\tau(t)) \text{ and } \tau'(t) = 1 - \tfrac{\pi}{2}\cos(\tfrac{\pi}{2}t) \text{ for } t \geq 0.$$

Note that τ' is oscillatory and $Q(t) := \min\{1, 1 - \tfrac{\pi}{2}\cos(\tfrac{\pi}{2}t)\}$ for $t \geq 0$. Obviously, Q is a periodic function with a period of 4. Further, $\int_0^4 Q(\eta)\mathrm{d}\eta = 2$, which shows that $\int^\infty Q(\eta)\mathrm{d}\eta = \infty$. Then, all the assumptions of Theorem 2.5 holds. Hence, every solution of (2.5) oscillates.

Theorem 2.7. *Let* $-1 \leq p(t) \leq 0$ *for* $t \geq t_0$. *Assume that (C1) and (A1)–(A5) hold. Then, every unbounded solution of* (1.1) *oscillates.*

Proof. Without loss of generality, suppose the contrary that x is an eventually positive unbounded solution of (1.1). Then, there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. Proceeding as in the proof of Lemma 2.1, we see rz' is nonincreasing and z is monotonic on $[t_2, \infty)$, where $t_2 \geq t_1$. We have the following two possible cases.

Case 1. Let $z(t) > 0$ for $t \geq t_2$. By Lemma 2.1, (2.1) holds for $t \geq t_3$. Clearly, $z(t) \leq x(t)$ for $t \geq t_3$ implies

$$(rz')'(t) + q(t)G(z(\sigma(t))) \leq 0 \text{ for } t \geq t_3, \tag{2.6}$$

where $t_4 \geq t_3$. Further, by Lemma 2.1 and Remark 2.2, there exists $\varepsilon > 0$ such that $z(t) \geq \varepsilon$ for $t \geq t_4$. Consequently, it follows from (2.6) that

$$(rz')'(t) + G(\varepsilon)q(t) \leq 0 \text{ for } t \geq t_4.$$

Integrating the last inequality over $[t_4, t) \subset [t_4, \infty)$, we have

$$G(\varepsilon)\int_{t_4}^t q(\eta)\mathrm{d}\eta \leq r(t_4)z'(t_4) \text{ for } t \geq t_4.$$

This contradicts (A5).

Case 2. Let $z(t) < 0$ for $t \geq t_2$. As x is unbounded, there exists $T \geq t_2$ such that $x(T) = \max\{x(\eta) : t_2 \leq \eta \leq T\}$. Then, from (1.4), we have $x(T) \leq z(T) + x(\tau(T)) < x(T)$, which is a contradiction.

The case where x is an eventually negative unbounded solution is similar and we omit it here. Thus, the proof is complete. $\qquad\square$

Theorem 2.8. *Let* $-1 < -p \leq p(t) \leq 0$ *for* $t \geq t_0$, *where* p *is a constant. Assume that (C1) and (A1)–(A5) hold. Then, every bounded solution of* (1.1) *either oscillates or converges to zero asymptotically.*

Proof. Without loss of generality, let x be an eventually positive bounded solution of (1.1). Then, there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. By Lemma 2.3, there exists $t_2 \geq t_1$ such that $z'(t) > 0$ for $t \geq t_2$. Consequently, we have the following two possible cases.

Case 1. Let $z(t) > 0$ for $t \geq t_2$. Proceeding as in Case 1 in the proof of Theorem 2.7, we get a contradiction.

Case 2. Let $z(t) < 0$ for $t \geq t_2$. Then, $\lim_{t\to\infty} z(t)$ exits. Thus, we have

$$0 \geq \lim_{t\to\infty} z(t) = \limsup_{t\to\infty} z(t) = \limsup_{t\to\infty}[x(t) + p(t)x(\tau(t))]$$

$$\geq \limsup_{t\to\infty}[x(t) - px(\tau(t))] \geq \limsup_{t\to\infty} x(t) + \liminf_{t\to\infty}[-px(\tau(t))]$$

$$= (1 - p)\limsup_{t\to\infty} x(t),$$

i.e., $\limsup_{t\to\infty} x(t) = 0$ (since $0 < p < 1$) and hence $\lim_{t\to\infty} x(t) = 0$.

The case where x is an eventually negative bounded solution is omitted since it can be dealt similarly. This completes the proof. $\square$

Combining Theorem 2.5 and Theorem 2.7, we have the following corollary.

Corollary 2.9. *Let $-1 < -p \le p(t) \le 0$ for $t \ge t_0$, where p is a constant. Assume that (C1) and (A1)–(A5) hold. Then, every solution of (1.1) either oscillates or converges to zero asymptotically.*

Theorem 2.10. *Let $-p_1 \le p(t) \le -p_2 < -1$ for $t \ge t_0$, where p_1 and p_2 are constants. Assume that (C1) and (A1)–(A5) hold. Then, every bounded solution of (1.1) either oscillates or converges to zero asymptotically.*

Proof. Without loss of generality, let x be an eventually positive bounded solution of (1.1). Then, z defined by (1.4) is also bounded. By Lemma 2.3, z is increasing. Hence, we have the following two possible cases.

Case 1. Let $z(t) > 0$ for $t \ge t_2$. Proceeding as in Case 1 in the proof of Theorem 2.7, we get a contradiction.

Case 2. Let $z(t) < 0$ for $t \ge t_2$. In this case, $\lim_{t\to\infty} z(t)$ exists as a nonpositive finite value. We claim that $\lim_{t\to\infty} z(t) = 0$. Otherwise, $\lim_{t\to\infty} z(t) < 0$, i.e., there exists $\varepsilon > 0$ such that $z(t) < -\varepsilon$ for $t \ge t_1$. Then, we have $z(t) \ge p(t)x(\tau(t)) \ge -p_1 x(\tau(t))$ for $t \ge t_1$, which implies $x(t) \ge -\frac{1}{p_1} z(\tau^{-1}(t)) \ge \frac{\varepsilon}{p_1}$ for $t \ge t_1$. Consequently, (2.2) becomes

$$(rz')'(t) + q(t)G\left(\frac{\varepsilon}{p_1}\right) \le 0 \text{ for } t \ge t_1.$$

Integrating the last inequality over the interval $[t_1, t) \subset [t_1, \infty)$, we get

$$G\left(\frac{\varepsilon}{p_1}\right) \int_{t_2}^{t} q(\eta)\mathrm{d}\eta \le r(t_2)z'(t_2) \text{ for } t \ge t_1.$$

This contradicts (A5). Therefore, $\lim_{t\to\infty} z(t) = 0$. Hence,

$$\begin{aligned}
0 = \lim_{t\to\infty} z(t) &= \liminf_{t\to\infty} z(t) \le \liminf_{t\to\infty}[x(t) - p_2 x(\tau(t))] \\
&\le \limsup_{t\to\infty} x(t) + \liminf_{t\to\infty}[-p_2 x(\tau(t))] \\
&\le (1 - p_2) \limsup_{t\to\infty} x(t),
\end{aligned}$$

which implies that $\limsup_{t\to\infty} x(t) = 0$ (since $p_2 > 1$). Thus, $\liminf_{t\to\infty} x(t) = 0$ and hence $\lim_{t\to\infty} x(t) = 0$.

Therefore, any nonoscillatory solution x of (1.1) converges to zero. This completes the proof of the theorem. $\square$

Example 2.11. Consider the differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[t\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - 3x(\mathrm{e}^{-\pi}t)]\right] + \frac{4}{t}x(t) = 0 \text{ for } t \ge 1, \tag{2.7}$$

where $r(t) := t$, $p(t) :\equiv -3$, $\tau(t) := \mathrm{e}^{-\pi}t$, $q(t) := \frac{4}{t}$, $\sigma(t) := t$ and $G(u) := u$ for $t \ge 1$ and $u \in \mathbb{R}$. It can be easily shown that Theorem 2.10 applies to (2.7). Thus, every bounded solution oscillates or converges to zero asymptotically. Obviously, $x(t) = \sin(\ln(t))$ for $t \ge 1$ is an oscillating solution.

2.2. Oscillation under the condition (C2)

Remark 2.12. If we set

$$R(t) := \int_t^\infty \frac{1}{r(\eta)} \mathrm{d}\eta \text{ for } t \geq t_0, \tag{2.8}$$

then (C2) implies that $R(t) \to 0$ as $t \to \infty$.

Lemma 2.13. *Assume that (C2) and (A1)–(A3) hold. If x is an eventually positive solution of* (1.1) *such that the companion function z defined by* (1.4) *is eventually decreasing and positive, then there exists $\varepsilon > 0$ such that z satisfies*

$$\varepsilon R(t) \leq z(t) \text{ for all large } t, \tag{2.9}$$

where R is defined in (2.8).

Proof. Suppose that $x(t), z(t) > 0$ and $z'(t) < 0$ for $t \geq t_1$, where $t \geq t_0$. By (A3), we may assume without loss of generality that $x(\sigma(t)) > 0$ for $t \geq t_1$. From (1.1) and (A2), we get (2.2). Consequently, rz' is nonincreasing on $[t_1, \infty)$. Therefore, $r(s)z'(s) \leq r(t)z'(t)$ for $s \geq t \geq t_1$, which implies

$$z'(s) \leq \frac{r(t)z'(t)}{r(s)} \text{ for } s \geq t \geq t_1.$$

Consequently,

$$z(s) \leq z(t) + r(t)z'(t) \int_t^s \frac{1}{r(\eta)} \mathrm{d}\eta \text{ for } s \geq t \geq t_1.$$

As rz' is nonincreasing, we can find a constant $\varepsilon > 0$ such that $r(t)z'(t) \leq -\varepsilon$ for $t \geq t_1$. As a result $z(s) \leq z(t) - \varepsilon \int_t^s \frac{1}{r(\eta)} \mathrm{d}\eta$ for $s \geq t \geq t_1$. By letting $s \to \infty$, we get $0 \leq z(t) - \varepsilon R(t)$ for $t \geq t_1$, which proves (2.9). $\square$

Theorem 2.14. *Let $0 \leq p(t) \leq p$ for $t \geq t_0$, where p is a constant. Assume that (C2), (A1)–(A4) and (A6)–(A9) hold. Further, assume*
(A10)

$$\int_{t_0}^\infty \frac{1}{r(\eta)} \int_{t_0}^\eta Q(\zeta)G(\varepsilon R(\sigma(\zeta))) \mathrm{d}\zeta \mathrm{d}\eta = \infty \text{ for every } \varepsilon > 0$$

and

$$\int_{t_0}^\infty \frac{1}{r(\eta)} \int_{t_0}^\eta Q(\zeta)G(\varepsilon R(\sigma(\zeta))) \mathrm{d}\zeta \mathrm{d}\eta = -\infty \text{ for every } \varepsilon < 0,$$

where Q is defined in (A9).
Then, every solution of (1.1) *is oscillatory.*

Proof. Without loss of generality, assume the contrary that x is an eventually positive solution of (1.1). Proceed as in the proof of Lemma 2.1 to obtain (2.2) for $t \geq t_1$, i.e., rz' is nonincreasing on $[t_2, \infty)$, where $t_2 \geq t_1$. Recall that z is positive on $[t_2, \infty)$. Thus, we have the following two cases.

Case 1. Let $z'(t) > 0$ for $t \geq t_2$. Then, we proceed as in Theorem 2.5 to get a contradiction.

Case 2. Let $z'(t) < 0$ for $t \geq t_2$. By Lemma 2.13, we have (2.9) for $t \geq t_3$, where $\varepsilon > 0$ and $t_3 \geq t_2$. Using (2.9) in (2.4), we have

$$w'(t) + \lambda Q(t)G(\varepsilon R(\sigma(t))) \leq 0 \text{ for } t \geq t_3,$$

where $t_3 \geq t_2$. Integrating the last inequality over the interval $[t_3, t) \subset [t_3, \infty)$, we obtain

$$\lambda \int_{t_3}^t Q(\eta)G(\varepsilon R(\sigma(\eta))) \mathrm{d}\eta \leq -w(t) \leq -(1 + G(p))r(t)z'(t),$$

which implies

$$\frac{\lambda}{1 + G(p)} \frac{1}{r(t)} \int_{t_3}^{t} Q(\eta) G(\varepsilon R(\sigma(\eta))) \mathrm{d}\eta \leq -z'(t) \text{ for } t \geq t_3.$$

Again integrating the last inequality over the interval $[t_3, t) \subset [t_3, \infty)$, we obtain

$$\frac{\lambda}{1 + G(p)} \int_{t_3}^{t} \frac{1}{r(\eta)} \int_{t_3}^{\eta} Q(\zeta) G(\varepsilon R(\sigma(\zeta))) \mathrm{d}\zeta \mathrm{d}\eta \leq z(t_3) \text{ for } t \geq t_3,$$

which contradicts (A10).

The case where x is eventually negative can be dealt similarly, and we omit the details here. This completes the proof. □

Example 2.15. Consider the differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{t}\frac{\mathrm{d}}{\mathrm{d}t}[x(t) + 3\mathrm{e}^{-t}x(t-3)]\right] + \mathrm{e}^{3t}(x(t-1))^{3} = 0 \text{ for } t \geq 3, \tag{2.10}$$

where $r(t) := \mathrm{e}^{t}$, $R(t) := \mathrm{e}^{-t}$, $p(t) := 3\mathrm{e}^{-t}$, $\tau(t) := t - 3$, $q(t) := \mathrm{e}^{3t}$, $\sigma(t) := t - 1$ and $G(u) := u^{3}$ for $t \geq 3$ and $u \in \mathbb{R}$. Then, all the assumptions of Theorem 2.14 holds. Hence, every solution of (2.10) oscillates.

Theorem 2.16. *Let* $-1 \leq p(t) \leq 0$ *for* $t \geq t_0$. *Assume that (C2) and (A1)–(A5) hold. Furthermore, assume that*

(A11)

$$\int_{t_0}^{\infty} \frac{1}{r(\eta)} \int_{t_0}^{\eta} q(\zeta) G(\varepsilon R(\sigma(\zeta))) \mathrm{d}\zeta \mathrm{d}\eta = \infty \text{ for every } \varepsilon > 0$$

and

$$\int_{t_0}^{\infty} \frac{1}{r(\eta)} \int_{t_0}^{\eta} q(\zeta) G(\varepsilon R(\sigma(\zeta))) \mathrm{d}\zeta \mathrm{d}\eta = -\infty \text{ for every } \varepsilon < 0.$$

Then, every unbounded solution of (1.1) *oscillates.*

Proof. Without loss of generality, let x be an eventually positive unbounded solution of (1.1). Then, there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. Proceeding as in the proof of Lemma 2.1, we see that z and z' are of single sign on $[t_2, \infty)$, where $t_2 \geq t_1$. Consequently, we have the following two possible cases.

Case 1. Let $z(t) > 0$ for $t \geq t_2$. Note that in this case, we have $z(t) \leq x(t)$ for $t \geq t_2$.

 (a) Let $z'(t) > 0$ for $t \geq t_2$. We easily get (2.6). Then, proceeding as in Case 1 in the proof of Theorem 2.7, we get a contradiction.

 (b) Let $z'(t) < 0$ for $t \geq t_2$. By Lemma 2.13, we have (2.9) for $t \geq t_3$, where $\varepsilon > 0$ and $t_3 \geq t_2$. Using $z(t) \leq x(t)$ for $t \geq t_2$ and (2.2), we get

$$(rz')'(t) + q(t)G(\varepsilon R(\sigma(t))) \leq 0 \text{ for } t \geq t_3,$$

where $t_3 \geq t_2$. The rest of the proof follows similar to Case 2 in the proof of Theorem 2.14.

Case 2. Let $z(t) < 0$ for $t \geq t_2$. The proof is similar to Case 2 in the proof of Theorem 2.7. The proof is therefore completed. □

Theorem 2.17. *Let* $-1 < -p \leq p(t) \leq 0$ *for* $t \geq t_0$, *where* p *is a constant. Assume that (C2), (A1)–(A5) and (A11) hold. Then, every bounded solution of* (1.1) *either oscillates or converges to zero asymptotically.*

Proof. Without loss of generality, let x be an eventually positive bounded solution of (1.1). Then, there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. Proceeding as in the proof of Lemma 2.1, we see that z and z' are of single sign on $[t_2, \infty)$, where $t_2 \geq t_1$. Consequently, we have the following two possible cases.

Case 1. Let $z(t) > 0$ for $t \geq t_2$. In this case, we proceed as in Case 1 in the proof of Theorem 2.16 and get a contradiction.

Case 2. Let $z(t) < 0$ for $t \geq t_2$. Recalling that z is monotonic, we follow the steps in Case 2 in the proof of Theorem 2.8 and see that $\lim_{t\to\infty} x(t) = 0$.

Hence, the proof of the theorem is complete. □

Combining Theorem 2.16 and Theorem 2.17, we have the following corollary.

Corollary 2.18. *Let* $-1 < -p \leq p(t) \leq 0$ *for* $t \geq t_0$, *where* p *is a constant. Assume that (C2), (A1)–(A5) and (A11) hold. Then, every solution of* (1.1) *either oscillates or converges to zero asymptotically.*

Theorem 2.19. *Let* $-p_1 \leq p(t) \leq -p_2 < -1$ *for* $t \geq t_0$, *where* p_1 *and* p_2 *are constants. Assume that (C2), (A1)–(A5) and (A11) hold. Further, assume that*

(A12) $\int_{t_0}^{\infty} \frac{1}{r(\eta)} \int_{t_0}^{\eta} q(\zeta) \mathrm{d}\zeta \mathrm{d}\eta = \infty$.

Then, every bounded solution of (1.1) *either oscillates or converges to zero asymptotically.*

Proof. Without loss of generality, let x be an eventually positive bounded solution of (1.1). Then, there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. Proceeding as in the proof of Lemma 2.1, we see that z and z' are of single sign on $[t_2, \infty)$, where $t_2 \geq t_1$. Consequently, we have the following two possible cases.

Case 1. Let $z(t) > 0$ for $t \geq t_2$. In this case, we proceed as in Case 1 in the proof of Theorem 2.16 and get a contradiction.

Case 2. Let $z(t) < 0$ for $t \geq t_2$. In this case, $\lim_{t\to\infty} z(t)$ exists as a nonpositive finite value. We claim that $\lim_{t\to\infty} z(t) = 0$. Otherwise, $\lim_{t\to\infty} z(t) < 0$, i.e., there exists $\varepsilon > 0$ such that $z(t) < -\varepsilon$ for $t \geq t_2$. Then, we have $z(t) \geq p(t)x(\tau(t)) \geq -p_1 x(\tau(t))$ for $t \geq t_2$, which implies $x(t) \geq -\frac{1}{p_1} z(\tau^{-1}(t)) \geq \frac{\varepsilon}{p_1}$ for $t \geq t_2$. Consequently, (2.2) becomes

$$(rz')'(t) + q(t)G\left(\frac{\varepsilon}{p_1}\right) \leq 0 \text{ for } t \geq t_2.$$

Integrating the last inequality over the interval $[t_2, t) \subset [t_2, \infty)$, we get

$$-r(t_2)z'(t_2) + G\left(\frac{\varepsilon}{p_1}\right) \int_{t_2}^{t} q(\eta)\mathrm{d}\eta \leq -r(t)z'(t) \text{ for } t \geq t_2.$$

Again, integrating the last inequality over the interval $[t_2, t) \subset [t_2, \infty)$ after dividing through by r, we get

$$-r(t_2)z'(t_2) \int_{t_2}^{t} \frac{1}{r(\eta)}\mathrm{d}\eta + G\left(\frac{\varepsilon}{p_1}\right) \int_{t_2}^{t} \frac{1}{r(\eta)} \int_{t_2}^{\eta} q(\zeta)\mathrm{d}\zeta \mathrm{d}\eta \leq -z(t) \text{ for } t \geq t_2,$$

which contradicts (A12) by (C2). Therefore, $\lim_{t\to\infty} z(t) = 0$. For the rest of the proof, we follow the steps in the last part of Case 2 of Theorem 2.10 to get $\lim_{t\to\infty} x(t) = 0$.

Hence, the proof of the theorem is complete. □

3. Final comments

In this section, we will be giving some simple remarks to conclude the paper.

Remark 3.1. In Theorem 2.4–Theorem 2.19, G is allowed to be linear, sublinear or superlinear. A prototype of the function G satisfying (A2), (A4), (A6) and (A7) is

$$(1 + \alpha|u|^{\beta})|u|^{\gamma}\mathrm{sgn}(u) \text{ for } u \in \mathbb{R},$$

where $\alpha \geq 1$ or $\alpha = 0$ and $\beta, \gamma > 0$ are reals. For verifying (A6), we may take help of the well-known inequality (see [9, p. 292])

$$u^p + v^p \geq h(p)(u+v)^p \text{ for } u, v > 0, \text{ where } h(p) := \begin{cases} 1, & 0 \leq p \leq 1, \\ \dfrac{1}{2^{p-1}}, & p \geq 1. \end{cases}$$

Remark 3.2. If the nonlinear term G is an odd function (presented as in Remark 3.1), it suffices to verify only the first conditions in (A6), (A7), (A10) and (A11).

Remark 3.3. The results of this paper also hold for equations of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[r(t)\frac{\mathrm{d}}{\mathrm{d}t}[x(t) + p(t)x(\tau(t))]\right] + \sum_{i=1}^{m} q_i(t)G_i(x(\sigma_i(t))) = 0 \text{ for } t \geq t_0,$$

where r, p and τ satisfy previously mentioned assumptions and q_i, G_i, σ_i $(i = 1, 2, \cdots, m)$ satisfy the assumptions in (A1)–(A3). In order to extend Theorem 2.4–Theorem 2.19, we may assume that there exists an index i such that G_i, q_i, σ_i, Q_i fulfills (A4)–(A12).

We finalize the paper by presenting two examples, which show how Remark 3.3 can be applied.

Example 3.4. Consider the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{-4t}\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t) - \frac{3}{\mathrm{e}^2}x(t-1)\right]\right] + 8\big(x(t)\big)^3 + 16\mathrm{e}^{-5t}x(\tfrac{t}{2}) = 0 \text{ for } t \geq 1, \qquad (3.1)$$

where $r(t) := \mathrm{e}^{-4t}$, $p(t) :\equiv -\frac{3}{\mathrm{e}^2}$, $\tau(t) := t - 1$, $q_1(t) :\equiv 8$, $\sigma_1(t) := t$, $G_1(u) := u^3$, $q_2(t) := 16\mathrm{e}^{-5t}$, $\sigma_2(t) = \frac{t}{2}$ and $G_2(u) := u$ for $t \geq 1$ and $u \in \mathbb{R}$. All the assumptions of Corollary 2.9 can be verified with the index $i = 1$. Hence, due to Remark 3.3, every solution of (3.1) oscillates or converges to zero asymptotically. Clearly $x(t) = \mathrm{e}^{-2t}$ for $t \geq 1$ is a solution satisfying $\lim_{t\to\infty} x(t) = 0$.

Example 3.5. Consider the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{t^2}\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t) - \mathrm{e}^{-\pi}x(t-\pi)\right]\right]$$
$$+ 4\cosh(\pi)t\big[\mathrm{e}^{-\frac{\pi}{2}}(t+1)x(t-\tfrac{\pi}{2}) + x(t-\pi)\big] = 0 \text{ for } t \geq 2\pi, \qquad (3.2)$$

where $r(t) := \frac{1}{t^2}$, $R(t) := \frac{1}{t}$, $p(t) :\equiv \mathrm{e}^{-\pi}$, $\tau(t) := t - \pi$, $q_1(t) := 4\mathrm{e}^{-\frac{\pi}{2}}\cosh(\pi)t(t+1)$, $\sigma_1(t) = t - \frac{\pi}{2}$, $G_1(u) := u$, $q_2(t) := 4\cosh(\pi)t$, $\sigma_2(t) := t - \pi$ and $G_2(u) := u$ for $t \geq 2\pi$ and $u \in \mathbb{R}$. All the assumptions of Corollary 2.18 can be verified with the index $i = 2$. In particular, for (A11), we have

$$\int_{2\pi}^{\infty} \frac{1}{\eta} \int_{2\pi}^{\eta} 4\cosh(\pi)\zeta \frac{\varepsilon}{\zeta - \pi}\mathrm{d}\zeta\mathrm{d}\eta = \infty \text{ for any } \varepsilon > 0.$$

Hence, due to Remark 3.3, every solution of (3.2) oscillates, and such a solution is $x(t) = \mathrm{e}^t \sin(t)$ for $t \geq 2\pi$.

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