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Research Article

Analytic approximation of the transition density function under a multi-scale volatility model

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Abstract

The transition density function plays an important role in understanding and explaining the dynamics of the stochastic process. We propose an approach which can be used for the analytic approximation of the transition density related to a multi-scale stochastic volatility model. Using perturbation theory, we compute the leading-order term and the first-order correction terms. A numerical test also confirms the effectiveness of the model.

Mathematics Subject Classification (2010). 60G15

Keywords. Multi-scale stochastic volatility model, transition density function, perturbation theory

1. Introduction

A stochastic process provides a useful tool to analyze time-series data and wide applications in many fields such as physics, finance, biotechnology, telecommunication studies, and so on. Especially, the transition density function of a continuous-time process plays an important role in understanding and explaining the dynamics of the process. However, the transition density functions are unknown for general diffusion processes except for a few special cases (refer to Aït-Sahalia [2], Black and Scholes [3], Cox et al. [5], and Vasicek [11]). So, finding analytical approximations to them is an important as an alternative approach. The main advantage of the analytical approximation approaches compared to other numerical methods, such as finite-difference method and Fourier inversion et al., is that in general the first ones are much faster and precise at least under certain model parameter regime. In addition, analytic approximation formulas retain qualitative model information and preserve an explicit dependence of the results on the underlying parameters.

Fat-tailed distribution and volatility clustering are stylized facts in the area of financial modeling. Generally, the impact of shocks, which accounts for fat-tailed distributions, tends to be short-lived, while the effects of business cycles, which explain volatility clustering, are more lasting. A one-time scale model cannot reflect these facts, whereas a multi-scale model can. Particularly, two factors in volatility are needed in order to express a well-separated time scale and these not only control the persistence of the volatility but also revert rapidly to the mean and contribute to the volatility of volatility (refer to Adrian and Rosenberg [1], Chernov et al. [4], and Gallant et al. [8]).

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Based upon the observation that multi-scale stochastic volatility exists in financial time-series data, we incorporate a slowly varying process into the result of Fouque and Zhou [6] and formulate a two-factor stochastic volatility model which contains a slowly varying process representing one persistent factor for volatility and an ergodic process displaying rapidly moving fluctuations. Using perturbation theory, we obtain the approximate transition density under the multi-scale stochastic volatility model. Our result can easily be applied to perturbed Gaussian copula and to the valuation of FX quanto options to a third currency, though these remain topics for future research.

2. Main results

2.1. Problem formulation

We start with the process (X_t, Y_t, Z_t) which follows stochastic differential equations under a risk-neutral measure:

$$\begin{aligned} dX_t &= f(Y_t, Z_t)dW_t^{(X)}, \\ dY_t &= \frac{1}{\epsilon}(m - Y_t)dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}dW_t^{(Y)}, \\ dZ_t &= \delta g(Z_t)dt + \sqrt{\delta}h(Z_t)dW_t^{(Z)}, \end{aligned} \tag{2.1}$$

where $W_t^{(X)}$, $W_t^{(Y)}$, and $W_t^{(Z)}$ are standard Brownian motions where are correlated as follows:

$$d\langle W^{(X)}, W^{(Y)}\rangle_t = \rho_{XY}dt, \quad d\langle W^{(X)}, W^{(Z)}\rangle_t = \rho_{XZ}dt, \quad d\langle W^{(Y)}, W^{(Z)}\rangle_t = \rho_{YZ}dt.$$

Here, the correlation coefficients ρ_{XY}, ρ_{XZ}, and ρ_{YZ} satisfy $-1 \le \rho_{XY}, \rho_{XZ}, \rho_{YZ} \le 1$ and $1 + 2\rho_{XY}\rho_{XZ}\rho_{YZ} - \rho_{XY}^2 - \rho_{XZ}^2 - \rho_{YZ}^2 > 0$ in order to ensure positive definiteness of the covariance matrix of the standard Brownian motion and the parameters ϵ, ν, and δ are positive constants with the same order of $\epsilon \approx \delta \ll 1$ being small. We also assume that the usual Lipschitz and growth conditions for the coefficients $g(z)$ and $h(z)$ are satisfied. We do not specify the concrete form of f in that it will not play an essential role in the asymptotic method utilized in this paper. However, f must satisfy a bound above and below 0.

For a fixed time $T > 0$, our goal is to calculate the following transition density of (2.1) at time $t < T$:

$$u^{\epsilon,\delta} := \mathbb{P}\{X_T \in d\xi | X_t = x, Y_t = y, Z_t = z\},$$

where ξ is an arbitrary number.

2.2. Asymptotic method

Perturbation theory as developed by Fouque et al. [7] is a methodology which is utilized to find an approximated solution when the original problem is difficult to solve by separating it into more easily solvable, simple parts. If we apply the Feynman-Kac formula, we find that $u^{\epsilon,\delta}$ satisfies the following Kolmogorov backward equation

$$\begin{aligned} &\mathcal{L}^{\epsilon,\delta}u^{\epsilon,\delta}(t,x,y,z) = 0, \quad t < T, \\ &\mathcal{L}^{\epsilon,\delta} := \frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\epsilon}}\mathcal{M}_3, \\ &u^{\epsilon,\delta}(T,x,y,z) = \delta(\xi; x), \end{aligned} \tag{2.2}$$

where

$$\mathcal{L}_0 = (m-y)\frac{\partial}{\partial y} + \nu^2\frac{\partial^2}{\partial y^2}, \qquad \mathcal{L}_1 = \nu\sqrt{2}\rho_{XY}f(y,z)\frac{\partial^2}{\partial x\partial y},$$
$$\mathcal{L}_2 = \frac{\partial}{\partial t} + \frac{1}{2}f^2(y,z)\frac{\partial^2}{\partial x^2}, \qquad \mathcal{M}_1 = \rho_{XZ}f(y,z)h(z)\frac{\partial^2}{\partial x\partial z},$$
$$\mathcal{M}_2 = g(z)\frac{\partial}{\partial z} + \frac{1}{2}h^2(z)\frac{\partial^2}{\partial z^2}, \quad \mathcal{M}_3 = \nu\sqrt{2}\rho_{YZ}h(z)\frac{\partial^2}{\partial y\partial z}.$$

Here, $\delta(\xi;x)$ is the Dirac delta function of ξ with a spike at $\xi = x$. Note that $\mathcal{L}_0$ is the infinitesimal generator of the Ornstein-Uhlenbeck (OU) process Y_t. $\mathcal{L}_1$ contains the mixed partial derivative due to the correlation of the two Brownian motions $W^{(X)}$ and $W^{(Y)}$. $\mathcal{L}_2$ is the operator of a generalized version of the Brownian motion at the volatility level $f(y,z)$ in stead of constant volatility. $\mathcal{M}_1$ includes the mixed partial derivative due to the correlation of the two Brownian motions $W^{(X)}$ and $W^{(Z)}$. $\mathcal{M}_2$ is the infinitesimal generator of the process Z_t. Finally, $\mathcal{M}_3$ holds the mixed partial derivative due to the correlation of the two Brownian motions $W^{(Y)}$ and $W^{(Z)}$.

Before we solve the problem (2.2), we write a useful lemma about the centering (or solvability) condition on the Poisson equation related to the operator $\mathcal{L}_0$ as follows:

Lemma 2.1. *If solution to the Poisson equation*

$$\mathcal{L}_0\chi(y) + \psi(y) = 0 \tag{2.3}$$

exists, then the centering condition $\langle\psi\rangle = \int \psi(y)\frac{1}{\sqrt{2\pi\nu^2}}\exp\left[-\frac{(y-m)^2}{2\nu^2}\right] = 0$ *must be satisfied, where the notation* $\langle\cdot\rangle$ *is the average (or expectation) with respect to the invariant distribution of* Y_t*. If then, solutions of* (2.3) *are given cy the form*

$$\chi(y) = \int_0^\infty \mathbb{E}^y[\psi(Y_t) \mid Y_0 = y]dt.$$

Proof. Refer to Fouque et al. [7]. □

Applying the solution of problem (2.2) to an asymptotic method, we consider the asymptotic expansion as follows:

$$u^{\epsilon,\delta}(t,x,y,z) = \Sigma_{j=0}^{\infty}\delta^{j/2}u_j^{\epsilon}(t,x,y,z), \tag{2.4}$$
$$u_j^{\epsilon}(t,x,y,z) = \Sigma_{i=0}^{\infty}\epsilon^{i/2}u_{i,j}(t,x,y,z).$$

Therefore, $u^{\epsilon,\delta}$ is a series of the general term $\epsilon^{i/2}\delta^{j/2}u_{i,j}$. Plugging expansion (2.4) into (2.2) respectively leads to u_0^ϵ and u_1^ϵ which satisfies the problem

$$\left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)u_0^\epsilon = 0, \qquad t < T, \tag{2.5}$$
$$u_0^\epsilon(T,x,y,z) = \delta(\xi;x)$$

and

$$\left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)u_1^\epsilon = -\left(\mathcal{M}_1 + \frac{1}{\sqrt{\epsilon}}\mathcal{M}_3\right)u_0^\epsilon, \qquad t < T, \tag{2.6}$$
$$u_1^\epsilon(T,x,y,z) = 0.$$

Note that these are singular perturbation problems with respect to the small parameter ϵ.

Theorem 2.2. *The leading-order term* $u_{0,0}$ *is independent of* y *and satisfies the partial differential equation (PDE) problem*

$$\langle\mathcal{L}_2\rangle u_{0,0}(t,x,z) = 0 \tag{2.7}$$
$$u_{0,0}(T,x,z) = \delta(\xi;x),$$

where

$$\langle \mathcal{L}_2 \rangle = \frac{\partial}{\partial t} + \frac{1}{2}\langle f^2(\cdot, z)\rangle \frac{\partial^2}{\partial x^2}$$

and then one obtains the solution $u_{0,0}$ *of the PDE*

$$u_{0,0} = \frac{1}{\overline{\sigma}(z)\sqrt{2\pi(T-t)}} \exp\left[-\frac{(\xi - x)^2}{2\overline{\sigma}^2(z)(T-t)}\right], \tag{2.8}$$

where $\overline{\sigma}(z) = \sqrt{\langle f^2(\cdot, z)\rangle}$.

Proof. Applying the expansion (2.4) with $j = 0$ to (2.5) leads to

$$\begin{aligned} &\frac{1}{\epsilon}\mathcal{L}_0 u_{0,0} + \frac{1}{\sqrt{\epsilon}}\left(\mathcal{L}_0 u_{1,0} + \mathcal{L}_1 u_{0,0}\right) + \left(\mathcal{L}_0 u_{2,0} + \mathcal{L}_1 u_{1,0} + \mathcal{L}_2 u_{0,0}\right) \\ &\qquad + \sqrt{\epsilon}\left(\mathcal{L}_0 u_{3,0} + \mathcal{L}_1 u_{2,0} + \mathcal{L}_2 u_{1,0}\right) + \cdots = 0. \end{aligned} \tag{2.9}$$

Multiplying (2.9) by ϵ and then letting ϵ go to zero, we obtain

$$\mathcal{L}_0 u_{0,0} = 0. \tag{2.10}$$

Recalling that the operator $\mathcal{L}_0$ is the generator of the OU process Y_t, the solution $u_{0,0}$ of (2.10) must be a constant with respect to the y variable; $u_{0,0} = u_{0,0}(t, x, z)$. Also, we have $\mathcal{L}_0 u_{1,0} + \mathcal{L}_1 u_{0,0} = 0$. $\mathcal{L}_1 u_{0,0}$=0 holds since $u_{0,0}$ does not rely on the y variable. So, we have

$$\mathcal{L}_0 u_{1,0} = 0.$$

Then $u_{1,0}$ is also independent of the y variable; $u_{1,0} = u_{1,0}(t, x, z)$. Therefore, the first two terms $u_{0,0}$ and $u_{1,0}$ do not depend on the current level y of the fast scale volatility driving the process Y_t. One can continue to eliminate the terms of order $1, \sqrt{\epsilon}, \epsilon, \cdots$. From the order 1 terms, we get $\mathcal{L}_0 u_{2,0} + \mathcal{L}_1 u_{1,0} + \mathcal{L}_2 u_{0,0} = 0$. This PDE becomes

$$\mathcal{L}_0 u_{2,0} + \mathcal{L}_2 u_{0,0} = 0 \tag{2.11}$$

due to the y-independence of $u_{1,0}$. This is a Poisson equation for $u_{2,0}$ with respect to the operator $\mathcal{L}_0$ with the source term $\mathcal{L}_2 u_{0,0}$. Then, Lemma 2.1 applied to (2.11) leads to (2.8). □

Note that $u_{0,0}$ is identical to the transition density of the one-dimensional Brownian motion, where only the coefficient σ is replaced by $\overline{\sigma}(z)$. Next, we obtain analytic formulas for the correction terms $u_{1,0}$ and $u_{0,1}$, respectively.

Theorem 2.3. $u_{1,0}$ *is independent of the* y *variable and the first order correction term satisfies the PDE*

$$\begin{aligned} \langle \mathcal{L}_2 \rangle u_{1,0}(t, x, z) &= \mathcal{A} u_{0,0} \\ u_{1,0}(T, x, z) &= 0, \end{aligned}$$

where $\mathcal{A} = \langle \mathcal{L}_1 \mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle)\rangle$ *and then one obtains the solution of the PDE*

$$\sqrt{\epsilon} u_{1,0} = -(T-t)R(z)\frac{\partial^3}{\partial x^3} u_{0,0},$$

where the constant parameter $R(z) = \frac{\nu \rho_{XY}\sqrt{\epsilon}}{\sqrt{2}}\langle f(\cdot, z)\phi_y(\cdot, z)\rangle$. *Here,* $\phi(y, z)$ *is defined as* $\mathcal{L}_0 \phi = f^2(y, z) - \langle f^2(\cdot, z)\rangle$.

Proof. The order $\sqrt{\epsilon}$ terms in (2.9) lead to $\mathcal{L}_0 u_{3,0} + \mathcal{L}_1 u_{2,0} + \mathcal{L}_2 u_{1,0} = 0$ which is a Poisson equation for $u_{3,0}$ whose centering condition is given by

$$\langle \mathcal{L}_1 u_{2,0} + \mathcal{L}_2 u_{1,0}\rangle = 0. \tag{2.12}$$

Meanwhile, from (2.7) and (2.11) we get

$$u_{2,0} = -\mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle\mathcal{L}_2\rangle)u_{0,0} + c(t,x,z) \tag{2.13}$$

for an arbitrary function $c(t,x,z)$ independent of the y variable. Plugging (2.13) into (2.12), we derive a PDE for $u_{1,0}$ as follows:

$$\langle\mathcal{L}_2\rangle u_{1,0} = \mathcal{A}u_{0,0}.$$

We note that the operator $\mathcal{A}$ is the same as $\langle\mathcal{L}_1\mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle\mathcal{L}_2\rangle)\rangle$. Then, we obtain the result of Theorem 2.3 by direct computation. □

In order to obtain another first-order correction term, it is necessary to consider another singular perturbation problem (2.6). Applying expansion (2.4) with $j = 0$ and $j = 1$ to (2.6) leads to

$$\begin{aligned}\frac{1}{\epsilon}\mathcal{L}_0u_{0,1} &+ \frac{1}{\sqrt{\epsilon}}\left(\mathcal{L}_0u_{1,1} + \mathcal{L}_1u_{0,1}\right) + \left(\mathcal{L}_0u_{2,1} + \mathcal{L}_1u_{1,1} + \mathcal{L}_2u_{0,1}\right)\\ &+\sqrt{\epsilon}\left(\mathcal{L}_0u_{3,1} + \mathcal{L}_1u_{2,1} + \mathcal{L}_2u_{1,1}\right) + \cdots\\ = &-\frac{1}{\sqrt{\epsilon}}\mathcal{M}_3u_{0,0} - (\mathcal{M}_1u_{0,0} + \mathcal{M}_3u_{1,0}) - \sqrt{\epsilon}(\mathcal{M}_1u_{1,0} + \mathcal{M}_3u_{2,0}) - \cdots .\end{aligned} \tag{2.14}$$

Theorem 2.4. *$u_{0,1}$ is independent of the y variable and the another first-order correction term satisfies the PDE problem*

$$\langle\mathcal{L}_2\rangle u_{0,1}(t,x,z) = -\langle\mathcal{M}_1\rangle u_{0,0} \qquad \text{with} \qquad u_{0,1}(T,x,z) = 0,$$

where $\langle\mathcal{M}_1\rangle = \rho_{XZ}h(z)\langle f(\cdot,z)\rangle\frac{\partial^2}{\partial x\partial z}$ after which it becomes possible to obtain the solution of the PDE

$$\sqrt{\delta}u_{0,1} = \frac{T-t}{2}S(z)\frac{\partial^2}{\partial x\partial z}u_{0,0},$$

where $S(z) = \rho_{XZ}h(z)\sqrt{\delta}\langle f(\cdot,z)\rangle$.

Proof. Multiplying (2.14) by ϵ and then letting ϵ go to zero, we find the first two leading-order terms as follows:

$$\begin{aligned}&\mathcal{L}_0u_{0,1} = 0,\\ &\mathcal{L}_0u_{1,1} + \mathcal{L}_1u_{0,1} = -\mathcal{M}_3u_{0,0}.\end{aligned}$$

Because the operator $\mathcal{L}_0$ is the generator of the OU process Y_t, $u_{0,1}$ (the solution of $\mathcal{L}_0u_{0,1} = 0$) must be a constant with respect to the y variable. Because $\mathcal{M}_3$ has a derivative with respect to the y variable and $u_{0,0}$ does not rely on y, we obtain $\mathcal{M}_3u_{0,0} = 0$. Moreover, because each term of $\mathcal{L}_1$ has a derivative with respect to y, $\mathcal{L}_1u_{0,1} = 0$ holds. Thus, the equation $\mathcal{L}_0u_{1,1} + \mathcal{L}_1u_{0,1} = -\mathcal{M}_3u_{0,0}$ reduces to $\mathcal{L}_0u_{1,1} = 0$, meaning that $u_{1,1}$ does not depend on the y variable. Hence, the two terms $u_{0,1}$ and $u_{1,1}$ do not depend on the current level y of the fast-scale volatility driving process Y_t; $u_{0,1} = u_{0,1}(t,x,z)$ and $u_{1,1} = u_{1,1}(t,x,z)$. In this way, it becomes possible to continue to remove the terms of order 1, $\sqrt{\epsilon}$, ϵ and others. For the order 1 term, we have $\mathcal{L}_0u_{2,1}+\mathcal{L}_1u_{1,1}+\mathcal{L}_2u_{0,1} = -(\mathcal{M}_1u_{0,0}+\mathcal{M}_3u_{1,0})$. This PDE becomes $\mathcal{L}_0u_{2,1} + \mathcal{L}_2u_{0,1} + \mathcal{M}_1u_{0,0} = 0$ due to the y-independence of $u_{1,0}$ and $u_{1,1}$. This is a Poisson equation for $u_{2,1}$ with respect to the operator $\mathcal{L}_0$ in the y variable, which has a solution only if $\mathcal{L}_2u_{0,1} + \mathcal{M}_1u_{0,0}$ is centered with respect to the invariant distribution of Y_t. Because $u_{0,0}$ and $u_{0,1}$ do not depend on the variable y, we have

$$\langle\mathcal{L}_2\rangle u_{0,1} = -\langle\mathcal{M}_1\rangle u_{0,0}.$$

We note that the operator $\langle\mathcal{M}_1\rangle$ is the same as $\rho_{XZ}h(z)\langle f(\cdot,z)\rangle\frac{\partial^2}{\partial x\partial z}$. Then, we obtain the result of Theorem 2.4 by direct computation. □

As a result, one can approximate $u^{\epsilon,\delta}$ to the summation of the leading-order term $u_{0,0}$ and the first correction terms $u_{1,0}$ and $u_{0,1}$ as follows:

$$u^{\epsilon,\delta}(t,x,y,z) \approx u_{0,0}(t,x,z) + \sqrt{\epsilon}u_{1,0}(t,x,z) + \sqrt{\delta}u_{0,1}(t,x,z).$$

Note that all the original parameters are absorbed in the group parameters $R(z)$ and $S(z)$, respectively. Also, The present level $y(z)$ of the hidden process Y_t need not be specified in the present approximation. It is melted down into the group parameters in the averaged form. By a straightforward calculation, we obtain

$$\frac{\partial^3 u_{0,0}}{\partial x^3} = \left\{-\frac{3(\xi-x)}{\sqrt{2\pi}\overline{\sigma}^5(z)(T-t)^{5/2}} + \frac{(\xi-x)^3}{\sqrt{2\pi}\overline{\sigma}^7(z)(T-t)^{7/2}}\right\}\exp\left[-\frac{(\xi-x)^2}{2\overline{\sigma}^2(z)(T-t)}\right],$$

$$\frac{\partial^2 u_{0,0}}{\partial x \partial z} = \left\{-\overline{\sigma}'(z) + \frac{\overline{\sigma}(z)(\xi-x)^2}{(T-t)\overline{\sigma}^3(z)} - \frac{2\overline{\sigma}'(z)}{\overline{\sigma}(z)}\right\}\frac{(\xi-x)}{\sqrt{2\pi}\overline{\sigma}^3(z)(T-t)^{3/2}}\exp\left[-\frac{(\xi-x)^2}{2\overline{\sigma}^2(z)(T-t)}\right],$$

respectively. We can use $\sqrt{\epsilon}u_{1,0}$ and $\sqrt{\delta}u_{0,1}$ from this result to obtain an analytic approximation of the transition density.

In order to guarantee that our approximated transition density function is non-negative, we newly define the following:

$$\tilde{u}^{\epsilon,\delta} = \frac{1}{N}u_{0,0}\left(1 + \tanh\left(\frac{\sqrt{\epsilon}u_{1,0}}{u_{0,0}} + \frac{\sqrt{\delta}u_{0,1}}{u_{0,0}}\right)\right). \tag{2.15}$$

Here, the normalizing constant N must be introduced owing to the presence of the second partial derivative in slow-scale factor.

3. Numerical experiment

In this section, we illustrate the effectiveness of our approach (2.15) by showing a numerical result. The chosen parameter values are as follows:

$$R(z) = 0.02, \quad S(z) = 0.03, \quad T-t = 1, \quad \overline{\sigma}(z) = 0.5, \quad \overline{\sigma}'(z) = 0.9, \quad x = 0.$$

Figure 1 shows that constant volatility is the leading-order term (standard Gaussian), short-scale volatility is a combination of the leading-order term and the first correction term driven by the fast moving fluctuation (Fouque and Zhou model), and the multi-scale volatility is a combination of the leading-order term and the first correction terms under the multi-scale stochastic volatility model. Figure 1 shows that these three models do not take negative values at any point and that the short-scale volatility and multi-scale volatility shift to the right from the constant volatility. Furthermore, there is a small gap between the model by Fouque and Zhou and the multi-scale stochastic volatility model. However, this implies that the prices of financial derivatives with short-term maturity levels are ruled by a fast-scale volatility process, while those of financial derivatives with long-term maturity levels are dominated by a slow-scale volatility process in the area of financial modeling. Note that our numerical result shows that this picture is sensitive to the choice of the involved parameters and gives a lot of flexibility to the shape of the transition densities.

4. Final remarks

Stochastic processes are popular in modeling various economics and financial variables. The transition density function plays a key role in the analysis of continuous-time diffusion models. In this paper, we obtained the analytic approximation of the transition density of a multi-scale stochastic volatility model. A simulation result shows that our result could explain a financial point of view.

This paper offers various possible directions for further development. For example, our result can easily be applied to perturbed Gaussian copula and to the valuation of FX

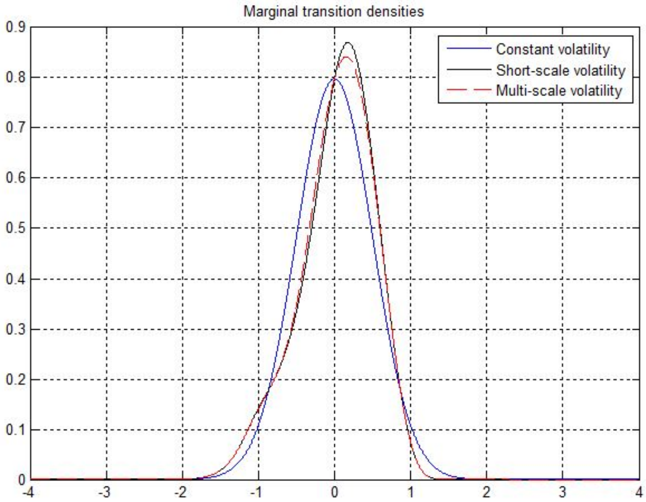

Figure 1. Multi-scale stochastic volatility effects on the transition density.

quanto options to a third currency. Also, this result can provide a very useful guide for credit risk management (refer to [9] and [10]). We leave these issues as future research topics.

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