Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 48 (3) (2019), 700 – 710
DOI : 10.15672/HJMS.2018.592

Research Article

Super (a, d)-star-antimagic graphs

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Abstract

A simple graph $G = (V, E)$ admitting an H-covering is said to be (a, d)-H-antimagic if there exists a bijection $f : V \cup E \to \{1, 2, \dots, |V| + |E|\}$ such that, for all subgraphs H' of G isomorphic to H, $wt_f(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e)$, form an arithmetic progression $a, a+d, \dots, a+(t-1)d$, where a is the first term, d is the common difference and t is the number of subgraphs in the H-covering. Then f is called an *(a, d)-H-antimagic labeling*. If $f(V) = \{1, 2, \dots, |V|\}$, then f is called *super (a, d)-H-antimagic labeling*. In this paper we investigate the existence of super (a,d)-star-antimagic labelings of a particular class of banana trees and construct a star-antimagic graph.

Mathematics Subject Classification (2010). 05C78, 05C70

Keywords. H-covering, (super) (a, d)-H-antimagic labeling, star, banana tree

1. Introduction

Let $G = (V, E)$ be a finite simple graph. A family of subgraphs $H_1, H_2, \dots, H_t$ of G is called an *edge-covering* of G if each edge of E belongs to at least one of the subgraphs H_i, $i = 1, 2, \dots, t$. Then the graph G admitting an H-covering is *(a, d)-H-antimagic* if there exists a bijection $f : V \cup E \to \{1, 2, \dots, |V| + |E|\}$ such that, for all subgraphs H' of G isomorphic to H, the H'-weights,

$$wt_f(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e),$$

form an arithmetic progression $a, a + d, \dots, a + (t - 1)d$, where $a > 0$ is the first term, $d \geq 0$ is the common difference and t is the number of subgraphs of G isomorphic to H. Such a labeling is called *super* if $f(V) = \{1, 2, \dots, |V|\}$. For $d = 0$ it is called *H-magic* and *H-supermagic*, respectively.

The notion of H-magic graphs was due to Gutiérrez and Lladó [3]. Inayah, Salman and Simanjuntak [4] introduced the concept of (a, d)-H-antimagic labeling. In [5] they proved that there exists a super (a, d)-H-antimagic total labeling for shackles of a connected graph

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Received: 12.01.2017; Accepted: 04.12.2017

H. Semaničová-Feňovčíková, Bača, Lascsáková, Miller and Ryan [8] proved that wheels W_n, $n \geq 3$ are super (a,d)-C_k-antimagic for every $k = 3, 4, \ldots, n-1, n+1$ and $d = 0, 1, 2$. For more information see [1,2] and [7].

2. Preliminaries and known results

We use the following notations. For two integers a, b, $a < b$, let $[a, b]$ denote the set of all integers from a to b. For any set $\mathbb{S}$, subset of integers $\mathbb{Z}$ we write, $\sum \mathbb{S} = \sum_{x \in \mathbb{S}} x$ and for an integer k, let $k + \mathbb{S} = \{k + x : x \in \mathbb{S}\}$. Thus $k + [a, b]$ is the set $\{x \in \mathbb{Z} : k + a \leq x \leq k + b\}$. It can be easily verified that $\sum(k + \mathbb{S}) = k|\mathbb{S}| + \sum \mathbb{S}$.

If $\mathbb{P} = \{X_1, X_2, \ldots, X_n\}$ is a partition of a set X of integers with the same cardinality then we say $\mathbb{P}$ is an *n-equipartition of X*. Also we denote the set of subsets sums of the parts of $\mathbb{P}$ by $\sum \mathbb{P} = \{\sum X_1, \sum X_2, \ldots, \sum X_n\}$.

Lemma 2.1. [3] *Let h and k be two positive integers. For each integer $0 \leq t \leq \lfloor h/2 \rfloor$ there is a k-equipartition $\mathbb{P}$ of $[1, hk]$ such that $\sum \mathbb{P}$ is an arithmetic progression of difference $d = h - 2t$.*

Lemma 2.2. [6] *If h is even, then there exists a k-equipartition $\mathbb{P} = \{X_1, X_2, \ldots, X_k\}$ of $X = [1, hk]$ such that $\sum X_r = h(hk+1)/2$ for $1 \leq r \leq k$.*

Arrange the numbers $X = [1, hk]$ in a matrix $\mathcal{A} = (a_{i,j})_{h \times k}$, where $a_{i,j} = (i-1)k + j$ for $1 \leq i \leq h$ and $1 \leq j \leq k$. For $1 \leq r \leq k$, define $X_r = \{a_{i,r} : 1 \leq i \leq h/2\} \cup \{a_{i,k-r+1} : h/2 + 1 \leq i \leq h\}$. Then,

$$\begin{aligned}\sum X_r &= \sum_{i=1}^{\frac{h}{2}} a_{i,r} + \sum_{i=\frac{h}{2}+1}^{h} a_{i,k-r+1} \\ &= \sum_{i=1}^{\frac{h}{2}} \{(i-1)k + r\} + \sum_{i=\frac{h}{2}+1}^{h} \{(i-1)k + k - r + 1\} = \tfrac{h(hk+1)}{2}.\end{aligned}$$

Lemma 2.3. [6] *Let h and k be two positive integers such that h is even and $k \geq 3$ is odd. Then there exists a k-equipartition $\mathbb{P} = \{X_1, X_2, \ldots, X_k\}$ of $X = [1, hk]$ such that $\sum X_r = (h-1)(hk+k+1)/2 + r$ for $1 \leq r \leq k$. Thus, $\sum \mathbb{P}$ is a set of consecutive integers given by $\sum \mathbb{P} = (h-1)(hk+k+1)/2 + [1, k]$.*

Let us arrange the set of integers $X = \{1, 2, 3, \ldots, hk\}$ in a matrix $\mathcal{A} = (a_{i,j})_{h \times k}$, where $a_{i,j} = (i-1)k + j$ for $1 \leq i \leq h$ and $1 \leq j \leq k$. We construct a k-equipartition $Y_1, Y_2, \ldots, Y_k$ using the first $(h-1)$ rows of the matrix as $Y_r = \{a_{i,r} : 1 \leq i \leq h/2\} \cup \{a_{i,k-r+1} : h/2 + 1 \leq i \leq h - 1\}$. For $1 \leq r \leq k$, we define $X_r = Y_{\sigma(r)} \cup \{(h-1)k + \pi(r)\}$, where σ and π denote the permutations of $\{1, 2, \ldots, k\}$ given by

$$\sigma(r) = \begin{cases} \frac{k-2r+1}{2} & \text{for } 1 \leq r \leq \frac{k-1}{2} \\ \frac{3k-2r+1}{2} & \text{for } \frac{k+1}{2} \leq r \leq k \end{cases}$$

and

$$\pi(r) = \begin{cases} 2r & \text{for } 1 \leq r \leq \frac{k-1}{2} \\ 2r - k & \text{for } \frac{k+1}{2} \leq r \leq k. \end{cases}$$

Then it can be verified that $\sum X_r = (h-1)(hk+k+1)/2 + r$ for $1 \leq r \leq k$.

Lemma 2.4. [6] *Let h and k be two positive integers and let h be odd. Then there exists a k-equipartition $\mathbb{P} = \{X_1, X_2, \ldots, X_k\}$ of $X = [1, hk]$ such that $\sum X_r = (h-1)(hk+k+1)/2 + r$ for $1 \leq r \leq k$. Thus, $\sum \mathbb{P}$ is a set of consecutive integers given by $\sum \mathbb{P} = (h-1)(hk+k+1)/2 + [1, k]$.*

Arrange the numbers $X = [1, hk]$ in a matrix $\mathcal{A} = (a_{i,j})_{h\times k}$, where $a_{i,j} = (i-1)k+j$ for $1 \le i \le h$ and $1 \le j \le k$. For $1 \le r \le k$, define $X_r = \{a_{i,r} : 1 \le i \le (h+1)/2\} \cup \{a_{i,k-r+1} : (h+3)2 \le i \le h\}$. Then $\sum \mathbb{P} = (h-1)(hk+k+1)/2 + [1,k]$.

3. Super star-antimagic graphs

In this section we prove that a particular class of banana trees are super star-antimagic. The star S_n, $n \ge 1$ is a graph isomorphic to the complete bipartite graph $K_{1,n}$. A banana tree $Bt(n_1, n_2, \ldots, n_k)$ is the tree obtained by joining a vertex v to one leaf vertex of each star in a family of disjoint stars $S_{n_1}, S_{n_2}, \ldots, S_{n_k}$. If $n_1 = n_2 = \cdots = n_k = n$ we will use the notation Bt_n^k instead of $Bt(n, n, \ldots, n)$.

Theorem 3.1. *The banana tree Bt_k^k, $k \ge 3$ admits a super (a,d)-S_k-antimagic labeling for $d \in \{0, 2, 4, \ldots, k+1\}$ if k is odd and $d \in \{1, 3, 5, \ldots, k+1\}$ if k is even.*

Proof. Let the vertex set and the edge set of the banana tree Bt_k^k be

$$V(Bt_k^k) = \{v, v_i, v_i^j : i = 1, 2, \ldots, k; j = 1, 2, \ldots, k\},$$
$$E(Bt_k^k) = \{v_i v_i^j : i = 1, 2, \ldots, k; j = 1, 2, \ldots, k\} \cup \{v v_1^j : j = 1, 2, \ldots, k\}.$$

Evidently, the graph Bt_k^k admits a S_k-covering consisting of $k+1$ stars isomorphic to S_k. We denote the k-stars by the symbols $\{S_k^1, S_k^2, \ldots, S_k^k, S_k^{k+1}\}$ such that the vertex set of S_k^i, $i = 1, 2, \ldots, k$, is $V(S_k^i) = \{v_i, v_j^i : j = 1, 2, \ldots, k\}$ and its edge set is $E(S_k^i) = \{v_i v_j^i : j = 1, 2, \ldots, k\}$. For the star S_k^{k+1} holds $V(S_k^{k+1}) = \{v, v_i : i = 1, 2, \ldots, k\}$ and $E(S_k^{k+1}) = \{vv^i : i = 1, 2, \ldots, k\}$.

Let us distinguish two cases.

Case i: k is odd.

Since $k+1$ is even, by Lemma 2.3 then there exists a k-equipartition $\mathbb{P} = \{X_1, X_2, \ldots, X_k\}$ of $X = [1, (k+1)k]$ such that

$$\sum X_i = \tfrac{k(k+1)^2}{2} + i \quad \text{for } 1 \le i \le k. \tag{3.1}$$

It can be easily verified by the definition of X_i in Lemma 2.3 that for $1 \le i \le k$

$$\left(\tfrac{k+1}{2} - 1\right) k + \sigma(i) \in X_i,$$

where σ is the permutation on $\{1, 2, \ldots, k\}$ given by

$$\sigma(i) = \begin{cases} \frac{k-2i+1}{2} & \text{for } 1 \le i \le \frac{k-1}{2}, \\ \frac{3k-2i+1}{2} & \text{for } \frac{k+1}{2} \le i \le k. \end{cases}$$

We construct a new set of integers X_{k+1} by choosing one particular element from each X_i, $i = 1, 2, \ldots, k$, together with k^2+k+1 as follows:

$$X_{k+1} = \left\{\left(\tfrac{k+1}{2} - 1\right) k + \sigma(i) : 1 \le i \le k\right\} \cup \left\{k^2 + k + 1\right\}.$$

Then

$$\begin{aligned} \sum X_{k+1} &= \sum_{i=1}^{k} \left[\left(\tfrac{k+1}{2} - 1\right) k + \sigma(i)\right] + k^2 + k + 1 \\ &= \tfrac{k^2(k-1)}{2} + \tfrac{k(k+1)}{2} + k^2 + k + 1 = \tfrac{k(k+1)^2}{2} + k + 1. \end{aligned} \tag{3.2}$$

From (3.1) and (3.2) we have

$$\sum X_i = \tfrac{k(k+1)^2}{2} + i \quad \text{for } 1 \le i \le k+1.$$

By Lemma 2.1, for each integer $0 \le t \le \lfloor k/2 \rfloor$ there is a $(k+1)$-equipartition $\mathbb{R}^t = \{R_1^t, R_2^t, \ldots, R_{k+1}^t\}$ of $[1, k(k+1)]$ such that for $1 \le i \le k+1$ holds $\sum R_i^t = \Delta_t + di$, where $d = k - 2t$ and $\Delta_t = \sum R_1^t - d$. Hence we have a $(k+1)$-equipartition $\mathbb{Q}^t = \{Y_1^t, Y_2^t, \dots, Y_{k+1}^t\}$ of the set $[k^2+k+2, 2k^2+2k+1]$ such that

$$\sum Y_i^t = k(k^2+k+1) + \Delta_t + di \quad \text{for } 1 \le i \le k+1.$$

For each $0 \le t \le \lfloor k/2 \rfloor$ we define a total labeling f_t, $f_t : V(Bt_k^k) \cup E(Bt_k^k) \to [1, 2k^2+2k+1]$ as follows:

$$\begin{aligned} f_t(E(S_k^i)) &= Y_i^t && \text{for } 1 \le i \le k+1,\\ f_t(v) &= k^2+k+1, &&\\ f_t(V(S_k^i)) &= X_i && \text{for } 1 \le i \le k+1, \end{aligned}$$

with the restriction that for $i = 1, 2, \dots, k$ it holds

$$f_t(v_1^i) = \left(\tfrac{k+1}{2} - 1\right) k + \sigma(i).$$

Then for $1 \le i \le k+1$ we get

$$\begin{aligned} wt_{f_t}(S_k^i) &= \sum f_t(V(S_k^i)) + \sum f_t(E(S_k^i)) = \sum X_i + \sum Y_i^t\\ &= \tfrac{k(k+1)^2}{2} + i + k(k^2+k+1) + \Delta_t + di = a_t + (d+1)i, \end{aligned}$$

where $a_t = k(k+1)^2/2 + k(k^2+k+1) + \Delta_t$.

Since k is odd and $d = k - 2t$ then for $0 \le t \le \lfloor k/2 \rfloor$ we have $d \in \{1, 3, 5, \dots, k\}$. Hence the banana tree Bt_k^k admits super (a, d^*)-S_k-antimagic labeling for $d^* \in \{2, 4, 6, \dots, k+1\}$.

To prove that the banana tree Bt_k^k admits a super S_k-magic labeling also, we define a total labeling $g : V(Bt_k^k) \cup E(Bt_k^k) \to [1, 2k^2+2k+1]$ as follows:

$$\begin{aligned} g(E(S_k^i)) &= f_{\left\lfloor \frac{k}{2} \right\rfloor}(E(S_k^i)) = Y_i^{\left\lfloor \frac{k}{2} \right\rfloor} && \text{for } 1 \le i \le k+1,\\ g(v) &= k^2+k+1, &&\\ g(V(S_k^i)) &= X_{k+2-i} && \text{for } 1 \le i \le k+1, \end{aligned}$$

with the restriction that for $i = 1, 2, \dots, k$

$$g(v_1^i) = \left(\tfrac{k+1}{2} - 1\right) k + \sigma(i).$$

Then for $1 \le i \le k+1$

$$\begin{aligned} wt_g(S_k^i) &= \sum g(V(S_k^i)) + \sum g(E(S_k^i)) = \sum X_{k+2-i} + \sum Y_i^{\left\lfloor \frac{k}{2} \right\rfloor}\\ &= \tfrac{k(k+1)^2}{2} + k + 2 - i + k(k^2+k+1) + \Delta_{\left\lfloor \frac{k}{2} \right\rfloor} + i\\ &= \tfrac{k(k+1)^2}{2} + k + 2 + k(k^2+k+1) + \Delta_{\left\lfloor \frac{k}{2} \right\rfloor}. \end{aligned}$$

Case ii: k is even.
Since $k+1$ is odd, by Lemma 2.4, there exists a k-equipartition $\mathbb{P} = \{X_1, X_2, \dots, X_k\}$ of $X = [1, (k+1)k]$ such that

$$\sum X_i = \tfrac{k(k+1)^2}{2} + i \quad \text{for } 1 \le i \le k. \tag{3.3}$$

It can be easily verified by the definition of X_i in Lemma 2.4 that for $1 \le i \le k/2$

$$\left(\tfrac{k+2}{2} - 1\right) k + i \in X_i$$

and for $k/2 + 1 \le i \le k$

$$\left(\tfrac{k}{2} - 1\right) k + i \in X_i.$$

We construct a new set of integers X_{k+1} by choosing one particular element from each X_i for $i = 1, 2, \dots, k$ together with $k^2 + k + 1$ as follows:

$$X_{k+1} = \left\{ \left(\tfrac{k+2}{2} - 1 \right) k + j : 1 \le j \le \tfrac{k}{2} \right\} \cup \left\{ \left(\tfrac{k}{2} - 1 \right) k + j : \tfrac{k}{2} + 1 \le j \le k \right\} \cup \{k^2 + k + 1\}.$$

Then

$$\begin{aligned} \sum X_{k+1} &= \sum_{j=1}^{\frac{k}{2}} \left[\left(\tfrac{k+2}{2} - 1 \right) k + j \right] + \sum_{j=\frac{k}{2}+1}^{k} \left[\left(\tfrac{k}{2} - 1 \right) k + j \right] + k^2 + k + 1 \\ &= \tfrac{k^2(k-1)}{2} + \tfrac{k(k+1)}{2} + k^2 + k + 1 = \tfrac{k(k+1)^2}{2} + k + 1. \end{aligned} \tag{3.4}$$

From (3.3) and (3.4) we have

$$\sum X_i = \tfrac{k(k+1)^2}{2} + i \quad \text{for } 1 \le i \le k + 1.$$

By Lemma 2.1, for each integer $0 \le t \le \lfloor k/2 \rfloor$ there is a $(k+1)$-equipartition $\mathbb{R}^t = \{R_1^t, R_2^t, \dots, R_{k+1}^t\}$ of $[1, k(k+1)]$ such that for $1 \le i \le k+1$ holds $\sum R_i^t = \Delta_t + di$, where $d = k - 2t$ and $\Delta_t = \sum R_1^t - d$. Hence we have a $(k+1)$-equipartition $\mathbb{Q}^t = \{Y_1^t, Y_2^t, \dots, Y_{k+1}^t\}$ of the set $[k^2 + k + 2, 2k^2 + 2k + 1]$ such that

$$\sum Y_i^t = k(k^2 + k + 1) + \Delta_t + di \quad \text{for } 1 \le i \le k + 1.$$

For each $0 \le t \le k/2$ we define a total labeling $f_t : V(Bt_k^k) \cup E(Bt_k^k) \to [1, 2k^2 + 2k + 1]$ as follows:

$$\begin{aligned} f_t(E(S_k^i)) &= Y_i^t && \text{for } 1 \le i \le k+1, \\ f_t(v) &= k^2 + k + 1, \\ f_t(V(S_k^i)) &= X_i && \text{for } 1 \le i \le k+1, \end{aligned}$$

with the restriction that

$$f_t(v_1^i) = \begin{cases} \left(\frac{k+2}{2} - 1 \right) k + i & \text{for } 1 \le i \le \frac{k}{2} \\ \left(\frac{k}{2} - 1 \right) k + i & \text{for } \frac{k}{2} + 1 \le i \le k. \end{cases}$$

Then for $1 \le i \le k+1$ we get

$$\begin{aligned} wt_{f_t}(S_k^i) &= \sum f_t(V(S_k^i)) + \sum f_t(E(S_k^i)) = \sum X_i + \sum Y_i^t \\ &= \tfrac{k(k+1)^2}{2} + i + k(k^2 + k + 1) + \Delta_t + di = a_t + (d+1)i, \end{aligned}$$

where $a_t = k(k+1)^2/2 + k(k^2 + k + 1) + \Delta_t$.

Since k is even and $d = k - 2t$ for $0 \le t \le k/2$ we have $d \in \{0, 2, 4, 6, \dots, k\}$. Hence the banana tree Bt_k^k admits a super (a, d^*)-S_k-antimagic labeling for $d^* \in \{1, 3, 5, \dots, k+1\}$. □

Figure 1 illustrates a super $(a, 6)$-S_5-antimagic labeling of the banana tree Bt_5^5 and a super $(a, 3)$-S_6-antimagic labeling of the banana tree Bt_6^6.

By the symbol St_n^p, $n \ge 1$, $p \ge 0$, we denote the graph obtained from the star S_n by attaching p pendant edges to every vertex of degree 1 in S_n. Let us denote the vertices and edges in St_n^p such that

$$\begin{aligned} V(St_n^p) &= \{v, v_i, v_i^j : i = 1, 2, \dots, n; j = 1, 2, \dots, p\}, \\ E(St_n^p) &= \{v_i v_i^j : i = 1, 2, \dots, n; j = 1, 2, \dots, p\} \cup \{v v_i : i = 1, 2, \dots, n\}. \end{aligned}$$

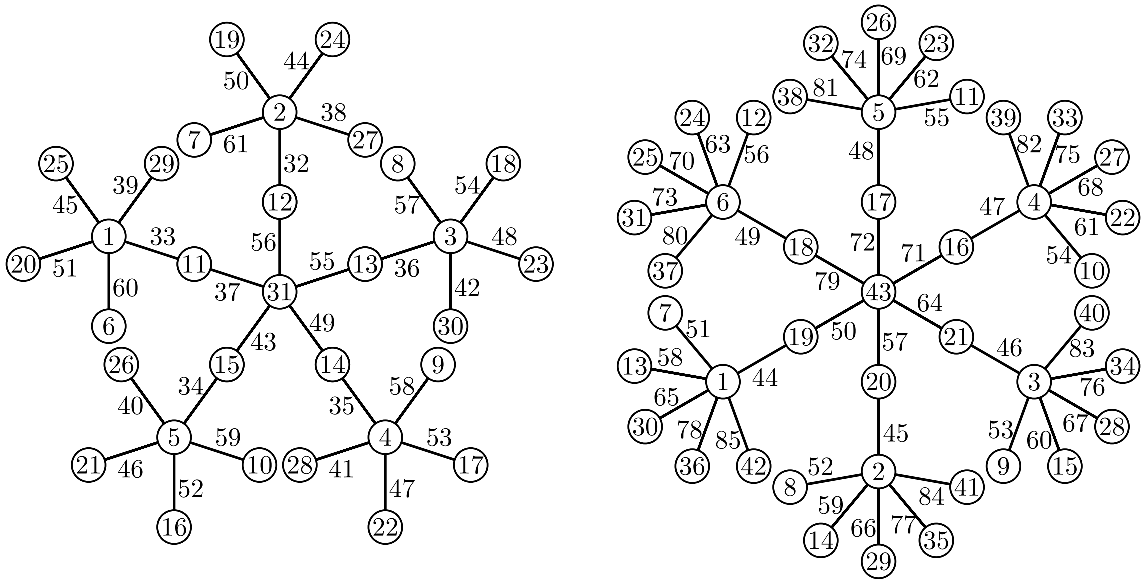

Figure 1. A super $(a,6)$-S_5-antimagic labeling of the banana tree Bt_5^5 and a super $(a,3)$-S_6-antimagic labeling of the banana tree Bt_6^6.

Evidently, the graph St_n^p admits a S_{p+1}-covering. If $p \geq n$ then the S_{p+1}-covering of St_n^p consists of n stars isomorphic to S_{p+1}. Let us denote the $(p+1)$-stars by the symbols $\{S_{p+1}^1, S_{p+1}^2, \dots, S_{p+1}^n\}$ such that the edge set of S_{p+1}^i, $i = 1, 2, \dots, n$, is

$$V(S_{p+1}^i) = \{v, v_i, v_i^j : j = 1, 2, \dots, p\}$$

and its edge set is

$$E(S_{p+1}^i) = \{vv_i, v_i v_i^j : j = 1, 2, \dots, p\}.$$

If $p = n-1$ then the graph St_n^{n-1} contains $n+1$ subgraphs isomorphic to S_n. We denote them $\{S_n^1, S_n^2, \dots, S_n^{n+1}\}$, where for $i = 1, 2, \dots, n$, is $V(S_n^i) = \{v, v_i, v_i^j : j = 1, 2, \dots, n-1\}$ and $E(S_n^i) = \{vv_i, v_i v_j^i : j = 1, 2, \dots, n-1\}$. Moreover, $V(S_n^{n+1}) = \{v, v_i : i = 1, 2, \dots, n\}$ and $E(S_n^{n+1}) = \{vv_i : i = 1, 2, \dots, n\}$.

Theorem 3.2. *Let n, p be positive integers, $p \geq n$. Then the graph St_n^p admits a super (a,d)-S_{p+1}-antimagic labeling for $d \in \{0, 2, \dots, 2(p+1)\}$. Moreover, if n and p are odd, $n \geq 3$, then differences $d \in \{1, 3, \dots, p+2\}$ are also feasible.*

Proof. The graph St_n^p has $(n(p+1)+1)$ vertices and $n(p+1)$ edges. If $p \geq n$ then the graph St_n^p contains n subgraphs isomorphic to S_{p+1}.

By Lemma 2.1, for each integer $0 \leq t \leq \lfloor (p+1)/2 \rfloor$ there is a n-equipartition $\mathbb{Q} = \{Q_1^t, Q_2^t, \dots, Q_n^t\}$ of $[1, n(p+1)]$ such that $\sum Q_i^t = \Delta_t + di$, where $d = p+1-2t$ and $\Delta_t = \sum Q_1^t - d$.

Hence we have a n-equipartition $\mathbb{Y}^{t_1} = \{Y_1^{t_1}, Y_2^{t_1}, \dots, Y_n^{t_1}\}$ of the set $[2, n(p+1)+1]$ such that

$$\sum Y_i^{t_1} = n + \Delta_{t_1} + (p+1-2t_1)i \quad \text{for } 1 \leq i \leq n,$$

where $0 \leq t_1 \leq \lfloor (p+1)/2 \rfloor$.

Moreover, we have a n-equipartition $\mathbb{Z}^{t_2} = \{Z_1^{t_2}, Z_2^{t_2}, \dots, Z_n^{t_2}\}$ of the set $[n(p+1)+2, 2n(p+1)+1]$ such that

$$\sum Z_i^{t_2} = n(n(p+1)+1) + \Delta_{t_2} + (p+1-2t_2)i \quad \text{for } 1 \leq i \leq n,$$

where $0 \leq t_2 \leq \lfloor (p+1)/2 \rfloor$.

We define a total labeling $f : V(St_n^p) \cup E(St_n^p) \to [1, 2n(p+1)+1]$ such that

$$\begin{aligned} f(v) &= 1, & \\ f(V(S_{p+1}^i) - \{v\}) &= Y_i^{t_1} & \text{for } 1 \le i \le n, \\ f(E(S_{p+1}^i)) &= Z_i^{t_2} & \text{for } 1 \le i \le n. \end{aligned}$$

Then for $1 \le i \le n$ we get

$$\begin{aligned} wt_f(S_{p+1}^i) =& \sum f(V(S_{p+1}^i)) + \sum f(E(S_{p+1}^i)) \\ =& f(v) + \sum f(V(S_{p+1}^i) - \{v\}) + \sum f(E(S_{p+1}^i)) \\ =& 1 + \sum Y_i^{t_1} + \sum Z_i^{t_2} = 1 + (n + \Delta_{t_1} + (p+1-2t_1)i) \\ &+ \big(n(n(p+1)+1) + \Delta_{t_2} + (p+1-2t_2)i\big) \\ =& a + (2p+2-2t_1-2t_2)i, \end{aligned}$$

where $a = n^2(p+1) + 2n + 1 + \Delta_{t_1} + \Delta_{t_2}$.

Thus, the weights of stars S_{p+1} for arithmetic sequence with difference $d = (2p+2-2t_1-2t_2)$. As $0 \le t_1 \le \lfloor (p+1)/2 \rfloor$ and $0 \le t_2 \le \lfloor (p+1)/2 \rfloor$ we get that for p odd

$$d \in \{0, 2, \dots, 2(p+1)\}$$

and for p even

$$d \in \{2, 4, \dots, 2(p+1)\}.$$

Moreover, for p even we define a total labeling $f_0 : V(St_n^p) \cup E(St_n^p) \to [1, 2n(p+1)+1]$ such that

$$\begin{aligned} f_0(v) &= 1, & \\ f_0(V(S_{p+1}^i) - \{v\}) &= Y_i^0 & \text{for } 1 \le i \le n, \\ f_0(E(S_{p+1}^i)) &= Z_{n+1-i}^0 & \text{for } 1 \le i \le n. \end{aligned}$$

For $1 \le i \le n$ we get

$$\begin{aligned} wt_{f_0}(S_{p+1}^i) =& \sum f_0(V(S_{p+1}^i)) + \sum f_0(E(S_{p+1}^i)) \\ =& f_0(v) + \sum f_0(V(S_{p+1}^i) - \{v\}) + \sum f_0(E(S_{p+1}^i)) \\ =& 1 + \sum Y_i^0 + \sum Z_{n+1-i}^0 = 1 + (n + \Delta_0 + 2(p+1)i) \\ &+ (n(n(p+1)+1) + \Delta_0 + 2(p+1)(n+1-i)) \\ =& n^2(p+1) + 2n + 1 + 2(p+1)(n+1). \end{aligned}$$

Thus also for p even we have a S_{p+1}-supermagic labeling of St_n^p.

Let n and p are odd, $n \ge 3$. Then according to Lemma 2.3 there exists a n-equipartition $\mathbb{P} = \{P_1, P_2, \dots, P_n\}$ of $X = [1, n(p+1)]$ such that $\sum P_i = p((p+1)n + n + 1)/2 + i$ for $1 \le i \le n$.

Hence we have a n-equipartition $\mathbb{T} = \{T_1, T_2, \dots, T_n\}$ of the set $[2, n(p+1)+1]$ such that

$$\sum T_i = n + \tfrac{p((p+1)n+n+1)}{2} + i \quad \text{for } 1 \le i \le n.$$

We define a total labeling $g : V(St_n^p) \cup E(St_n^p) \to [1, 2n(p+1)+1]$ such that

$$\begin{aligned} g(v) &= 1, & \\ g(V(S_{p+1}^i) - \{v\}) &= Y_i^{t_1} & \text{for } 1 \le i \le n, \\ g(E(S_{p+1}^i)) &= T_i & \text{for } 1 \le i \le n. \end{aligned}$$

The star weights of S^i_{p+1}, $1 \le i \le n$, are

$$\begin{aligned} wt_g(S^i_{p+1}) &= \sum g(V(S^i_{p+1})) + \sum g(E(S^i_{p+1})) \\ &= g(v) + \sum g(V(S^i_{p+1}) - \{v\}) + \sum g(E(S^i_{p+1})) \\ &= 1 + \sum Y_i^{t_1} + \sum T_i = 1 + (n + \Delta_{t_1} + (p+1-2t_1)i) \\ &\quad + \Big(n + \tfrac{p((p+1)n+n+1)}{2} + i\Big) \\ &= \Big(2n + \tfrac{p((p+1)n+n+1)}{2} + \Delta_{t_1} + 1\Big) + (p+2-2t_1)i. \end{aligned}$$

As $0 \le t_1 \le (p+1)/2$ we get that

$$p + 2 - 2t_1 \in \{1, 3, \dots, p+2\}.$$

This concludes the proof. □

Theorem 3.3. *The graph St_n^{n-1} admits a super (a,d)-S_n-antimagic labeling for $d \in \{0, 2(n-2)\}$.*

Proof. The graph St_n^{n-1} has (n^2+1) vertices and n^2 edges and contains $n+1$ subgraphs isomorphic to S_n.

We define a total labeling h, $f : V(St_n^{n-1}) \cup E(St_n^{n-1}) \to \{1, 2, \dots, 2n^2+1\}$ such that

$$\begin{aligned} h(v) &= 1, \\ \Big\{h(u) : u \in V(St_n^{n-1}) - \{v\}\Big\} &= \{2, 3, \dots, n^2+1\}, \\ h(vv_i) &= 2n^2 + 3 - h(v_i) \quad \text{for } 1 \le i \le n, \\ h(v_i v_i^j) &= 2n^2 + 3 - h(v_i^j) \quad \text{for } 1 \le i \le n,\ 1 \le j \le n-1. \end{aligned}$$

For the weights of the stars S^i_n, $1 \le i \le n$ we get

$$\begin{aligned} wt_h(S^i_n) &= h(v_i) + h(v) + \sum_{j=1}^{n-1} h(v_i^j) + \sum_{j=1}^{n-1} h(v_i v_i^j) + h(vv_i) \\ &= h(v_i) + 1 + \sum_{j=1}^{n-1} h(v_i^j) + \sum_{j=1}^{n-1} \Big(2n^2 + 3 - h(v_i^j)\Big) \\ &\quad + (2n^2 + 3 - h(v_i)) = n(2n^2+3) + 1. \end{aligned} \tag{3.5}$$

Moreover,

$$\begin{aligned} wt_h(S_n^{n+1}) &= h(v) + \sum_{i=1}^{n} h(v_i) + \sum_{i=1}^{n} h(vv_i) \\ &= 1 + \sum_{i=1}^{n} h(v_i) + \sum_{i=1}^{n} \Big(2n^2 + 3 - h(v_i)\Big) = n(2n^2+3) + 1. \end{aligned} \tag{3.6}$$

Using (3.5) and (3.6) we proved that h is a S_n-supermagic labeling of St_n^{n-1}.

We distinguish two cases to obtain the difference $2(n-2)$.

Case i: n is odd.

We define a total labeling f of St_n^{n-1} as follows:

$$\begin{aligned} f(v) &= 1, \\ f(v_i) &= i + 1 \quad \text{for } 1 \le i \le n, \\ f(v_i^j) &= \begin{cases} nj + 1 + i & \text{for } 1 \le i \le n,\ 1 \le j \le n-2, \\ n^2 + 2 - i & \text{for } 1 \le i \le n,\ j = n-1, \end{cases} \end{aligned}$$

$$f(v_iv_i^j) = \begin{cases} n(n-1)+nj+1+i & \text{for } 1 \le i \le n, \\ & \quad i \ne \frac{n+3}{2},\ 2 \le j \le n-1, \\ 2n^2 - \frac{n-1}{2} & \text{for } i = \frac{n+3}{2},\ j=1, \end{cases}$$

$$f(vv_i) = \begin{cases} 2n^2+2-i & \text{for } 1 \le i \le n,\ i \ne \frac{n+3}{2}, \\ n^2 + \frac{n-1}{2} + 3 & \text{for } i = \frac{n+3}{2}. \end{cases}$$

We find the S_n-weights of the stars in the covering.

$$\begin{aligned} wt_f(S_n^{n+1}) &= f(v) + \sum_{i=1}^{n} f(v_i) + \sum_{i=1}^{n} f(vv_i) \\ &= 1 + \sum_{i=1}^{n} [i+1] + \sum_{\substack{i=1 \\ i \ne \frac{n+3}{2}}}^{n} \left[2n^2+2-i\right] + n^2 + \tfrac{n-1}{2} + 3 \\ &= 2n^3 - n^2 + 4n + 3. \end{aligned} \tag{3.7}$$

For $1 \le i \le n$, $i \ne (n+3)/2+2$ is

$$\begin{aligned} wt_f(S_n^i) &= f(v_i) + f(v) + \sum_{j=1}^{n-1} f(v_i^j) + \sum_{j=1}^{n-1} f(v_iv_i^j) + f(vv_i) \\ &= (i+1) + 1 + \left(\sum_{j=1}^{n-2} [nj+1+i] + n^2+2-i\right) \\ &\quad + \left(\sum_{j=1}^{n-1} [n(n-1)+nj+1+i]\right) + \left(2n^2+2-i\right) \\ &= 2n^3 - n^2 + 4n + 3 + 2(n-2)i \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} wt_f\left(S_n^{\frac{n+3}{2}}\right) &= f\left(v_{\frac{n+3}{2}}\right) + f(v) + \sum_{j=1}^{n-1} f\left(v_{\frac{n+3}{2}}^j\right) + \sum_{j=1}^{n-1} f\left(v_{\frac{n+3}{2}} v_{\frac{n+3}{2}}^j\right) \\ &\quad + f\left(vv_{\frac{n+3}{2}}\right) = \left(\tfrac{n-1}{2} + 2 + 1\right) + 1 \\ &\quad + \left(\sum_{j=1}^{n-2} \left[nj + 1 + \tfrac{n-1}{2} + 2\right] + n^2 + 2 - \tfrac{n-1}{2} - 2\right) \\ &\quad + \left(\sum_{j=2}^{n-1} \left[n(n-1) + nj + 1 + \tfrac{n-1}{2} + 2\right] + 2n^2 - \tfrac{n-1}{2}\right) \\ &\quad + \left(n^2 + \tfrac{n-1}{2} + 3\right) = 2n^3 - n^2 + 4n + 3 + 2(n-2)\tfrac{n+3}{2}. \end{aligned} \tag{3.9}$$

From (3.7), (3.8) and (3.9) we proved that the graph St_n^{n-1} admits a super $(a, 2(n-2))$-S_n-antimagic labeling when n is odd.

Case ii: n is even.

We define a total labeling g of St_n^{n-1} as follows:

$$g(v) = 1,$$

$$g(v_i) = i+1 \quad \text{for } 1 \le i \le n,$$

$$g(v_i^j) = \begin{cases} nj+1+i & \text{for } 1 \le i \le n,\ 1 \le j \le n-2, \\ n^2+2-i & \text{for } 1 \le i \le n,\ j = n-1, \end{cases}$$

$$
\begin{aligned}
g(v_iv_i^j) &= n(n-1)+nj+1+i \quad \text{for } 1 \le i \le n,\ 1 \le j \le n-2,\\
g(v_{\frac{n}{2}}v_{\frac{n}{2}}^{n-1}) &= 2n(n-1)+1+\tfrac{n}{2},\\
g(v_nv_n^{n-1}) &= (2n-1)n+1,\\
g(v_iv_i^{n-1}) &= n(n-1)+nj+1+i \quad \text{for } 1 \le i \le n-1,\ i \ne \tfrac{n}{2},\\
g(vv_{\frac{n}{2}}) &= 2n^2+2-\tfrac{n}{2},\\
g(vv_n) &= 2n^2+2-n,\\
g(vv_i) &= 2n(n-1)+1+i \quad \text{for } 1 \le i \le n-1,\ i \ne \tfrac{n}{2}.
\end{aligned}
$$

We find the S_n-weights of the stars in the covering.

$$
\begin{aligned}
wt_g(S_n^{n+1}) =& g(v)+\sum_{i=1}^{n} g(v_i)+\sum_{i=1}^{n} g(vv_i) = 1+\sum_{i=1}^{n}[i+1]\\
&+ \sum_{\substack{i=1\\ i\ne\frac{n}{2},\ i\ne n}}^{n} [2n(n-1)+1+i]+2n^2+2-\tfrac{n}{2}+2n^2+2-n\\
=& 2n^3-n^2+4n+3. \qquad (3.10)
\end{aligned}
$$

For $1 \le i \le n-1$, $i \ne n/2$ we have

$$
\begin{aligned}
wt_g(S_n^i) =& g(v_i)+g(v)+\sum_{j=1}^{n-1} g(v_i^j)+\sum_{j=1}^{n-1} g(v_iv_i^j)+g(vv_i)\\
=& (i+1)+1+\left(\sum_{j=1}^{n-2}[nj+1+i]+n^2+2-i\right)\\
&+\left(\sum_{j=1}^{n-2}[n(n-1)+nj+1+i]+2n^2+2-i\right)\\
&+(2n(n-1)+1+i) = 2n^3-n^2+4n+3+2(n-2)i \qquad (3.11)
\end{aligned}
$$

and

$$
\begin{aligned}
wt_g(S_n^{\frac{n}{2}}) =& g(v_{\frac{n}{2}})+g(v)+\sum_{j=1}^{n-1} g(v_{\frac{n}{2}}^j)+\sum_{j=1}^{n-1} g(v_{\frac{n}{2}}v_{\frac{n}{2}}^j)+g(vv_{\frac{n}{2}})\\
=& \left(\tfrac{n}{2}+1\right)+1+\left(\sum_{j=1}^{n-2}\left[nj+1+\tfrac{n}{2}\right]+n^2+2-\tfrac{n}{2}\right)\\
&+\left(\sum_{j=1}^{n-2}\left[n(n-1)+nj+1+\tfrac{n}{2}\right]+2n(n-1)+1+\tfrac{n}{2}\right)\\
&+\left(2n^2+2-\tfrac{n}{2}\right) = 2n^3-n^2+4n+3+n(n-2). \qquad (3.12)
\end{aligned}
$$

$$
\begin{aligned}
wt_g(S_n^n) =& g(v_n)+g(v)+\sum_{j=1}^{n-1} g(v_n^j)+\sum_{j=1}^{n-1} g(v_nv_n^j)+g(vv_n)\\
=& (n+1)+1+\left(\sum_{j=1}^{n-2}[nj+1+n]+n^2+2-n+2\right)\\
&+\left(\sum_{j=1}^{n-2}[n(n-1)+nj+1+n]+2n(n-1)+1+n\right)
\end{aligned}
$$

$$+\left(2n^2+2-n\right)=2n^3-n^2+4n+3+2(n-2)n. \qquad (3.13)$$

According to (3.10) - (3.13) we have that for n even the graph St_n^{n-1} is super $(a, 2(n-2))$-S_n-antimagic. □

Figure 2 illustrates a super $(248, 6)$-S_5-antimagic labeling of graph St_5^4.

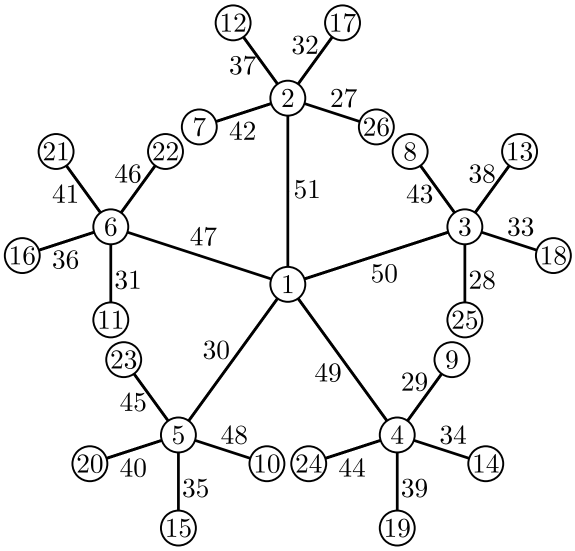

Figure 2. A super $(248, 6)$-S_5-antimagic labeling of graph St_5^4.

Acknowledgment. This work was supported by the Slovak Science and Technology Assistance Agency under the contract No. APVV-15-0116 and by VEGA 1/0233/18.

References

[1] M. Bača, Z. Kimáková, A. Semaničová-Feňovčíková and M.A. Umar, *Tree-antimagicness of disconnected graphs*, Math. Probl. Eng. **2015**, Article ID: 504251, 1–4, 2015.

[2] M. Bača, M. Miller, J. Ryan and A. Semaničová-Feňovčíková, *On H-antimagicness of disconnected graphs*, Bull. Aust. Math. Soc. **94**, 201–207, 2016.

[3] A. Gutiérrez and A. Lladó, *Magic coverings*, J. Combin. Math. Combin. Comput. **55**, 43–56, 2005.

[4] N. Inayah, A.N.M. Salman and R. Simanjuntak, *On (a, d)-H-antimagic coverings of graphs*, J. Combin. Math. Combin. Comput. **71**, 273–281, 2009.

[5] N. Inayah, R. Simanjuntak, A.N.M. Salman and K.I.A. Syuhada, *On (a, d)-H-antimagic total labelings for shackles of a connected graph H*, Australas. J. Combin. **57**, 127–138, 2013.

[6] P. Jeyanthi and P. Selvagopal, *Supermagic coverings of some simple graphs*, Int. J. Math. Combin. **1**, 33–48, 2011.

[7] T.K. Maryati, A.N.M. Salman, E.T. Baskoro, J. Ryan and M. Miller, *On H-supermagic labelings for certain shackles and amalgamations of a connected graph*, Utilitas Math. **83**, 333–342, 2010.

[8] A. Semaničová-Feňovčíková, M. Bača, M. Lascsáková, M. Miller and J. Ryan, *Wheels are cycle-antimagic*, Electron. Notes Discrete Math. **48**, 11–18, 2015.