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Research Article

On graded 2-absorbing and graded weakly 2-absorbing ideals

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Abstract

In this paper, we introduce and study graded 2-absorbing and graded weakly 2-absorbing ideals of a graded ring which are different from 2-absorbing and weakly 2-absorbing ideals. We give some properties and characterizations of these ideals and their homogeneous components. We investigate graded (weakly) 2-absorbing ideals of $R_1 \times R_2$ where R_1 and R_2 are two graded rings.

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1. Introduction

Throughout this paper, all rings are assumed to be commutative with identity elements. The concept of 2-absorbing ideal was introduced by Badawi in [4] as a generalization of the notion of prime ideal and studied in [1], [10], [11]. Let R be a ring. A proper ideal I of R is called a 2-absorbing ideal of R if whenever a, b, $c \in R$ with $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Weakly prime ideals are also generalizations of prime ideals. Recall from [2] that a proper ideal I of R is called a weakly prime ideal if whenever $0 \neq ab \in I$, then $a \in I$ or $b \in I$. The concept of weakly prime ideal was generalized to the concept of weakly 2-absorbing ideal in [5]. A proper ideal I of R is said to be a weakly 2-absorbing ideal of R if whenever $0 \neq abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$.

In this paper, we introduce and study graded 2-absorbing and graded weakly 2-absorbing ideals of graded rings. First, we recall some basic properties of graded rings and modules which will be used in the sequel. We refer to [7] and [8] for these basic properties and more information on graded rings and modules. Let G be a multiplicative group and e denote the identity element of G. A ring R is called a graded ring (or G-graded ring) if there exist additive subgroups R_g of R indexed by the elements $g \in G$ such that $R = \oplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. If the inclusion is an equality, then the ring R is called strongly graded. The elements of R_g are called homogeneous of degree g and all

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the homogeneous elements are denoted by $h(R)$, i.e. $h(R) = \cup_{g\in G} R_g$. If $x \in R$, then x can be written uniquely as $\Sigma_{g\in G} x_g$, where x_g is called homogeneous component of x in R_g. Moreover, R_e is a subring of R and $1 \in R_e$. Also, if $r \in R_g$ and r is a unit, then $r^{-1} \in R_{g-1}$. A G-graded ring $R = \oplus_{g\in G} R_g$ is called a crossed product if R_g contains a unit for every $g \in G$. Note that a G-crossed product $R = \oplus_{g\in G} R_g$ is a strongly graded ring (see [8, 1.1.2. Remark]).

Let $R = \underset{g\in G}{\oplus} R_g$ be a G-graded ring. An ideal I of R is said to be a graded ideal if $I = \oplus_{g\in G}(I \cap R_g) := \oplus_{g\in G} I_g$. If I is a graded ideal of R, then the quotient ring R/I is a G-graded ring. Indeed, $R/I = \underset{g\in G}{\oplus} (R/I)_g$ where $(R/I)_g = \{x + I : x \in R_g\}$. A proper graded ideal P of R is said to be a graded prime ideal (or gr-prime ideal) of R if whenever a and b are homogeneous elements of R such that $ab \in P$, then either $a \in P$ or $b \in P$. A graded ideal I of R is said to be graded maximal ideal of R if $I \neq R$ and if J is a graded ideal of R such that $I \subseteq J \subseteq R$, then $I = J$ or $J = R$. Let R_1 and R_2 be G-graded rings and $R = R_1 \times R_2$. Then R is a G-graded ring with $h(R) = h(R_1) \times h(R_2)$.

Let $R = \underset{g\in G}{\oplus} R_g$ be a G-graded ring. A right R-module M is said to be a graded R-module (or G-graded R-module) if there exists a family of additive subgroups $\{M_g\}_{g\in G}$ of M such that $M = \oplus_{g\in G} M_g$ and $M_g R_h \subseteq M_{gh}$ for all $g, h \in G$. Also if an element of M belongs to $\cup_{g\in G} M_g = h(M)$, then it is called homogeneous. Note that M_g is an R_e-module for every $g \in G$. So, if $I = \oplus_{g\in G} I_g$ is a graded ideal of R, then I_g is an R_e-module for every $g \in G$.

In this article, we define graded (weakly) 2-absorbing ideals of a graded ring. We show that the set of all graded 2-absorbing ideals and the set of all 2-absorbing graded ideals need not to be equal in a graded ring (see Example 2.2). According to our definition, every graded prime ideal is a graded 2-absorbing ideal. But we show that not every graded 2-absorbing ideal is a graded prime ideal (see Example 2.3). Various properties of graded (weakly) 2-absorbing ideals and their homogeneous components are considered. We also define the concept of g-2-absorbing ideal for $g \in G$ and prove that if $I = \underset{g\in G}{\oplus} I_g$ a graded weakly 2-absorbing ideal of R, then for each $g \in G$, either I is a g-2-absorbing ideal of R or $I_g^3 = (0)$ (see Theorem 3.4). We give a number of results concerning graded (weakly) 2-absorbing ideals of $R_1 \times R_2$ where R_1 and R_2 are two graded rings (see Theorem 2.12 and Theorems 3.8-3.11).

2. Graded 2-absorbing ideals

Definition 2.1. Let R be a G-graded ring and I be a proper graded ideal of R. I is said to be a graded 2-absorbing ideal of R if whenever $r, s, t \in h(R)$ with $rst \in I$, then $rs \in I$ or $rt \in I$ or $st \in I$.

Clearly, every 2-absorbing graded ideal of a graded ring R is also a graded 2-absorbing ideal. But the next example shows that not every graded 2-absorbing ideal of a graded ring is a 2-absorbing ideal.

Example 2.2. Let $R = \mathbb{Z}[i]$ and $G = \mathbb{Z}_2$. Then R is a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$. Let $I = 6R$. Then I is not a 2-absorbing ideal of R. Because $6 = (1+i)(1-i)3 \in I$ but $(1 + i)(1 - i) = 2 \notin I$, $(1 - i)3 \notin I$ and $(1 + i)3 \notin I$. However an easy computation shows that I is a graded 2-absorbing ideal of R.

It is also clear that every graded prime ideal of a graded ring R is a graded 2-absorbing ideal. But the next example shows that not every graded 2-absorbing ideal is a graded prime ideal.

Example 2.3. Let F be a field and $R = F[x, y]$. R is a $\mathbb{Z}$-graded ring with $\deg(x) = \deg(y) = 1$. Let $Q = (x^2, xy)$. Then Q is a graded 2-absorbing ideal of R which is not a graded prime ideal of R.

In [14], the concept of 2-absorbing ideal of a ring was extended to the notion of 2-absorbing submodule of a module. Let R be a ring and M be an R-module. A proper submodule N of M is called 2-absorbing, if whenever a, $b \in R$, $m \in M$ and $abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$.

Theorem 2.4. *Let R be a G-graded ring and $I = \oplus_{g\in G} I_g$ be a graded ideal of R. Then the following hold.*

(1) *If I is a graded 2-absorbing ideal of R, then I_g is a 2-absorbing submodule of the R_e-module R_g for every $g \in G$ with $I_g \neq R_g$.*

(2) *If R is a crossed product and I_e is a 2-absorbing ideal of R_e, then I is a graded 2-absorbing ideal of R.*

Proof. (1) Let $g \in G$ and $I_g \neq R_g$. Assume that $r, s \in R_e$ and $t \in R_g$ with $rst \in I_g$. Since I is a graded 2-absorbing ideal of R, we have, $rs \in I$ or $rt \in I$ or $st \in I$. If $rs \in I$, then $rs \in (I_g :_{R_e} R_g)$. If $st \in I$ or $rt \in I$, then $st \in I_g$ or $rt \in I_g$, respectively. This shows that I_g is a 2-absorbing R_e-submodule of R_g.

(2) Clearly, $I \neq R$. First we show that if I_e is a 2-absorbing ideal of R_e, then I_g is a 2-absorbing submodule of the R_e-module R_g for every $g \in G$. Let $g \in G$. If $I_g = R_g$, then it can be easily seen that $I_e = R_e$, a contradiction. So $I_g \neq R_g$. Let $a, b \in R_e$, $c \in R_g$ such that $abc \in I_g$. Let d be a unit in $R_{g^{-1}}$. Then $ab(cd) \in I_e$. Since I_e is a 2-absorbing ideal of R_e, we have $ab \in I_e$ or $a(cd) \in I_e$ or $b(cd) \in I_e$. If $ab \in I_e$, then $ab \in (I_g :_{R_e} R_g)$. If $a(cd) \in I_e$ or $b(cd) \in I_e$, then $ac \in I$ or $bc \in I$, respectively.

Now, let $r, s, t \in h(R)$ with $rst \in I$. There exist $g, h, \sigma \in G$ such that $r \in R_g$, $s \in R_h$ and $t \in R_\sigma$. Also, $R_{g^{-1}}$ contains a unit, say r' and $R_{h^{-1}}$ contains a unit, say s'. It follows that $(rr')(ss')t \in I_\sigma$. Since I_σ is a 2-absorbing submodule of the R_e-module R_σ, we have $(rr')t \in I_\sigma$ or $(ss')t \in I_\sigma$ or $(rr')(ss') \in (I_\sigma :_{R_e} R_\sigma)$. If $(rr')t \in I_\sigma$ or $(ss')t \in I_\sigma$, then $rt \in I$ or $st \in I$, respectively. If $(rr')(ss') \in (I_\sigma :_{R_e} R_\sigma)$, then $rs(r's')R_\sigma \in I_\sigma$. Since R is strongly graded, $rs(r's') \in I_e$, this implies that $rs \in I$. Thus I is a graded 2-absorbing ideal of R. □

The graded radical of a graded ideal I, denoted by $Gr(I)$, is the set of all $x = \sum_{g\in G} x_g \in R$ such that for each $g \in G$ there exists $n_g \in \mathbb{Z}^+$ with $x_g^{n_g} \in I$. Note that, if r is a homogeneous element, then $r \in Gr(I)$ if and only if $r^n \in I$ for some $n \in \mathbb{Z}^+$ [13].

Lemma 2.5. *Let R be a G-graded ring and I be a graded 2-absorbing ideal of R. Then $Gr(I)$ is a graded 2-absorbing ideal of R and $a^2 \in I$ for every $a \in h(Gr(I))$.*

Proof. Let $a \in h(Gr(I))$. Then $a^k \in I$ for some $k \in \mathbb{N}$. Since I is graded 2 absorbing, $a^2 \in I$.

Now, let $r, s, t \in h(R)$, and $rst \in Gr(I)$, then $(rst)^n \in I$ for some $n \in \mathbb{Z}^+$. Since I is a graded 2-absorbing ideal, we have $(rst)^2 = r^2s^2t^2 \in I$ and hence $r^2s^2 = (rs)^2 \in I$ or $r^2t^2 = (rt)^2 \in I$ or $s^2t^2 = (st)^2 \in I$. Thus, $rs \in Gr(I)$ or $rt \in Gr(I)$ or $st \in Gr(I)$. Therefore $Gr(I)$ is graded 2-absorbing ideal. □

Proposition 2.6. *Let R be a G-graded ring and I be a graded ideal of R such that $Gr(I) \neq I$ and $Gr(I)$ is a graded prime ideal of R. If $(I : x)$ is a graded prime ideal of R for all $x \in h(Gr(I)) - h(I)$, then I is a graded 2-absorbing ideal of R.*

Proof. Let $r, s, t \in h(R)$ with $rst \in I$. Since $I \subseteq Gr(I)$ and $Gr(I)$ is a graded prime ideal, we have $r \in Gr(I)$ or $s \in Gr(I)$ or $t \in Gr(I)$. We may assume that $r \in Gr(I)$. If $r \in I$, then $rs \in I$ and we are done. So, assume that $r \in Gr(I) - I$. Then by assumption $(I : r)$ is a graded prime ideal and since $st \in (I : r)$, either $s \in (I : r)$ or $t \in (I : r)$. Hence $rs \in I$ or $tr \in I$. Thus, I is graded 2-absorbing ideal. □

Recall that a proper graded ideal I of a graded ring R is said to be a graded irreducible ideal if whenever J_1 and J_2 are graded ideals of R with $I = J_1 \bigcap J_2$, then either $I = J_1$ or $I = J_2$ [13].

Theorem 2.7. *Let R be a G-graded ring and I be a graded irreducible ideal of R such that $Gr(I) = P$ is a graded prime ideal of R. If $P^2 \subseteq I$ and $(I : x) = (I : x^2)$ for all $x \in h(R) - P$, then I is a graded 2-absorbing ideal of R.*

Proof. Suppose that $P^2 \subseteq I$ and $(I : x) = (I : x^2)$ for all $x \in h(R) - P$. Let $a, b, c \in h(R)$ with $abc \in I$. Assume that $ab \notin I$. Since $P^2 \subseteq I$, either $a \notin P$ or $b \notin P$. So, we may assume that $(I : a) = (I : a^2)$. Let $J_1 = I + Rac$ and $J_2 = I + Rbc$. Then J_1 and J_2 are graded ideals of R containing I. Now, we show that $I = J_1 \bigcap J_2$. Let $s \in J_1 \bigcap J_2$. Then we can write $s = i_1 + r_1ac = i_2 + r_2bc$ for some $i_1, i_2 \in I$ and for some $r_1, r_2 \in R$ and then $as = ai_1 + r_1a^2c = ai_2 + r_2abc$. Since $abc \in I$, $as \in I$ and hence $r_1a^2c \in I$. Since $(I : a) = (I : a^2)$, $r_1ac \in I$ and hence $s \in I$. Thus $I = J_1 \bigcap J_2$. Since I is graded irreducible, either $I = J_1$ or $I = J_2$ and hence either $ac \in I$ or $bc \in I$. Therefore, I is a graded 2-absorbing ideal of R. □

Recall that a proper graded ideal I of a graded ring R is said to be a graded primary ideal if whenever $r, s \in h(R)$ with $rs \in I$, either $r \in I$ or $s \in Gr(I)$ [13].

Lemma 2.8. [13, Lemma 1.8] *Let R be a G-graded ring and I be a graded primary ideal of R. Then $P = Gr(I)$ is a graded prime ideal of R, and we say that I is a graded P-primary ideal of R.*

Lemma 2.9. [13, Corollary 1.12] *Let R be a G-graded ring and M be a graded maximal ideal of R. Then, for any positive integer n, M^n is a graded M-primary ideal of R.*

Proposition 2.10. *Let R be a G-graded ring and I a graded primary ideal of R such that $(Gr(I))^2 \subseteq I$. Then I is a graded 2-absorbing ideal of R.*

Proof. Let $r, s, t \in h(R)$ with $rst \in I$. Assume $st \notin I$. If $r \in I$, then we are done, so assume that $r \notin I$. Since I is a graded primary ideal, $r \in Gr(I)$ and $st \in Gr(I)$. Since $Gr(I)$ is a graded prime ideal, $r, s \in Gr(I)$ or $r, t \in Gr(I)$. Since $(Gr(I))^2 \subseteq I$, $rs \in I$ or $rt \in I$. □

Corollary 2.11. *Let R be a G-graded ring and M a graded maximal ideal of R. Then M^2 is a graded 2-absorbing ideal of R.*

Proof. By Lemma 2.9, M^2 is a graded primary ideal of R such that $Gr(M^2) = M$. Hence M^2 is a graded 2-absorbing ideal by Proposition 2.10. □

Theorem 2.12. *Let R_1 and R_2 be two graded rings, and let I be a proper graded ideal of R_1. Then I is a graded 2-absorbing ideal of R_1 if and only if $I \times R_2$ is a graded 2-absorbing ideal of $R_1 \times R_2$.*

Proof. Suppose that I is a graded 2-absorbing ideal of R_1 and let $(a_1, b_1)(a_2, b_2)(a_3, b_3) = (a_1a_2a_3, b_1b_2b_3) \in I \times R_2$ for some $a_1, a_2, a_3 \in h(R_1)$ and $b_1, b_2, b_3 \in h(R_2)$. Since $a_1a_2a_3 \in I$ and I is a graded 2-absorbing ideal of R_1, we have $a_1a_2 \in I$ or $a_1a_3 \in I$ or $a_2a_3 \in I$. Hence, $(a_1, b_1)(a_2, b_2) \in I \times R_2$ or $(a_1, b_1)(a_3, b_3) \in I \times R_2$ or $(a_2, b_2)(a_3, b_3) \in I \times R_2$. Thus $I \times R_2$ is a graded 2-absorbing ideal of $R_1 \times R_2$. Conversely, suppose that $I \times R_2$ is a graded 2-absorbing ideal of $R_1 \times R_2$, and let $a_1a_2a_3 \in I$ for some $a_1, a_2, a_3 \in h(R_1)$. Since $(a_1, 1)(a_2, 1)(a_3, 1) = (a_1a_2a_3, 1) \in I \times R_2$ and $I \times R_2$ is a graded 2-absorbing ideal of $R_1 \times R_2$, we have $(a_1, 1)(a_2, 1) = (a_1a_2, 1) \in I \times R_2$ or $(a_1, 1)(a_3, 1) = (a_1a_3, 1) \in I \times R_2$ or $(a_2, 1)(a_3, 1) = (a_2a_3, 1) \in I \times R_2$ and hence, $a_1a_2 \in I$ or $a_1a_3 \in I$ or $a_2a_3 \in I$. Thus I is a graded 2-absorbing ideal of R_1. □

3. Graded weakly 2-absorbing ideals

Definition 3.1. Let R be a G-graded ring and I be a proper graded ideal of R. I is said to be a graded weakly 2-absorbing ideal of R if whenever $r, s, t \in h(R)$ with $0 \neq rst \in I$, then $rs \in I$ or $rt \in I$ or $st \in I$.

Clearly, a graded 2-absorbing ideal of a graded ring R is a graded weakly 2-absorbing ideal. However, since (0) is a graded weakly 2-absorbing ideal of R (by definition), (0) need not to be a graded 2-absorbing ideal of R.

Proposition 3.2. *Let I, P be graded ideals of R with $I \subseteq P$ and $P \neq R$. Then the following hold.*

(1) *If P is a graded weakly 2-absorbing ideal of R, then P/I is a graded weakly 2-absorbing ideal of R/I.*

(2) *If I and P/I are graded weakly 2-absorbing ideals of R and R/I, respectively, then P is a graded weakly 2-absorbing ideal of R.*

Proof. (1) Clearly, $P/I \neq R/I$. Let $0 \neq (r + I)(s + I)(t + I) = rst + I \in P/I$, where $r, s, t \in h(R)$. Since $rst + I \neq 0$, we have $rst \neq 0$. Since $0 \neq rst \in P$ and P is a graded weakly 2-absorbing ideal of R, we conclude that $rs \in P$ or $rt \in P$ or $st \in P$. Hence $(r+I)(s+I) \in P/I$ or $(r+I)(t+I) \in P/I$ or $(s+I)(t+I) \in P/I$. Thus P/I is a graded weakly 2-absorbing ideal of R/I.

(2) Clearly, $P \neq R$. Let $0 \neq rst \in P$, where $r, s, t \in h(R)$. Hence $(r + I)(s + I)(t + I) = rst + I \in P/I$. If $rst \in I$, then $rs \in I \subseteq P$ or $rt \in I \subseteq P$ or $st \in I \subseteq P$. So we may assume $rst \notin I$ and hence $rst + I \neq 0$. Since P/I is a graded weakly 2-absorbing ideal of R/I, we have $(r + I)(s + I) = rs + I \in P/I$ or $(r + I)(t + I) = rt + I \in P/I$ or $(s + I)(t + I) = st + I \in P/I$. Hence $rs \in P$ or $rt \in P$ or $st \in P$. Thus P is a graded weakly 2-absorbing ideal of R. □

Definition 3.3. Let $R = \oplus_{g\in G} R_g$ be a graded ring, $I = \oplus_{g\in G} I_g$ be a graded ideal of R and $g \in G$. We say that I is a (weakly) g-2-absorbing ideal of R if $I_g \neq R_g$ and whenever $r, s, t \in R_g$ with $(0 \neq rst \in I)$ $rst \in I$, then $rs \in I$ or $rt \in I$ or $st \in I$.

Theorem 3.4. *Let $R = \oplus_{g\in G} R_g$ be a graded ring and $I = \bigoplus_{g\in G} I_g$ be a graded weakly 2-absorbing ideal of R. Then, for each $g \in G$, either I is a g-2-absorbing ideal of R or $I_g^3 = (0)$.*

Proof. It is enough to show that if $I_g^3 \neq (0)$ for $g \in G$, then I is a g-2-absorbing ideal of R. Let $rst \in I$ where $r, s, t \in R_g$. If $0 \neq rst$, then $rs \in I$ or $st \in I$ or $rt \in I$ by the hypothesis. So we may assume that $rst = 0$. Suppose first that $rsI_g \neq (0)$, then there exists $i \in I_g$ such that $rsi \neq 0$. Hence $0 \neq rs(t + i) = rsi \in I$. Since I is a graded weakly 2-absorbing ideal of R, we have $rs \in I$ or $r(t + i) \in I$ or $s(t + i) \in I$, and hence $rs \in I$ or $rt \in I$ or $st \in I$. So we can assume that $rsI_g = (0)$. Similarly, we can assume that $rtI_g = (0)$ and $stI_g = (0)$. If $rI_g^2 \neq (0)$, then there exist $a, b \in I_g$ such that $rab \neq 0$. Hence $0 \neq r(s + a)(t + b) = rab \in I$. Since I is a graded weakly 2-absorbing ideal of R, we have $r(s + a) \in I$ or $r(t + b) \in I$ or $(s + a)(t + b) \in I$ and hence $rs \in I$ or $rt \in I$ or $st \in I$. So we can assume that $rI_g^2 \neq (0)$. Similarly, we can assume that $sI_g^2 \neq (0)$ and $tI_g^2 \neq (0)$. Since $I_g^3 \neq (0)$, there exist $i_1, i_2, i_3 \in I_g$ such that $i_1 i_2 i_3 \neq 0$. Hence $0 \neq (r + i_1)(s + i_2)(t + i_3) = i_1 i_2 i_3 \in I_g$. Since I is a graded weakly 2-absorbing ideal of R, we get that $(r + i_1)(s + i_2) \in I$ or $(s + i_2)(t + i_3) \in I$ or $(r + i_1)(t + i_3) \in I$ and hence $rs \in I$ or $st \in I$ or $rt \in I$. Therefore, I is a g-2-absorbing ideal of R. □

Corollary 3.5. *Let $R = \oplus_{g\in G} R_g$ be a graded ring and $I = \bigoplus_{g\in G} I_g$ be a graded weakly 2-absorbing ideal of R such that I is not a g-2-absorbing ideal of R for every $g \in G$. Then $Gr(I) = Gr(0)$.*

Proof. Clearly, $Gr(0) \subseteq Gr(I)$. By Theorem 3.4, $I_g^3 = (0)$ for every $g \in G$. This implies that $Gr(I) \subseteq Gr(0)$. □

Proposition 3.6. *Let $R = \oplus_{g\in G} R_g$ be a graded ring, $P = \oplus_{g\in G} P_g$ be a graded weakly 2-absorbing ideal of R and $g \in G$. Then, for $a, b \in R_g$ with $ab \in R_{g^2} - P_{g^2}$, we have $(P_{g^2} :_{R_e} ab) = (P_g :_{R_e} a)$ or $(P_{g^2} :_{R_e} ab) = (P_g :_{R_e} b)$ or $(P_{g^2} :_{R_e} ab) = (0 :_{R_e} ab)$ or $(P_{g^2} :_{R_e} ab)^3 \subseteq (P_g :_{R_e} a) \cap (P_g :_{R_e} b) \cap (0 :_{R_e} ab)$.*

Proof. First, we show that $(P_{g^2} :_{R_e} ab) = (P_g :_{R_e} a) \cup (P_g :_{R_e} b) \cup (0 :_{R_e} ab)$. Clearly, $(P_g :_{R_e} a) \cup (P_g :_{R_e} b) \cup (0 :_{R_e} ab) \subseteq (P_{g^2} :_{R_e} ab)$. Let $c \in (P_{g^2} :_{R_e} ab)$. Then $cab \in P_{g^2}$. If $cab = 0$, then $c \in (0 :_{R_e} ab)$. If $cab \neq 0$, then we have $ca \in P$ or $cb \in P$ by the hypothesis. It follows that $c \in (P_g :_{R_e} a) \cup (P_g :_{R_e} b)$.Thus $(P_{g^2} :_{R_e} ab) = (P_g :_{R_e} a) \cup (P_g :_{R_e} b) \cup (0 :_{R_e} ab)$. According to [6, Theorem 1] and its proof $(P_{g^2} :_{R_e} ab)$ is contained in the union of any two of these ideals or $(P_{g^2} :_{R_e} ab)^3 \subseteq (P_g :_{R_e} a) \cap (P_g :_{R_e} b) \cap (0 :_{R_e} ab)$. In the first case, $(P_{g^2} :_{R_e} ab)$ is contained in one of these ideals and this implies that $(P_{g^2} :_{R_e} ab) = (P_g :_{R_e} a)$ or $(P_{g^2} :_{R_e} ab) = (P_g :_{R_e} b)$ or $(P_{g^2} :_{R_e} ab) = (0 :_{R_e} ab)$. □

Recall that a ring in which every finitely generated ideal is principal is called a Bezout ring.

Corollary 3.7. *Let $R = \oplus_{g\in G} R_g$ be a graded ring such that R_e is a Bezout ring, $P = \oplus_{g\in G} P_g$ be a graded weakly 2-absorbing ideal and $g \in G$. Then, for $a, b \in R_g$ with $ab \in R_{g^2} - P_{g^2}$, we have $(P_{g^2} :_{R_e} ab) = (P_g :_{R_e} a)$ or $(P_{g^2} :_{R_e} ab) = (P_g :_{R_e} b)$ or $(P_{g^2} :_{R_e} ab) = (0 :_{R_e} ab)$.*

Proof. In the proof of Proposition 3.6, we showed that $(P_{g^2} :_{R_e} ab) = (P_g :_{R_e} a) \cup (P_g :_{R_e} b) \cup (0 :_{R_e} ab)$. By [12, Proposition 1.1], $(P_{g^2} :_{R_e} ab)$ is equal to one of these ideals. □

Theorem 3.8. *Let R_1 and R_2 be two graded rings, and let I_1 and I_2 be non-zero proper graded ideals of R_1 and R_2, respectively. If $I_1 \times I_2$ is a graded weakly 2-absorbing ideal of $R_1 \times R_2$, then I_1 and I_2 are graded prime ideals of $R_1$and R_2, respectively.*

Proof. Suppose that $I_1 \times I_2$ is a graded weakly 2-absorbing ideal of $R_1 \times R_2$. We show that I_1 is a graded prime ideal of R_1. Let $r, s \in h(R_1)$ with $rs \in I_1$ and let $0 \neq i_2 \in h(I_2)$. Hence $(0, 0) \neq (1, i_2)(r, 1)(s, 1) = (rs, i_2) \in I_1 \times I_2$. Since $I_1 \times I_2$ is a graded weakly 2-absorbing ideal of $R_1 \times R_2$ and $(r, 1)(s, 1) = (rs, 1) \notin I_1 \times I_2$, we conclude that either $(1, i_2)(r, 1) = (r, i_2) \in I_1 \times I_2$ or $(1, i_2)(s, 1) = (s, i_2) \in I_1 \times I_2$, and hence either $r \in I_1$ or $s \in I_1$. Thus I_1 is a graded prime ideal of R_1. Similarly, one can show that I_2 is a graded prime ideal of R_2. □

Theorem 3.9. *Let R_1 and R_2 be two graded rings, and let I_1 and I_2 be non-zero proper graded ideals of R_1 and R_2, respectively. Then $I_1 \times I_2$ is a graded weakly 2-absorbing ideal of $R_1 \times R_2$ if and only if $I_1 \times I_2$ is a graded 2-absorbing ideal of $R_1 \times R_2$.*

Proof. Suppose that $I_1 \times I_2$ is a graded weakly 2-absorbing ideal of $R_1 \times R_2$. We show that $I_1 \times I_2$ is a graded 2-absorbing ideal of $R_1 \times R_2$. Suppose that $(a_1, b_1)(a_2, b_2)(a_3, b_3) = (a_1a_2a_3, b_1b_2b_3) \in I_1 \times I_2$ for some $a_1, a_2, a_3 \in h(R_1)$ and for some $b_1, b_2, b_3 \in h(R_2)$. By Theorem 3.8, we conclude that I_1 and I_2 are graded prime ideals of R_1 and R_2, respectively. Since I_1 is a graded prime ideal of R_1 and $a_1a_2a_3 \in I_1$, we have $a_1 \in I_1$ or $a_2 \in I_1$ or $a_3 \in I_1$. We may assume $a_1 \in I_1$. Since I_2 is a graded prime ideal of R_2 and $b_1b_2b_3 \in I_2$, we have $b_1 \in I_2$ or $b_2 \in I_2$ or $b_3 \in I_2$. We may assume $b_2 \in I_2$. Hence $(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2) \in I_1 \times I_2$. Thus $I_1 \times I_2$ is a graded 2-absorbing ideal of $R_1 \times R_2$. The converse is clear. □

Let R be a G-graded ring and P be a proper graded ideal of R. Recall from [3] that P is said to be a graded weakly prime ideal of R if whenever $a, b \in h(R)$ and $0 \neq ab \in P$, then either $a \in P$ or $b \in P$.

Theorem 3.10. *Let R_1 and R_2 be two graded rings, and let I be a nonzero proper graded ideal of R_1. Then $I \times (0)$ is a graded weakly 2-absorbing ideal of $R_1 \times R_2$ if and only if I is a graded weakly prime ideal of R_1 and (0) is a graded prime ideal of R_2.*

Proof. Suppose that $I \times (0)$ is a graded weakly 2-absorbing ideal of $R_1 \times R_2$. First, we show that I is a graded weakly prime ideal of R_1. Let $r, s \in h(R_1)$ with $0 \neq rs \in I$. Hence $(0,0) \neq (r,1)(s,1)(1,0) = (rs,0) \in I \times (0)$. Since $I \times (0)$ is a graded weakly 2-absorbing ideal of $R_1 \times R_2$ and $(r,1)(s,1) = (rs,1) \notin I \times (0)$, we conclude that either $(r,1)(1,0) = (r,0) \in I \times (0)$ or $(s,1)(1,0) = (s,0) \in I \times (0)$ and hence either $r \in I$ or $s \in I$. Thus I is a graded weakly prime ideal of R_1. Now, we show that (0) is a graded prime ideal of R_2. Let $r, s \in h(R_2)$ with $rs \in (0)$, and let $0 \neq i \in h(I)$. Hence $(0,0) \neq (i, rs) = (i,1)(1,r)(1,s) \in I \times (0)$. Since $I \times (0)$ is a graded weakly 2-absorbing ideal of $R_1 \times R_2$ and $(1,r)(1,s) = (1,rs) \notin I \times (0)$, we conclude that $(i,1)(1,r) = (i,r) \in I \times (0)$ or $(i,1)(1,s) = (i,s) \in I \times (0)$ and hence either $r \in (0)$ or $s \in (0)$. Thus (0) is a graded prime ideal of R_2. Conversely, assume that I is a graded weakly prime ideal of R_1 and (0) is a graded prime ideal of R_2. We show that $I \times (0)$ is a graded weakly 2-absorbing ideal of $R_1 \times R_2$. Suppose that $(0,0) \neq (a_1,b_1)(a_2,b_2)(a_3,b_3) = (a_1a_2a_3, b_1b_2b_3) \in I \times (0)$ for some $a_1, a_2, a_3 \in h(R_1)$ and for some $b_1, b_2, b_3 \in h(R_2)$. Since I is a graded weakly prime ideal of R_1 and $0 \neq a_1a_2a_3 \in I$, we conclude that at least one of the a_i's is in I, say a_1. Since (0) is a graded prime ideal of R_2 and $b_1b_2b_3 \in (0)$, we conclude that at least one of the b_i's is in (0), say $b_2 = 0$. Hence $(a_1,b_1)(a_2,b_2) = (a_1a_2, 0) \in I \times (0)$. Thus $I \times (0)$ is a graded weakly 2-absorbing ideal of $R_1 \times R_2$. □

Theorem 3.11. *Let R_1 and R_2 be two graded rings, and let I_1 be a nonzero proper graded ideal of R_1, and I_2 be a proper graded ideal of R_2. Then $I_1 \times I_2$ is a graded weakly 2-absorbing ideal of $R_1 \times R_2$ that is not a graded 2-absorbing ideal if and only if $I_2 = (0)$ is a graded prime ideal of R_2 and I_1 is a graded weakly prime ideal of R_1 that is not a graded prime ideal.*

Proof. Assume that $I_1 \times I_2$ is a graded weakly 2-absorbing ideal of $R_1 \times R_2$ that is not a graded 2-absorbing ideal. Theorem 3.9 implies that $I_2 = (0)$. By Theorem 3.10, $I_2 = (0)$ is a graded prime ideal of R_2 and I_1 is a graded weakly prime ideal of R_1. Now suppose that I_1 is a graded prime ideal of R_1. Then $I_1 \times I_2$ is a graded 2-absorbing ideal by the proof of Theorem 3.9 which contradicts the assumption. Thus I_1 is not a graded prime ideal of R_1. Conversely, suppose that I_1 is a graded weakly prime ideal of R_1 that is not a graded prime ideal and $I_2 = (0)$ is a graded prime ideal of R_2. By Theorem 3.10, $I_1 \times I_2$ is a graded weakly 2-absorbing ideal of $R_1 \times R_2$. Now, we show that $I_1 \times (0)$ is not a graded 2-absorbing ideal of $R_1 \times R_2$. Since I_1 is a graded weakly prime ideal of R_1, that is not a graded prime ideal, we conclude that there exist $r, s \in h(R)$ such that $rs = 0 \in I_1$ and neither $r \in I_1$ nor $s \in I_1$. We get that $(r,1)(s,1)(1,0) = (rs,0) \in I_1 \times (0)$ but $(r,1)(s,1) = (rs,1) \notin I_1 \times (0)$ and $(r,1)(1,0) = (r,0) \notin I_1 \times (0)$ and $(s,1)(1,0) = (s,0) \notin I_1 \times (0)$. This shows that $I_1 \times (0)$ is not a graded 2-absorbing ideal of $R_1 \times R_2$. □

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