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Research Article

Dual neighborhood systems and polars in locally convex cones

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Abstract

In this paper, we define dual (abstract) neighborhood systems for locally convex cones. Also we consider three types of different polars and study some relations of them with bounded sets in locally convex cones.

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1. Introduction

The theory of locally convex cones uses an order theoretical concept or a convex quasiuniform structure to introduce a topological structure on a cone. In this paper we use the former. These cones developed in [4]. We shall review some of the main concepts and refer to [4] and [9] for details. For recent researches see [1–3, 6, 10].

A *cone* is a set $\mathcal{P}$ endowed with an addition and a scalar multiplication for non-negative real numbers. The addition is associative and commutative, and there is a neutral element $0 \in \mathcal{P}$. For the scalar multiplication the usual associative and distributive properties hold. We have $1a = a$ and $0a = 0$ for all $a \in \mathcal{P}$. A *preordered cone* (*ordered cone*) is a cone with a preorder, that is a reflexive transitive relation $\leq$ such that $a \leq b$ implies $a + c \leq b + c$ and $\alpha a \leq \alpha b$ for all $a, b, c \in \mathcal{P}$ and $\alpha \geq 0$.

The extended real numbers $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a natural example of an ordered cone with the usual order and algebraic operations in $\overline{\mathbb{R}}$, in particular $0 \cdot (+\infty) = 0$.

A subset $\mathcal{V}$ of the preordered cone $\mathcal{P}$ is called an *(abstract) 0-neighborhood system*, if the following properties hold:

(i) $0 < v$ for all $v \in \mathcal{V}$;
(ii) for all $u, v \in \mathcal{V}$ there is a $w \in \mathcal{V}$ with $w \leq u$ and $w \leq v$;
(iii) $u + v \in \mathcal{V}$ and $\alpha v \in \mathcal{V}$ whenever $u, v \in \mathcal{V}$ and $\alpha > 0$.

The elements v of $\mathcal{V}$ define upper, resp. lower, neighborhoods for the elements a of $\mathcal{P}$ by

$$v(a) = \{b \in \mathcal{P} \mid b \leq a + v\}, \quad \text{resp.} \quad (a)v = \{b \in \mathcal{P} \mid a \leq b + v\},$$

creating the upper, resp. lower, topologies on $\mathcal{P}$. Their common refinement is called *symmetric* topology. We denote the neighborhoods of the symmetric topology as $v(a) \cap (a)v$ or $v^s(a)$ for $a \in \mathcal{P}$ and $v \in \mathcal{V}$. For technical reasons we require that the elements of $\mathcal{P}$

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to be *bounded below*, i.e. for every $a \in \mathcal{P}$ and $v \in \mathcal{V}$ we have $0 \leq a + \lambda v$ for some $\lambda > 0$. An element a of $(\mathcal{P}, \mathcal{V})$ is called bounded if it is also *upper bounded*, i.e. for every $v \in \mathcal{V}$ there is a $\lambda > 0$ such that $a \leq \lambda v$. A full locally convex cone $(\mathcal{P}, \mathcal{V})$ is an ordered cone $\mathcal{P}$ that contains an (abstract) neighborhood system $\mathcal{V}$. Finally, a locally convex cone $(\mathcal{P}, \mathcal{V})$ is a subcone of a full locally convex cone not necessarily containing the (abstract) neighborhood system $\mathcal{V}$.

For cones $\mathcal{P}$ and $\mathcal{Q}$ a mapping $T : \mathcal{P} \to \mathcal{Q}$ is called a *linear operator* if $T(a + b) = T(a) + T(b)$ and $T(\alpha a) = \alpha T(a)$ hold for $a, b \in \mathcal{P}$ and $\alpha \geq 0$. If both $(\mathcal{P}, \mathcal{V})$ and $(\mathcal{Q}, \mathcal{W})$ are locally convex cones, the operator T is called *(uniformly) continuous* if for every $w \in \mathcal{W}$ one can find $v \in \mathcal{V}$ such that $T(a) \leq T(b) + w$ whenever $a \leq b + v$ for $a, b \in \mathcal{P}$. A linear functional on $\mathcal{P}$ is a linear operator $\mu : \mathcal{P} \to \overline{\mathbb{R}}$.

The polar v° of a neighborhood $v \in \mathcal{V}$ consists of all linear functionals μ on $\mathcal{P}$ satisfying $\mu(a) \leq \mu(b) + 1$ whenever $a \leq b + v$ for $a, b \in \mathcal{P}$. A linear functional μ on $(\mathcal{P}, \mathcal{V})$ is (uniformly) continuous if there is $v \in \mathcal{V}$ such that $\mu \in v^\circ$. The continuous linear functionals on a locally convex cone $(\mathcal{P}, \mathcal{V})$ (into $\overline{\mathbb{R}}$) form a cone with the usual addition and scalar multiplication of functions. This cone is called the *dual cone* of $\mathcal{P}$ and denoted by $\mathcal{P}^*$.

Endowed with the (abstract) 0-neighborhood system $\mathcal{V} = \{\varepsilon \in \mathbb{R} : \varepsilon > 0\}$, $\overline{\mathbb{R}}$ is a full locally convex cone. For $a \in \mathbb{R}$ the intervals $(-\infty, a + \varepsilon]$ are the upper and the intervals $[a - \varepsilon, +\infty]$ are the lower neighborhoods, while for $a = +\infty$ the entire cone $\overline{\mathbb{R}}$ is the only upper neighborhood, and $\{+\infty\}$ is open in the lower topology. The symmetric topology is the usual topology on $\mathbb{R}$ with as an isolated point $+\infty$.

2. Dual neighborhood systems

In this section we define dual (abstract) neighborhood systems for dual pair $(\mathcal{P}, \mathcal{P}^*)$ and investigate the closure of a convex set in these systems.

Definition 2.1 ([4, II.3.1])**.** A dual pair $(\mathcal{P}, \mathcal{Q})$ consists of two cones $\mathcal{P}$ and $\mathcal{Q}$ with a bilinear mapping $(a, x) \to \langle a, x \rangle : \mathcal{P} \times \mathcal{Q} \to \overline{\mathbb{R}}$.

The weak topology $\sigma(\mathcal{P}, \mathcal{P}^*)$ on a locally convex cone $(\mathcal{P}, \mathcal{V})$ is generated by its dual cone in the following way:
For an element $a \in \mathcal{P}$, an upper (symmetric) neighborhood $v_A(a)$ (resp $v_A^s(a)$), corresponding to a finite subset $A = \{\mu_1, ..., \mu_n\}$ of $\mathcal{P}^*$, is given by

$$v_A(a) = \{b \in \mathcal{P} : \mu_i(b) \leq \mu_i(a) + 1 \text{ for all } \mu_i \in A\},$$

respectively

$$v_A^s(a) = \left\{ b \in \mathcal{P} : \begin{array}{lll} |\mu_i(b) - \mu_i(a)| \leq 1 & \text{if} & \mu_i(a) < +\infty \\ \mu_i(b) = +\infty & \text{if} & \mu_i(a) = +\infty \end{array} \right\}.$$

Endowed with these neighborhoods, $\mathcal{P}$ forms again a locally convex cone (see I.4.6 in [9]). We say that $\mathcal{V}$ is finer than $\mathcal{W}$ (or $\mathcal{W}$ is coarser than $\mathcal{V}$) if the identity mapping $i : (\mathcal{P}, \mathcal{V}) \to (\mathcal{Q}, \mathcal{W})$ is (uniformly) continuous, that is, for every $w \in \mathcal{W}$ one can find $v \in \mathcal{V}$ such that $a \leq b + v$ implies $a \leq b + w$ (see [7]). In this case, each of three (upper, lower and symmetric) topologies induced by $\mathcal{V}$ is finer than the one induced by $\mathcal{W}$. We denote by $\mathcal{V}_\sigma$ the coarsest (abstract) neighborhood system on $\mathcal{P}$ that makes all $\mu \in \mathcal{P}^*$ continuous.

Definition 2.2. Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone. Any (abstract) neighborhood system on $\mathcal{P}$ under which the dual is $\mathcal{P}^*$ is called a *dual (abstract) neighborhood system* of $(\mathcal{P}, \mathcal{P}^*)$.

It is easy to see that $(\mathcal{P}, \mathcal{V}_\sigma)^* = (\mathcal{P}, \mathcal{V})^*$.

Example 2.3. Let $\mathcal{V} = \{0\}$ and $\mathcal{W} = \{\varepsilon > 0 : \varepsilon \in \mathbb{R}\}$ be two neighborhood systems for $\overline{\mathbb{R}}_+ = \{a \in \overline{\mathbb{R}} : a \geq 0\}$. Then the identity mapping $i : (\overline{\mathbb{R}}_+, \mathcal{V}) \to (\overline{\mathbb{R}}_+, \mathcal{W})$ is continuous. Thus $\mathcal{V}$ is finer than $\mathcal{W}$. It is clear that $\mathcal{V} = \{0\}$ and $\mathcal{W} = \{\varepsilon > 0 : \varepsilon \in \mathbb{R}\}$ are not dual (abstract) neighborhood systems, indeed the dual cone of $\overline{\mathbb{R}}_+$ under $\mathcal{W}$ consists of all positive reals and the functional $\bar{0}$ such that $\bar{0}(a) = 0$ for all $a \in \mathbb{R}$ and $\bar{0}(+\infty) = +\infty$ but the dual cone of $\overline{\mathbb{R}}_+$, endowed with the neighborhood system $\mathcal{V} = \{0\}$, is the positive reals together with $\bar{0}$, but further include the element $+\infty$, acting as $+\infty(0) = 0$ and $+\infty(a) = +\infty$ for all $0 \neq a \in \overline{\mathbb{R}}_+$. Therefore $(\overline{\mathbb{R}}_+, \mathcal{V})^* \neq (\overline{\mathbb{R}}_+, \mathcal{W})^*$.

We shall mention a result deals with the separation of convex sets by continuous linear functionals:

Corollary 2.4 ([8, Corollary 4.6])**.** *Let A be a non-empty convex subset of a locally convex cone $(\mathcal{P}, \mathcal{V})$ such that $0 \in A$.*

(i) *If A is closed with respect to the lower topology on $\mathcal{P}$, then for every element $b \notin A$ in $\mathcal{P}$ there exists a monotone linear functional $\mu \in \mathcal{P}^*$ such that $\mu(a) \leq 1 \leq \mu(b)$ for all $a \in A$ and indeed $1 < \mu(b)$ if b is bounded above.*

(ii) *If A is closed with respect to the upper topology on $\mathcal{P}$, then for every element $b \notin A$ in $\mathcal{P}$ there exists a monotone linear functional $\mu \in \mathcal{P}^*$ such that $\mu(b) < -1 \leq \mu(a)$ for all $a \in A$.*

Proposition 2.5. *Let A be a convex subset of a locally convex cone $(\mathcal{P}, \mathcal{V})$ such that $0 \in A$. Then the closure of A with respect to the upper topology, $\overline{A}^u$, is the same for every dual (abstract) neighborhood system of the dual pair.*

Proof. Let $\mathcal{W}$ be any dual (abstract) neighborhood system of the dual pair $(\mathcal{P}, \mathcal{P}^*)$. Since $\mathcal{W}$ is finer than $\mathcal{V}_\sigma$, we have $\overline{A}^u(\mathcal{W}) \subseteq \overline{A}^u(\mathcal{V}_\sigma)$. Now, let $b \notin \overline{A}^u(\mathcal{W})$. By Corollary 2.4(ii), there exists a monotone linear functional $\mu \in \mathcal{P}^*$ such that $\mu(b) < -1$ and $\mu(a) \geq -1$ for all $a \in \overline{A}^u(\mathcal{W})$. We set $B = \{\mu\}$. Then for every $\delta > 0$,

$$v_B(b) = \{x \in \mathcal{P} : \mu(x) \leq \mu(b) + \delta\}$$

is an upper neighborhood of b in the weak topology $\sigma(\mathcal{P}, \mathcal{P}^*)$ which does not meet A. For, if there is $x \in v_B(b) \cap A$, then $\mu(x) \geq -1$ and $\mu(x) \leq \mu(b) + \delta$. Thus $\mu(b) + \delta \geq -1$ for every $\delta > 0$, which gives a contradiction. Hence $b \notin \overline{A}^u(\mathcal{V})$. □

Proposition 2.6. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone and A be a convex subset of $\mathcal{P}$ such that $0 \in A$. If all elements of $\mathcal{P} \setminus A$ are bounded, then the closure of A with respect to the lower topology, $\overline{A}^l$, is the same for every dual (abstract) neighborhood system of the dual pair.*

Proof. Let $\mathcal{W}$ be any dual (abstract) neighborhood system of the dual pair $(\mathcal{P}, \mathcal{P}^*)$. Since $\mathcal{W}$ is finer than $\mathcal{V}_\sigma$, so $\overline{A}^l(\mathcal{W}) \subseteq \overline{A}^l(\mathcal{V}_\sigma)$. If $b \notin \overline{A}^l(\mathcal{W})$, then by Corollary 2.4(i), there exists a monotone linear functional $\mu \in \mathcal{P}^*$ such that $\mu(b) \geq 1$ and $\mu(a) \leq 1$ for all $a \in \overline{A}^l(\mathcal{W})$. Since b is bounded, $\mu(b) > 1$. We set $B = \{\mu\}$. Then

$$(b)v_B = \{x \in \mathcal{P} : \mu(b) \leq \mu(x) + \delta\}$$

for each $\delta > 0$ is a lower neighborhood of b in the weak topology $\sigma(\mathcal{P}, \mathcal{P}^*)$ which does not meet A. Indeed, if there is $x \in (b)v_B \cap A$, then $\mu(x) \leq 1$ and $\mu(b) \leq \mu(x) + \delta$. Therefore $\mu(b) \leq 1 + \delta$. This is a contradiction, because $\delta > 0$ was arbitrarily chosen. □

3. Polars in locally convex cones

Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone. The polar of a subset F of $\mathcal{P}^2 = \mathcal{P} \times \mathcal{P}$ is defined in [1] as follows:

$$F^\circ = \{\mu \in \mathcal{P}^* : \mu(a) \leq \mu(b) + 1, \text{ for all } (a, b) \in F\}.$$

For every neighborhood $v \in \mathcal{V}$, indeed, we have $v^\circ = (\tilde{v})^\circ$, where

$$\tilde{v} = \{(a, b) \in \mathcal{P} \times \mathcal{P} \ : \ a \leq b + v\}.$$

Let A be a subset of a locally convex cone $(\mathcal{P}, \mathcal{V})$. Then its lower polar is defined to be

$$A_\circ = \{\mu \in \mathcal{P}^* : \ \mu(a) \le 1, \quad \text{for all} \ \ a \in A\},$$

and its upper polar is defined to be

$$A^\circ = \{\mu \in \mathcal{P}^* : \ \mu(a) \ge 1, \ \ \text{for all} \ \ a \in A\},$$

(see [5]). It is straightforward to see that $A_\circ \cap A^\circ \subseteq (A \times A)^\circ$, and if $0 \in A$, we have $A^\circ = \emptyset$ and $(A \times A)^\circ \subseteq A_\circ$.

Proposition 3.1. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone. If $A \subseteq \mathcal{P}$ ($F \subseteq \mathcal{P}^2$), then $A^\circ = (\overline{A})^\circ$ and $A_\circ = (\overline{A})_\circ$ ($F^\circ = (\overline{F})^\circ$), where the closure can be taken with respect to each of the upper, lower and symmetric (product) topologies.*

Proof. Since all $\mu \in \mathcal{P}^*$ are (uniformly) continuous, then they are continuous under the mentioned three topologies. □

Proposition 3.2. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone. For a subset $A \subseteq \mathcal{P}$ ($F \subseteq \mathcal{P}^2$), $(co(A))_\circ = A_\circ$ and $(co(A))^\circ = A^\circ$ ($(co(F))^\circ = F^\circ$), where $co(A)$ is the convex envelope of A.*

Proof. We show the lower polar case. Since $A \subseteq co(A)$, $(co(A))_\circ \subseteq A_\circ$. Let $\mu \in A_\circ$ and $x \in co(A)$. Then there exists $k \in \mathbb{N}$ and $\alpha_1, ..., \alpha_k \in \mathbb{R}_+$ and $x_1, ..., x_k \in A$ such that $\sum_{i=1}^k \alpha_i = 1$ and $x = \sum_{i=1}^k \alpha_i x_i$. Now $\mu(x) = \sum_{i=1}^k \alpha_i \mu(x_i) \le \sum_{i=1}^k \alpha_i = 1$. Hence $\mu \in (co(A))_\circ$. □

Corollary 3.3. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone. If $A \subseteq \mathcal{P}$ ($F \subseteq \mathcal{P}^2$), $(\overline{co(A)})_\circ = A_\circ$ and $(\overline{co(A)})^\circ = A^\circ$ ($(\overline{co(F)})^\circ = F^\circ$).*

Proof. It follows from Proposition 3.1 and Proposition 3.2. □

Proposition 3.4. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone. Then for a subset F of $\mathcal{P}^2$, F° is $\sigma(\mathcal{P}^*, \mathcal{P})$-symmetric closed.*

Proof. Suppose $\mu_0 \in \overline{F^\circ}^s(\mathcal{V}_\sigma)$ and $(a_1, a_2) \in F$. Let $B = \{a_1, a_2\}$ and $\varepsilon > 0$ be arbitrary. Then

$$v_B^s(\mu_0) = \left\{ \mu \in \mathcal{P}^* : \begin{array}{lll} |\mu(a_i) - \mu_0(a_i)| \le \varepsilon & \text{if} & \mu_0(a_i) < +\infty \\ \mu(a_i) = +\infty & \text{if} & \mu_0(a_i) = +\infty \end{array} \right\},$$

for $i = 1, 2$, is a symmetric neighborhood of μ_0 in the topology $\sigma(\mathcal{P}^*, \mathcal{P})$. Hence this neighborhood meets F° and so there exists $\mu \in v_B^s(\mu_0) \cap F^\circ$. Thus $\mu(a_1) \le \mu(a_2) + 1$. If $\mu_0(a_i) < +\infty$, then $|\mu(a_i) - \mu_0(a_i)| \le \varepsilon$ for $i = 1, 2$. Now $\mu_0(a_1) \le \mu(a_1) + \varepsilon \le \mu(a_2) + 1 + \varepsilon \le \mu_0(a_2) + 1 + 2\varepsilon$. Arbitrariness of $\varepsilon > 0$ implies $\mu_0 \in F^\circ$. If $\mu_0(a_1) = +\infty$, then $\mu(a_1) = +\infty$. Hence $\mu(a_2) = +\infty$ and so $\mu_0(a_2) = +\infty$ and $\mu_0 \in F^\circ$. If $\mu_0(a_2) = +\infty$, then $\mu_0(a_1) \le \mu_0(a_2) + 1$, i.e. $\mu_0 \in F^\circ$. □

Example 3.5. Consider the cone $\overline{\mathbb{R}}_+$ with the neighborhood system $\mathcal{V} = \{0\}$. Let $F = \{(+\infty, 1)\}$ be a subset of $\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+$. We have $F^\circ = \{0, +\infty\}$ (see Example 2.3). If we set $\mu_0 = \alpha$ for $\alpha \ge 0$, then every upper neighborhood of μ_0 contains the functional 0 and so $\mu_0 \in \overline{F^\circ}^u(\mathcal{V}_\sigma)$. Therefore F° is not $\sigma(\mathcal{P}^*, \mathcal{P})$-upper closed. Also every lower neighborhood of μ_0 contains the functional $+\infty$ and so $\mu_0 \in \overline{F^\circ}^l(\mathcal{V}_\sigma)$. Hence F° is not $\sigma(\mathcal{P}^*, \mathcal{P})$-lower closed.

Proposition 3.6. *For a subset A of the locally convex cone $(\mathcal{P}, \mathcal{V})$, $A_\circ$ is $\sigma(\mathcal{P}^*, \mathcal{P})$-lower closed.*

Proof. Let $\mu_0 \in \overline{A_\circ}^l(\mathcal{V}_\sigma)$ and $a \in A$. We set $B = \{a\}$. Then $(\mu_0)v_B = \{\mu \in \mathcal{P}^* : \mu_0(a) \le \mu(a) + \varepsilon\}$ for each $\varepsilon > 0$, is a lower neighborhood of μ_0 in the topology $\sigma(\mathcal{P}^*, \mathcal{P})$. Therefore there exists $\mu \in (\mu_0)v_B \cap A_\circ$. Thus $\mu(a) \le 1$ for all $a \in A$ and $\mu_0(a) \le \mu(a) + \varepsilon \le 1 + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, hence $\mu_0 \in A_\circ$. □

Example 3.7. Consider the locally convex cone $(\overline{\mathbb{R}}, \mathcal{V})$, where $\mathcal{V} = \{\varepsilon > 0 : \varepsilon \in \mathbb{R}\}$. Let $Conv(\overline{\mathbb{R}}) = \{A : A \text{ is a non-empty convex subset of } \overline{\mathbb{R}}\}$ endowed with the usual addition and multiplication of sets by non-negative scalars, that is $\alpha A = \{\alpha a : a \in A\}$ and $A + B = \{a + b : a \in A \text{ and } b \in B\}$ for $A, B \in Conv(\overline{\mathbb{R}})$ and $\alpha \geq 0$. With the preorder

$$A \leq B \text{ iff for every } a \in A \text{ there is some } b \in B \text{ such that } a \leq b,$$

and with $\overline{\mathcal{V}} = \{\{\varepsilon\} : \varepsilon > 0 \text{ and } \varepsilon \in \mathbb{R}\}$ as (abstract) neighborhood system, $(Conv(\overline{\mathbb{R}}), \overline{\mathcal{V}})$ is a locally convex cone. With $\mu \in \overline{\mathbb{R}}^*$ we associate an element $\overline{\mu} \in (Conv(\overline{\mathbb{R}}))^*$ defined by $\overline{\mu}(A) = \sup\{\mu(a) : a \in A\}$ for every $A \in Conv(\overline{\mathbb{R}})$. All $\overline{\mu}$ of this type belong to $(Conv(\overline{\mathbb{R}}))^*$, but not all elements of $(Conv(\overline{\mathbb{R}}))^*$ are of this type (see [4], II.2.16). We recall that $\overline{\mathbb{R}}^* = [0, +\infty) \cup \{\overline{0}\}$ (see Example 2.3). Let $A = \{(-\infty, b)\}$ be a subset of $Conv(\overline{\mathbb{R}})$. For positive real number b we have,

$$A^\circ \supseteq \{\overline{\alpha} \ : \alpha \geq \frac{1}{b}\} \text{ and } A_\circ \supseteq \{\overline{\alpha} \ : 0 \leq \alpha \leq \frac{1}{b}\} \cup \{\overline{0}\},$$

where $\overline{\alpha}$ is the functional associated with $\alpha \in \overline{\mathbb{R}}^*$. For $b < 0$, any $\overline{\mu}$ as defined above does not belong to A° and also in this case $A_\circ \supseteq \{\overline{0}\} \cup \{\overline{\alpha} : \ \alpha \geq 0\}$.

Let $B = \{(a, +\infty]\}$. Then $B^\circ \supseteq \{\overline{0}\} \cup \{\overline{\alpha} : \ \alpha > 0\}$ and $B_\circ$ contains only the functional 0 of all $\overline{\mu}$ defined as above.

Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone and A be a subset of $\mathcal{P}$. The lower polar $A_{\circ\circ}$ of $A_\circ$ in $\mathcal{P}$ is called the *lower bipolar* of A in $\mathcal{P}$, that is

$$A_{\circ\circ} = \{a \in \mathcal{P} : \ \mu(a) \leq 1, \quad \text{for all} \quad \mu \in A_\circ\}.$$

Proposition 3.8. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone and A be a subset of $\mathcal{P}$ containing 0. If all elements of $\mathcal{P} \setminus \overline{co}^l(A)$ are bounded, then the lower bipolar $A_{\circ\circ}$ of A is the lower closed convex envelope of A.*

Proof. Let B be the $\sigma(\mathcal{P}, \mathcal{P}^*)$-lower closed convex envelope of A. By Proposition 3.6, $A_{\circ\circ}$ is a $\sigma(\mathcal{P}, \mathcal{P}^*)$-lower closed convex set containing A. Thus $B \subseteq A_{\circ\circ}$. If $a \notin B$, then by Corollary 2.4(i), there exists a monotone linear functional $\mu \in \mathcal{P}^*$ such that $\mu(a) \geq 1$ and $\mu(b) \leq 1$ for all $b \in B$. Then $\mu \in B_\circ$ and so $\mu \in A_\circ$, since $A \subseteq B$. On the other hand a is bounded and therefore $\mu(a) > 1$. Then $a \notin A_{\circ\circ}$. Hence $A_{\circ\circ} \subseteq B$. Thus $A_{\circ\circ}$ is the $\sigma(\mathcal{P}, \mathcal{P}^*)$-lower closed convex envelope of A, and by Proposition 2.6, is the lower closed convex envelope of A. □

Proposition 3.9. *For a subset A of the locally convex cone $(\mathcal{P}, \mathcal{V})$, A° is $\sigma(\mathcal{P}^*, \mathcal{P})$-upper closed.*

Proof. Let $\mu_0 \in \overline{A^\circ}^u(\mathcal{V}_\sigma)$ and $a \in A$. We set $B = \{a\}$. Then $v_B(\mu_0) = \{\mu \in \mathcal{P}^* : \mu(a) \leq \mu_0(a) + \varepsilon\}$ for each $\varepsilon > 0$, is an upper neighborhood of μ_0 in the weak topology and so $A^\circ \cap v_B(\mu_0) \neq \emptyset$. Therefore there is $\mu \in \mathcal{P}^*$ such that $\mu(a) \geq 1$ for all $a \in A$ and $1 \leq \mu(a) \leq \mu_0(a) + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, thus $\mu_0 \in A^\circ$. □

We endow $\mathcal{P}^*$ with the topology $w(\mathcal{P}^*, \mathcal{P})$ of pointwise convergence on the elements of $\mathcal{P}$, considered as functions on $\mathcal{P}^*$ with values in $\overline{\mathbb{R}}$ with its usual topology. The $\sigma(\mathcal{P}^*, \mathcal{P})$-symmetric topology is finer than $w(\mathcal{P}^*, \mathcal{P})$ (for more informations see [4], II.3.4). As in locally convex topological vector spaces, the polar v° of a neighborhood $v \in \mathcal{V}$ is $w(\mathcal{P}^*, \mathcal{P})$-compact and convex (see [4, Proposition II.2.4]).

Theorem 3.10. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone and A be a subset of $\mathcal{P}$. Then the upper polar A° of A is $w(\mathcal{P}^*, \mathcal{P})$-compact.*

Proof. For every μ in A° we have $\mu(a) \geq 1$ for all $a \in A$. On the other hand A° is a $w(\mathcal{P}^*, \mathcal{P})$-closed subset of $\prod_{a \in A}[1, +\infty]$ and hence similar to [4], Proposition II.2.4 is $w(\mathcal{P}^*, \mathcal{P})$-compact. □

Suppose that $(\mathcal{P}, \mathcal{V})$ is a locally convex cone. A subset A of $\mathcal{P}$ is called *bounded above* (*below*) if for every $v \in \mathcal{V}$ there is $\lambda \geq 0$ such that $a \leq \lambda v$ $(0 \leq a + \lambda v)$ for all $a \in A$. The set A is called *bounded* if it is both bounded below and above (see [9, I.4.24]). We shall say that $A \subseteq \mathcal{P}$ is *weakly bounded*, if it is bounded in locally convex cone $(\mathcal{P}, \mathcal{V}_\sigma)$. Similar to the case of topological vector spaces:

Theorem 3.11. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone. A subset A of $\mathcal{P}$ is weakly bounded above if and only if $A_\circ$ is an absorbent subset of $\mathcal{P}^*$.*

Proof. Let A be a weakly bounded above subset of $\mathcal{P}$ and let $\mu \in \mathcal{P}^*$. Let $v_B \in \mathcal{V}_\sigma$ such that $\mu \in v_B^\circ$. There is $\lambda > 0$ such that $a \leq \lambda v_B$ for all $a \in A$. Thus $\mu(a) \leq \lambda$ for all $a \in A$. Hence $\frac{\mu}{\lambda} \in A_\circ$, i.e. $\mu \in \lambda A_\circ$. Conversely, let $v_B \in \mathcal{V}_\sigma$. Then there is $n \in \mathbb{N}$ such that $B = \{\mu_1, \mu_2, ..., \mu_n\}$. Now for every μ_i there is $\lambda_i > 0$ such that $\mu_i \in \lambda_i A_\circ$. We set $\lambda = max\{\lambda_i : i = 1, .., n\}$. Since $A_\circ$ is convex and containing the functional 0, then $\mu_i \in \lambda A_\circ$. Thus $\frac{\mu_i}{\lambda} \in A_\circ$ for every $\mu_i \in B$. Therefore $\frac{\mu_i}{\lambda}(a) \leq \mu_i(0) + 1$ for every $\mu_i \in B$ and every $a \in A$. Hence $\frac{a}{\lambda} \in v_B(0)$ for every $a \in A$, that is $a \leq \lambda v_B$ for every $a \in A$. □

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