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Research Article

A note on weak almost limited operators

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Abstract

Let us recall that an operator $T : E \to F$, between two Banach lattices, is said to be weak* Dunford-Pettis (resp. weak almost limited) if $f_n(Tx_n) \to 0$ whenever (x_n) converges weakly to 0 in E and (f_n) converges weak* to 0 in F' (resp. $f_n(Tx_n) \to 0$ for all weakly null sequences $(x_n) \subset E$ and all weak* null sequences $(f_n) \subset F'$ with pairwise disjoint terms). In this note, we state some sufficient conditions for an operator $R : G \to E$(resp. $S : F \to G$), between Banach lattices, under which the product TR (resp. ST) is weak* Dunford-Pettis whenever $T : E \to F$ is an order bounded weak almost limited operator. As a consequence, we establish the coincidence of the above two classes of operators on order bounded operators, under a suitable lattice operations' sequential continuity of the spaces (resp. their duals) between which the operators are defined. We also look at the order structure of the vector space of weak almost limited operators between Banach lattices.

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Introduction

This note is a sequel to the recent works [9, 17] where the authors introduced and characterized the class of weak almost limited operators, and investigated their relationship with almost limited (resp. almost Dunford-Pettis, weak* Dunford-Pettis) operators. Here, we extend some results to order bounded operators between Banach lattices using some new lattice approximations established for weak almost limited operators (Sec. 3). These lattice approximations allowed us to investigate the product of weak almost limited operators by some order type operators recenlty introduced (Sec. 4). Consequently, the w*-conterpart of a result noted by W. Wnuk in [21, Proposition 6] is obtained in the last of the paper. The last section is devoted to some notes on the order structure of the vector space of weak almost limited operators between Banach lattices and some further results.

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1. Terminology

A Banach lattice is a Banach space $(E, \|\cdot\|)$ such that E is a vector lattice and for each $x, y \in E$, $|x| \leq |y|$ implies $\|x\| \leq \|y\|$. The positive cone of E will be denoted by E^+ and for a subset A of a Banach lattice, the solid hull of A will be denoted by $sol(A)$ and $A^+ := A \cap E^+$. A norm $\|\cdot\|$ of a Banach lattice E is order continuous if for each net (x_α) such that $x_\alpha \downarrow 0$ in E, (x_α) converges to 0 for the norm $\|\cdot\|$ where the notation $x_\alpha \downarrow 0$ means that (x_α) is decreasing and $\inf(x_\alpha) = 0$. A subset $A \subset E$ of a Banach lattice is order bounded if it is contained in some order interval. An operator $T : E \to F$ between two Banach lattices is an order bounded operator, if it maps order bounded subsets of E into an order bounded ones of F. It is positive if $T(E^+) \subset F^+$. The positive operators between two Banach lattices generate the vector space of all regular operators, i.e., operators that are written as a difference of two positive operators. The vector space of all continuous (resp. order bounded, resp. regular) operators from E to F will be denoted $\mathcal{L}(E, F)$ (resp. $\mathcal{L}_b(E, F)$, resp. $\mathcal{L}_r(E, F)$). Note that $\mathcal{L}_r(E, F) \subset \mathcal{L}_b(E, F)$ and this inclusion may be proper (see example of H. P. Lotz [2, Example 1.16]). The lattice operations in a Banach lattice E (resp. E') are said to be sequentially weakly (resp. weak*) continuous if for every weakly null sequence (x_n) in E (resp. weak* null sequence (f_n) in E'), $|x_n| \to 0$ for $\sigma(E, E')$ (resp. $|f_n| \to 0$ for $\sigma(E', E)$). We will use the notation $x_n \perp x_m$ to mean that the sequence (x_n) in a Banach lattice is disjoint, that is, $|x_n| \wedge |x_m| = 0$, $n \neq m$. Throughout this paper, X, Y will denote real Banach spaces, E, F will denote real Banach lattices, and we mean by operator between Banach spaces, a bounded linear mapping.

Let us recall that a subset A of a Banach space X is called limited [5], if every weak* null sequence (f_n) in X' converges uniformly to zero on A. In his paper [15], T. Leavelle considered the notion of *(L)-set* as a dual counterpart of limited sets; a subset B of a dual Banach space X' is called (L)-set, if every weakly null sequence (x_n) in X converges uniformly to zero on B, that is, $\sup_{f \in B} |f(x_n)| \to 0$. If the sequences (x_n) are taken disjoint in the latter defintion, then B is said to be almost (L)-set (see [3] for more details).

A Banach lattice E has the (positive) Schur property if (disjoint, or positive) weakly null sequences in E are norm null. The authors of [4] introduced the so-called *dual positive Schur property.* A Banach lattice E has the dual positive Schur property (abb. DPS) if any positive weak* null sequence in E' is norm null, equivalently, any positive weak* null sequence in E' consisting of pairwise disjoint terms is norm null [23, Proposition 2.3]. A Banach lattice E is said to have the *(weak) Dunford-Pettis* (abb. (w)DP*) property*, whenever $f_n(x_n) \to 0$ for every weakly null sequence $(x_n) \subset E$ and every (disjoint) weak* null sequence $(f_n) \subset E'$. The WDP* property is weaker than both the Dunford-Pettis* property and the positive Schur property of a Banach lattice, and stronger than the weak Dunford-Pettis property (that is, sequences of functionals in the preceding definition are taken disjoint weakly null).

Let us recall that an operator $T : X \to Y$ is called Dunford-Pettis if $\|Tx_n\| \to 0$ for every weakly null sequence $(x_n) \subset X$. It is called limited if $\|T' f_n\| \to 0$ for every weak* null sequence $(f_n) \subset Y'$. As weak versions of limited operators, several types of operators were recently introduced and studied. An operator $T : E \to F$ is

- *weak* Dunford-Pettis (w*DP)* [11], if $f_n(Tx_n) \to 0$ whenever (x_n) converges weakly to 0 in E and (f_n) converges weak* to 0 in F'.
- *almost limited* [18], if $\|T' f_n\| \to 0$ for every disjoint weak* null sequence $(f_n) \subset F'$.
- *weak almost limited (wa-limited)* [9], if $f_n(Tx_n) \to 0$ for all weakly null sequences $(x_n) \subset E$ and all weak* null sequences $(f_n) \subset F'$ with pairwise disjoint terms.

The class of wa-limited operators extends both the notions of w*DP operator, almost limited operator, and the WDP* property of a Banach lattice, since every w*DP (resp. almost limited) operator $T : E \to F$ is wa-limited, and a Banach lattice E has the WDP* property iff the identity operator on E is wa-limited. Also, clearly, a Banach lattice E has the DP* (resp. (positive) Schur) property if, and only if, the identity operator on E is a w*DP (resp. (almost) DP) operator. Furthermore, a σ-Dedekind complete Banach lattice E has the DPS property iff the identity operator on E is almost limited [18, Theorem 3.3]. Note that the class of wa-limited operators contains strictly that of w*DP operators as well as that of almost limited operators, that is, every w*DP (resp. almost limited) operator is wa-limited. But a wa-limited operator is not necessarily w*DP (resp. almost limited). For instance, the identity operator $I : \ell^1 \to \ell^1$ is weak almost limited as ℓ^1 has the Schur (WDP*) property. But, as ℓ^1 does not have the dual positive Schur property [23, Proposition 2.5], $I : \ell^1 \to \ell^1$ is not almost limited. On the other hand, the identity operator $I : L^1[0,1] \to L^1[0,1]$ is weak almost limited as $L^1[0,1]$ has the positive Schur (WDP*) property. But, as $L^1[0,1]$ does not have the DP* property $I : L^1[0,1] \to L^1[0,1]$ is not w*DP.

Finally, we refer the reader to [2, 19] for unexplained terminologies on Banach lattice theory and positive operators.

2. Characterization and lattice approximation of wa-limited operators

Our following main result shows that the sequences' disjointness condition in the definition of wa-limited operators can be reversed, extending the result obtained in [17, Theorem 2.7(5)] for positive operators.

Theorem 2.1. *Let $T : E \to F$ be an order bounded operator between two Banach lattices such that F is σ-Dedekind complete. Then, the following assertions are equivalent:*

(1) *T is a wa-limited operator.*
(2) *T' carries the solid hull of each weak* null sequence (f_n) of F' to an almost (L)-set in E'.*
(3) *$f_n(Tx_n) \to 0$ for every disjoint weakly null sequence $(x_n) \subset E$ and every weak* null sequence $(f_n) \subset F'$.*

Proof. (1) $\Rightarrow$ (2) Let $(f_n) \subset F'$ and $(x_n) \subset E$ be respectively a weak* null sequence and a disjoint weakly null sequence. Put $B = sol(\{f_n : n \in \mathbb{N}\})$. We proceed in two steps:

Step 1: We claim that $|Tx_n| \xrightarrow{w} 0$ in F. Let $f \in F'_+$. Since $T' : F' \to E'$ is order bounded [2, Theorem 1.73], there exists some $g \in E'_+$ such that $T'[-f, f] \subseteq [-g, g]$. For each n pick $|f_n| \leq f$ with $f(|Tx_n|) = f_n(Tx_n)$ (see [2, Theorem 1.23]). Thus, for each n we have

$$f(|T(x_n)|) = f_n(Tx_n) \leq |T'(f_n)|(|x_n|) \leq g(|x_n|) \to 0,$$

since $|x_n| \xrightarrow{w} 0$ (see [2, Theorem 4.34]). This shows that $|Tx_n| \xrightarrow{w} 0$ holds in F.

Step 2: We show that $g_n(Tx_n) \to 0$ for every disjoint sequence $(g_n) \subset B^+$. For such sequence, we have $g_n \xrightarrow{w^*} 0$ (see [13, Lemma 3.1]). As T is wa-limited we see that $g_n(Tx_n) \to 0$ as desired. Now, by the lattice embedding $F \hookrightarrow F''$ we have by Step 1 $|Tx_n| \xrightarrow{w} 0$ in F''. Then, since $(Tx_n)(f_n) = f_n(Tx_n) \to 0$ for every disjoint sequence $(f_n) \subset B^+$, by a particular case of [8, Theorem 2.4] we see that $\sup_{f \in B} f(|Tx_n|) \to 0$. Now, from

$$\sup_{g \in T'(B)} |g(x_n)| = \sup_{f \in B} |f(Tx_n)| \leq \sup_{f \in B} f(|T(x_n)|)$$

we get $\sup_{g \in T'(B)} |g(x_n)| \to 0$, i.e., $T'(B)$ is an almost (L)-set.

(2) $\Rightarrow$ (3) Let $(x_n) \subset E$, $(f_n) \subset F'$ be respectively a disjoint weakly null sequence and a weak* null sequence. Since the set $\{T'f_n : n \in \mathbb{N}\}$ is an almost (L)-set, thus $\sup_k |f_n(Tx_k)| \to 0$ as $n \to \infty$, and from the inequality $\sup_k |f_n(Tx_k)| \geq |f_n(Tx_n)|$ we see that $f_n(Tx_n) \to 0$.

(3) $\Rightarrow$ (1) Follows from [9, Theorem 2.4(3)]. $\square$

Clearly, by definition of a wa-limited operator the composition from the right of each wa-limited operator by an arbitrary operator is still wa-limited. For the composition from the left, the characterization (3) of the above theorem enable us to derive similar fact for order bounded operators.

Corollary 2.2. *Let E, F and G be a Banach lattices such that F and G are σ-Dedekind complete and let $T \in \mathcal{L}_b(E,F)$. If T is wa-limited, then so is $S \circ T$ for every operator $S : F \to G$. In particular, in $\mathcal{L}_b(F)$ the order bounded wa-limited operators form a two-sided ideal.*

Corollary 2.3. *Let E and F be two Banach lattices such that F is σ-Dedekind complete. If E' has order continuous norm and sequentially weak* continuous lattice operations, then an order bounded operator $T : E \to F$ is wa-limited if and only if it is limited.*

Proof. Let $(f_n) \subset F'$ be a weak* null sequence. Since $|T'f_n| \overset{w^*}{\to} 0$ holds in E', to show that $\|T'f_n\| \to 0$, it suffices by [8, Corollary 2.7] to show that $f_n(Tx_n) \to 0$ for each bounded and disjoint sequence $(x_n) \subset E^+$. For such sequence, since the norm of E' is order continuous, it follows by [19, Theorem 2.4.14] that $x_n \overset{w}{\to} 0$. Now, applying Theorem 2.1(3) we see by hypothesis that $f_n(Tx_n) \to 0$ as desired. $\square$

Next, we show in our following result that order bounded wa-limited operators satisfy some lattice approximations, extending [17, Theorem 2.5] which is stated for positive operators.

Theorem 2.4. *Let $T : E \to F$ be an order bounded wa-limited operator between two Banach lattices such that F is σ-Dedekind complete. Then, the following assertions hold:*

(1) *For each relatively weakly compact subset $A \subset E$ and each weak* null sequence $(f_n) \subset F'$, given $\varepsilon > 0$, there exists some $u \in E^+$ satisfying*
$$\left|f\left(T\left(|x|-u\right)^+\right)\right| \leq \varepsilon$$
for all $x \in sol(A)$ and all $f \in B$, where $B = sol\left(\{f_n : n \in \mathbb{N}\}\right)$.

(2) *For each relatively weakly compact subset $A \subset E$ and each weak* null sequence $(f_n) \subset F'$, given $\varepsilon > 0$, there exists some $g \in (F')^+$ satisfying*
$$\left|\left(|f|-g\right)^+(Tx)\right| \leq \varepsilon$$
for all $x \in sol(A)$ and all $f \in B$.

Proof. Note that the proof is similar for the two assertions, so we present only that of the first one. Assume by way of contradiction that there exist a relatively weakly compact subsets $A \subset E$, a weak* null sequence $(f_n) \subset F'$, and some $\varepsilon > 0$ such that for each $u \in E^+$ we have
$$\left|f\left(T\left(|x|-u\right)^+\right)\right| > \varepsilon$$
for at least $x \in sol(A)$ and $f \in B$. In particular, an easy inductive argument shows that there exist a sequences $(x_n) \subset sol(A)$, $(f_n) \subset B$ such that
$$\left|f_n\left(T\left(|x_{n+1}| - 4^n \sum_{i=1}^{n} |x_i|\right)^+\right)\right| > \varepsilon \tag{2.1}$$
holds for each n. Put $y = \sum_{n=1}^{\infty} 2^{-n}|x_n|$, $y_n = \left(|x_{n+1}| - 4^n \sum_{i=1}^{n} |x_i|\right)^+$ and $z_n = \left(|x_{n+1}| - 4^n \sum_{i=1}^{n} |x_i| - 2^{-n}y\right)^+$. From Lemma 4.35 of [2] the sequence (z_n) is disjoint. Also, since $0 \leq z_n \leq |x_n+1|$ holds, we see that $(z_n) \subset sol(A)$ and therefore $z_n \overset{w}{\to} 0$ (see [2, Theorem 4.34]). Now, it follows from Theorem 2.1 that $sup_{f\in B}|f(Tz_n)| \to 0$. In

particular, $f_n(Tz_n) \to 0$. On the other hand, we have $0 \le y_n - z_n \le 2^{-n}y$ from which we get $\|y_n - z_n\| \le 2^{-n}\|y\|$. In particular, we infer that $f_n(T(y_n - z_n)) \to 0$. Therefore, we see that

$$|f_n(Ty_n)| \le |f_n(T(y_n - z_n))| + |f_n(Tz_n)| \to 0,$$

which contradicts (2.1). This completes the proof. □

3. The product of wa-limited operators by some order type operators

Recently, some order type operators were introduced and studied. An operator $T : E \to X$ is said to be order limited [12], if T carries each order bounded subset of E to a limited one in X. The dual counterpart of an order limited operator is defined in [10] as follows: an operator $T : X \to E$ is called order (L)-Dunford-Pettis, if the adjoint T' carries each order bounded subset of E' to an (L)-set in X'. The following two sequential characterizations were established for the two latter types of operators (see [12, Theorem 3.3] and [10, Theorem 2.5]).

(*a*) $T : E \to X$ is order limited iff $|T'f_n| \xrightarrow{w^*} 0$ for every weak* null sequence $(f_n) \subset X'$.
(*b*) $T : X \to E$ is order (L)-Dunford-Pettis iff $|Tx_n| \xrightarrow{w} 0$ for every weakly null sequence $(x_n) \subset X$.

It follows for a Banach lattice E that the lattice operations in E (resp. E') are sequentially weakly (resp. weak*) continuous iff the identity operator on E is order (L)-Dunford-Pettis (resp. order limited).

We are now in position to state our following main result.

Theorem 3.1. *Let E, F and G be a Banach lattices such that F is σ-Dedekind complete. Then, for a wa-limited operator $T \in \mathcal{L}_b(E, F)$ the following statements hold:*

(1) *the product TR is a w*DP operator for every $R \in \mathcal{L}_b(G, E)$ such that R is order limited (resp. order (L)-Dunford-Pettis).*
(2) *the product ST is a w*DP operator for every $S \in \mathcal{L}_b(F, G)$ such that S is order (L)-Dunford-Pettis.*

Proof. (1) Let $(x_n) \subset G$ and $(f_n) \subset F'$ be respectively a weakly null sequence and a weak* null sequence. We shall see that $f_n(TR(x_n)) \to 0$. It suffices to show that $f_n(TRy_n) \to 0$ where $(y_n) \in \{(x_n^+);(x_n^-)\}$. To this end, let $\varepsilon > 0$. As T is a wa-limited operator then, TR is so (Corollary 2.2). Therefore, by Theorem 2.4, pick some $u \in E^+$ such that

$$\left|f_n\left(TR(y_n - u)^+\right)\right| < \varepsilon$$

holds for all n. Now, for every n we have

$$\begin{aligned} |f_nTRy_n| &\le \left|f_n\left(TR(y_n - u)^+\right)\right| + |f_n(TR(y_n \wedge u))| \\ &\le \varepsilon + |T'(f_n)(R(y_n \wedge u))|. \end{aligned}$$

Since $T'(f_n) \xrightarrow{w^*} 0$ we see from $(R(y_n \wedge u))_n \subset R([0,u])$ and the order limitedness of R that $T'(f_n)(R(y_n \wedge u)) \to 0$. Therefore, as $\varepsilon > 0$ is arbitrary, we conclude that $f_n(TRy_n) \to 0$ as desired.

Assume now that R is an order (L)-Dunford-Pettis operator. we shall show that $g_n(TRx_n) \to 0$ where $(g_n) \in \{(f_n^+);(f_n^-)\}$. To this end, let $\varepsilon > 0$. By Theorem 2.4, pick some $g \in F'_+$ such that

$$\left|(g_n - g)^+(TRx_n)\right| \le \varepsilon$$

holds for all n, and for every n we have

$$\begin{aligned} |g_n T R x_n| &\leq \left|(g_n - g)^+ (T R x_n)\right| + |(g_n \wedge g)(T R x_n)| \\ &\leq \varepsilon + |T'(g_n \wedge g)(R x_n)| \\ &\leq \varepsilon + |T'(g_n \wedge g)|(|R x_n|). \end{aligned}$$

Since T' is order bounded, there exists some $h \in E'_+$ with $|T'(g_n \wedge g)| \leq h$ for all n. Therefore, as R is order (L)-Dunford-Pettis it follows that $|T'(g_n \wedge g)|(|R x_n|) \leq h(|R x_n|) \to 0$. As $\varepsilon > 0$ is arbitrary, we conclude that $g_n(T R x_n) \to 0$ as desired.

(2) Using the second lattice approximation of Theorem 2.4, the proof is obtained by similar arguments as in (1) for the case R is order limited.

□

As consequense we have the following generalisation of [17, Corollary 2.11].

Corollary 3.2. *Let E and F be two Banach lattices such that F is σ-Dedekind complete. Then, an order bounded operator $T : E \to F$ is wa-limited if and only if it is w*DP, whenever one of the following holds:*

(i) E has sequentially weakly continuous lattice operations.
(ii) E' has sequentially weak continuous lattice operations.*
(iii) F has sequentially weakly continuous lattice operations.

Remark 3.3. In the above corollary:

(1) The conditions (ii) and (iii) present a reversed conditions compared to those of [17, Corollary 2.11].
(2) The condition (iii) is a generalisation of the assumption " F' has sequentially weak* continuous lattice operations " that is in [17, Corollary 2.11]. Indeed, since F is σ-Dedekind complete, the latter assumption is equivalent by [22, Theorem 6.6] to F is discrete with order continuous norm, and hence by [19, Proposition 2.5.23] F has sequentially weakly continuous lattice operations.

In case F is σ-Dedekind complete and F' has sequentially weak* continuous lattice operations, the conclusion of Corollary 3.2 is sharpened as follows.

Proposition 3.4. *Let E and F be two Banach lattices such that F is σ-Dedekind complete and F' has sequentially weak* continuous lattice operations. Then, an order bounded operator $T : E \to F$ is wa-limited if and only if it is Dunford-Pettis.*

Proof. Let $(x_n) \subset E$ be a weakly null sequence. It follows by Remark 3.3(2) that $|T x_n| \xrightarrow{w} 0$ holds in F. To show that $\|T x_n\| \to 0$, it suffices by [8, Corollary 2.6] to show that $f_n(T x_n) \to 0$ for each norm bounded and disjoint sequence $(f_n) \subset F'_+$. For such sequence, since the norm of F is order continuous (Remark 3.3(2)), it follows by [19, Corollary 2.4.3] that $f_n \xrightarrow{w^*} 0$. Now, as T is wa-limited we see that $f_n(T x_n) \to 0$ as desired. □

In particular, we obtain the w*-conterpart of a result noted by W. Wnuk in [21, Proposition 6].

Corollary 3.5. *For a discrete Banach lattice E with order continuous norm the following statements are equivalent:*

(1) E has the WDP property.*
(2) E has the DP property.*
(3) E has the positive Schur property.
(4) E has the Schur property.

4. Order structure of wa-limited operators and further results

Let E and F be two Banach lattices. The vector space of all wa-limited operators from E into F is denoted by $\mathcal{L}_{wal}(E,F)$. The space $\mathcal{L}_{wal}(E,F)$ is norm closed in $\mathcal{L}(E,F)$. Indeed, Let $(T_n) \subset \mathcal{L}_{wal}(E,F)$ such that $T_n \overset{\|\cdot\|}{\to} T$ for some $T \in \mathcal{L}(E,F)$, and let $x_n \overset{\sigma(E,E')}{\to} 0$ in E and $f_n \overset{\sigma(F',F)}{\to} 0$ in F' with $f_n \perp f_m$. If $M = \sup \|f_n\| \times \sup \|x_n\|$, by the inequality

$$\begin{aligned} |f_n(T(x_n))| &= |f_n(T - T_m)(x_n) + f_n(T_m(x_n))| \\ &\leq \|f_n\| \|T - T_m\| \|x_n\| + |f_n(T_m(x_n)| \\ &\leq M \|T - T_m\| + |f_n(T_m(x_n)|, \end{aligned}$$

we see that $\limsup |f_n(T(x_n))| \leq M \|T - T_m\|$ for every m, since $|f_n(T_m(x_n)| \to 0$ as $n \to \infty$. Letting $m \to \infty$, we get $\lim f_n(T(x_n)) = 0$, that is $T \in \mathcal{L}_{wal}(E,F)$, as claimed.

But in general $\mathcal{L}_{wal}(E,F)$ does not form a vector lattice (and hence is not an ideal in $\mathcal{L}_b(E,F)$), even if the range space F is Dedekind complete, as shown in the following example due to G. Ya. Lozanovsky [16].

Example 4.1. Let $T : L_2[0,1] \to c_0$ be the operator defined by

$$T(f) = \left(\int_0^1 f(x) \sin x dx, \int_0^1 f(x) \sin 2x dx, ... \right) \quad \text{for each} \quad f \in L_2[0,1].$$

Since $L_2[0,1]$ is reflexive the operator, T is a weakly compact operator, and hence the operator S, restriction of T to $C[0,1]$, is also weakly compact. By Theorems 5.82 and 5.85 in [2] S is a Dunford pettis operator, and hence it is wa-limited. But, S is not an order bounded operator. Indeed, for each $n \in \mathbb{N}$ let $f_n \in C[0,1]$ be defined by $f_n(x) = \sin nx$ for each x in $[0,1]$, and consider in $C[0,1]$ the order bounded set $A = \{f_n : n \in \mathbb{N}\}$. Since the nth term of the sequence $(S(f_n))$ is given by

$$\left| \int_0^1 \sin nx \sin nx dx \right| = \frac{1}{2} \left| 1 - \frac{\sin 2n}{2n} \right| \to \frac{1}{2},$$

it follows that the set $S(A)$ does not have any upper bound in c_0, and thus the modulus of S does not exist.

Although the modulus of an operator $T \in \mathcal{L}_{wal}(E,F)$ exists, this modulus need not be wa-limited, as shown in the following example based on the example of U. Krengel [14] (see also for details [24, Exercise 125.9]).

Example 4.2. Consider the real vector space $L_n = \mathbb{R}^{2^n}$ with its pointwise ordering and Euclidean norm. For each n, let $T_n : L_n \to L_n$ be the operator whose matrix is $2^{-n} A_n$, where the matrices A_n are given by

$$A_0 = (1) \quad \text{and} \quad A_{n+1} = \begin{pmatrix} A_n & A_n \\ -A_n & A_n \end{pmatrix}.$$

Note that for each n we have

$$\|T_n x\| = 2^{-\frac{n}{2}} \|x\| \text{ for all } x \in L_n \quad \text{and} \quad \||T_n|\| = 1. \qquad (*)$$

Since the space ℓ^2 can be written as a direct sum $\ell^2 = L_0 \oplus L_1 \oplus ...$, under the norm

$$\|(x_0, x_1, ...)\| = \left(\sum_{k=0}^{\infty} \|x_k\|^2 \right)^{\frac{1}{2}}, \quad x_k \in L_k,$$

define the operators S_n, $T : \ell^2 \to \ell^2$ by

$$S_n(x_0, x_1, ...) = (T_0 x_0, ..., T_n x_n, 0, 0, ...)$$

and

$$T(x_0, x_1, ...) = (T_0 x_0, T_1 x_1, ...).$$

From the first equality of $(*)$ it can be easly shown that $\|S_n - T\| \to 0$ as $n \to \infty$. Thus T is compact and hence wa-limited. Also, an easy computation shows that the modulus of T is given by

$$|T|(x_0, x_1, ...) = (|T_0| x_0, |T_1| x_1, ...).$$

However, since $\||T_n|\| = 1$, for each n fix $x_n \in L_n$ with $\|x_n\| = 1$ and $\||T_n|(x_n)\| = 1$. Let $\widetilde{x_n}$ denote the element of ℓ^2 whose nth component is x_n and having zero elsewhere. It follows that $\left\|\widetilde{x_n}\right\| = 1$ in ℓ^2, and for $n > m$ we have

$$\left\||T|(\widetilde{x_n}) - |T|(\widetilde{x_m})\right\| = \|(0, ..., 0, -|T_m|(x_m), 0, ..., 0, |T_n|(x_n), 0, 0, ...)\| = \sqrt{2}.$$

This shows that $|T|$ is not compact. Now, since ℓ^2 is reflexive then it is a Glefand-Phillips space (i.e. limited sets are relatively compact). It follows that $|T|$ is not a limited operator. Now, by Corollary 2.3 $|T|$ is not a wa-limited operator as required.

In [1] the idea of a generalized sublattice was introduced. There it is said that $(E, \leq)$ is a partially ordered vector spaces and F is a subspace of E , then F is a generalized sublattice of E if $(F, \leq)$ is a lattice and the supremum of x and y calculated in F is also their supremum in E for each $x, y \in F$.

$\mathcal{L}^r_{wal}(E, F)$ denotes the linear span of the positive wa-limited operators from E into F. The vector space $\mathcal{L}^r_{wal}(E, F)$ forms a vector lattice as shown in the following theorem whose proof is routine.

Theorem 4.3. *Let E and F be two Banach lattices such that F is Dedekind complete. Then, $\mathcal{L}^r_{wal}(E, F)$ is an ideal in $\mathcal{L}_b(E, F)$.*

Furthermore, the vector space $\mathcal{L}^r_{wal}(E, F)$ is not order closed in $\mathcal{L}_b(E, F)$ in general:

Example 4.4. Consider the operators $T_n : c_0 \to c_0$, $n \in \mathbb{N}$ defined by

$$T_n((\alpha_n)) = (\alpha_1, \alpha_2, ..., \alpha_n, 0, 0, 0...).$$

It is clear that $(T_n) \subset \mathcal{L}^r_{wal}(c_0)$ and $0 \leq T_n \uparrow Id_{c_0}$ is satisfied in $\mathcal{L}_b(c_0)$, where Id_{c_0} is the identity operator on c_0. But $Id_{c_0} \notin \mathcal{L}_{wal}(c_0)$.

Now, we are in position to look at the relationship between the vector spaces $\mathcal{L}_{wal}(E, F)$, $\mathcal{L}_b(E, F)$ and $\mathcal{L}(E, F)$.

Theorem 4.5. *Let E and F be two Banach lattices. Then, the following assertions hold:*

(1) *$\mathcal{L}(E, F) = \mathcal{L}_{wal}(E, F)$ if F is an AL-space.*
(2) *$\mathcal{L}_b(E, F) \subset \mathcal{L}_{wal}(E, F)$ if one of the following assertions is valid:*
   (a) *E is an AL space.*
   (b) *F is σ-Dedekind complete and either E or F is an AM space with unit.*

Proof. (1) Let $x_n \overset{\sigma(E,E')}{\to} 0$ in E and $f_n \overset{\sigma(F',F)}{\to} 0$ in F', $f_n \perp f_m$. By uniform boundedness principle and the fact that F' is an AM-space with unit, the set $\{f_n : n \in \mathbb{N}\}$ is order bounded in F'. On the other hand $\{T(x_n) : n \in \mathbb{N}\}$ is relatively weakly compact in F. It follows by [19, Theorem 2.5.3] that the order bounded disjoint sequence (f_n) converges uniformly to zero on the set $\{T(x_n)\}$. In particular $\lim f_n(T(x_n)) = 0$, which completes the proof.

(2.a) Let $T : E \to F$ be an order bounded operator and let $x_n \overset{\sigma(E,E')}{\to} 0$ in E, $x_n \perp x_m$ and $f_n \overset{\sigma(F',F)}{\to} 0$ in F'. It suffices by Theorem 2.1(3) to show that $\lim f_n(T(x_n)) = 0$. Since $T' : F' \to E'$ is $\sigma(F', F) - \sigma(E', E)$-continuous $T'(f_n) \overset{\sigma(E',E)}{\to} 0$, then by the uniform boundedness principle and the fact that E' is an AM–space with unit, it follows that the sequence $(T'(f_n))$ is order bounded in E'. So, there exists a positive element g in E' such that $|T'(f_n)| \leq g$. Hence, it follows by [2, Theorem 4.34] that

$$|f_n(T(x_n))| = |(T'f_n)(x_n)| \leq g(|x_n|) \to 0.$$

Therefore $\lim f_n(T(x_n)) = 0$ as desired.

(2.b) Let E be an AM-space with unit and let $x_n \overset{\sigma(E,E')}{\to} 0$ in E and $f_n \overset{\sigma(F',F)}{\to} 0$ in F', $f_n \perp f_m$. Since (x_n) is norm bounded in E then it is order bounded, and by the order boundedness of $T : E \to F$ it follows that $(T(x_n))$ is order bounded in F. So there exists a positive element y in F such that $|T(x_n)| \leq y$, and hence from the inequality $|f_n(T(x_n))| \leq |f_n|(y)$ and [13, Lemma 3.1] we see that $\lim f_n(T(x_n)) = 0$ as desired.

The proof for the case F is an AM-space with unit is obtained by similar arguments. □

Now, for two Banach lattices E and F, the regular norm of an operator $T : E \to F$ having a modulus is defined by

$$\|T\|_r = \||T|\|.$$

Also, Recall that a Banach lattice is said to have a Levi norm if every norm bounded upward directed set of positive elements has a supremum.

From the above theorem we get the following results dealing with the lattice structure of the vector space $\mathcal{L}_{wal}(E,F)$.

Corollary 4.6. *Let E and F be two Banach lattices such that E is a discrete AL-space. Then $\mathcal{L}_{wal}(E,F)$ is a vector lattice.*

Proof. It follows from Theorem 4.5 and Theorem 2.4 [20] that $\mathcal{L}(E,F) = \mathcal{L}_r(E,F) = \mathcal{L}_b(E,F) = \mathcal{L}_{wal}(E,F)$ is a vector lattice. □

Corollary 4.7. *Let E and F be two Banach lattices such that F is Dedekind complete. Then, $\mathcal{L}_{wal}(E,F)$ is a Banach lattice under the regular norm whenever one of the following holds:*

(i) *E is a discrete AL-space.*
(ii) *F is an AM-space with unit.*
(iii) *E is an AL-space and F has a Levi norm.*

Proof. (i) Follows from Theorem 4.74 in [2] by similar arguments of the proof of Corollary 4.6.

(ii) and (iii) In this case $\mathcal{L}(E,F) = \mathcal{L}_b(E,F)$ (see for the case (iii) [20, Theorem 2.8]), and by Theorem 4.5 we obtain $\mathcal{L}_{wal}(E,F) = \mathcal{L}_b(E,F) = \mathcal{L}(E,F)$. Now the result follows again from Theorem 4.74 in [2]. □

Let us recall that a Banach lattice E is said to have the Grothendieck property (or E is called a Grothendieck space), whenever

$$f_n \overset{\sigma(E',E)}{\to} 0 \text{ in } E' \text{ implies } f_n \overset{\sigma(E',E'')}{\to} 0 \text{ in } E'.$$

From [23], E is said to have the positive Grothendieck (PG) property, if sequences in the latter definition are restricted to those with positive terms. Clearly, the Grothendieck property implies the PG property. Every σ-Dedekind complete Banach lattice that is an AM-space with unit (resp. has the PG property) is a Grothendieck space (Corollary 2.5.17 and Theorem 5.3.13 [19]).

We introduce another weak version of Grothendieck property as follows:

Definition 4.8. A Banach lattice E is said to have the weak Grothendieck (WG) property, if for every sequence $(f_n) \subset E'$ with disjoint terms, $f_n \overset{\sigma(E',E'')}{\to} 0$ in E' whenever $f_n \overset{\sigma(E',E)}{\to} 0$ in E'.

Note that a similar property was considered by W. Wnuk [23, p 6]. Clearly, the Grothendieck property implies the WG property. Every AL-space E satisfy the WG property, since in this case disjoint weak* null sequences $(f_n) \subset E'$ are order bounded, and hence weakly null [2, p 192]. Therefore, ℓ^1 is an example of Banach lattice having the WG property without the (positive) Grothendieck property (see [23, p 6]). On the other hand, consider the Banach lattice c (of convergent sequences). If $0 \leq (\lambda_n) \overset{\sigma(c',c)}{\to} 0$ in $c' = \ell^1$ then from the equality

$$(\lambda_n)(e) = \sum_{n=1}^{\infty} \lambda_n = \|(\lambda_n)\|_1$$

where $e = (1, 1, 1, ...)$, we see that c has the PG property. However, c does not have the WG property. In fact, let $f_n \in c' = \ell^1$ be defined as follows $f_n = (0, \cdots, 0, 1_{(2n)}, -1_{(2n+1)}, 0, \cdots)$. Then, clearly (f_n) is a disjoint weak*null sequence in c'. If (f_n) were weakly null in c' then it would be norm null in the Schur space $c' = \ell^1$. This contradicts the fact that $\|f_n\| = 2$ (cf. [6, Example 2.1(2)]).

Note that if E is a Banach lattice such that E is σ-Dedekind complete (see [23, Proposition 1.4]) or the lattice operations of E' are weak* sequentially continuous then E satisfies the following property:

$$f_n \overset{\sigma(E',E)}{\to} 0 \text{ in } E',\ f_m \perp f_k \text{ implies } |f_n| \overset{\sigma(E',E)}{\to} 0. \tag{d}$$

However, the Banach lattice ℓ^∞/c_0 has the property (d) but it is not σ-Dedekind complete [23, Remark 1.5]. Also, the Dedekind complete Banach lattice ℓ^∞ has the property (d) but the lattice operations of $(\ell^\infty)'$ are not weak* sequentially continuous.

We have the following easy proposition dealing with the relationship between WG property and PG property.

Proposition 4.9. *For a Banach lattice E, the following statements hold:*

(1) *If E has the property (d) and the PG property then E has the WG property.*
(2) *If E has the WG property and E' has order continuous norm then E has the PG property.*

Proof. (1) Let $(f_n) \subset E'$ be a disjoint weak* null sequence. Since E has the property (d) then $|f_n| \overset{w^*}{\to} 0$. Therefore, $|f_n| \overset{w}{\to} 0$ by the PG property of E. Now, as $(f_n) \subset sol\{|f_n| : n \in \mathbb{N}\}$ it follows from [2, Theorem 4.34] that $f_n \overset{w}{\to} 0$ as well.

(2) Is a consequence of [19, Theorem 5.3.13 (iii)$\Rightarrow$ (ii)]. □

Banach lattices whose duals are separable and contain no isomorphic copy of ℓ^1 are examples of such Banach lattices with the WG property. The details follow.

Proposition 4.10. *If the dual of a Banach lattice E is separable and contains no isomorphic copy of ℓ^1, then E has both the weak and the positive Grothendieck properties.*

Proof. Assume by way of contradiction that there is some disjoint weak* null sequence $(f_n) \subset E'$ which does not converge weakly to zero. Thus, by passing to a subsequence, we may suppose that $|\varphi(f_n)| > \varepsilon$ for each n and for some $\varphi \in E''$ and $\varepsilon > 0$. Note that the sequence $\left(\overset{\wedge}{f_n}\right)$ is norm bounded and disjoint in E''' (where $\hat{x}$ is the image of x under the canonical mapping from the Banach lattice into its bidual). Now, we know by Odell-Rosenthal Theorem [7, Theorem 10 p 236] that the closed unit ball $B_{E'''}$ is weak* sequentially compact, and hence there exists a subsequence $\left(\overset{\wedge}{f_{n_k}}\right)$ with $\overset{\wedge}{f_{n_k}} \overset{\sigma(E''',E'')}{\to} \phi$ in E'''. Since E'' is Dedekind complete, Proposition 1.4 of [23] ensures that $\phi = 0$. But, the latter is impossible by the inequality

$$\left|\hat{f}_{n_k}(\varphi)\right| = |\varphi(f_{n_k})| > \varepsilon \ ,$$

and we are done.

For the rest of the proof, note that since E' is Dedekind complete then its norm is order continuous and hence the result follows from Proposition 4.9(2). □

We conclude this note by examining the following question: is an operator $T : E \to F$ a wa-limited operator when its second adjoint $T'' : E'' \to F''$ is one? The answer is negative in general, as the identity operator $Id_{c_0} \notin \mathcal{L}_{wal}(c_0)$ even if $(Id_{c_0})'' = Id_{\ell^\infty} \in \mathcal{L}_{wal}(\ell^\infty)$. The following theorem gives us a sufficient condition under which the answer of the preceding question is positive.

Theorem 4.11. *Let E and F be two Banach lattices such that F has the weak Grothendieck property. Then, each operator $T : E \to F$ is a weak almost limited operator whenever its second adjoint $T'' : E'' \to F''$ is one.*

Proof. Let $x_n \overset{\sigma(E,E')}{\to} 0$ in E and $f_n \overset{\sigma(F',F)}{\to} 0$ in F' with $f_n \perp f_m$. Since the canonical embedding $E \hookrightarrow E''$ is weakly continuous then $\hat{x}_n \overset{\sigma(E'',E''')}{\to} 0$ in E''. Note that from the weak Grothendieck property of F the sequence $(\hat{f}_n)$ is disjoint weakly null in F'''. Now, using $T'' : E'' \to F''$ is a wa-limited operator it follows that

$$|f_n(T(x_n))| = |(T'(f_n)(x_n)| = \left|\hat{x}_n(T'(f_n))\right| = \left|(T''(\hat{x}_n)(f_n)\right| = \left|\hat{f}_n(T''(\hat{x}_n))\right| \to 0.$$

Thus T must be a wa-limited operator as desired. □

As a consequence, we have the following necessary condition for a Banach lattice to satisfy the WG property in term of its WDP* property.

Corollary 4.12. *If a Banach lattice E has the weak Grothendieck property, then either E has the WDP* property or E'' does not have the WDP* property.*

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