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Research Article

Multiplication operators between mixed norm Lebesgue spaces

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Abstract

The boundedness, compactness and closed range of the multiplication operator defined on mixed norm Lebesgue spaces are characterized in this paper.

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1. Introduction

Mixed norm spaces are spaces of multivariable functions in which the norm takes advantage of the product structure in the domain. They were first named and formally studied in [4], as a tool to study generalizations of Sobolev's theorem regarding the continuity of certain potencial operators and the Hausdorff-Young theorem. Spaces of this type arise naturally in harmonic and functional analysis. See [11, 14] for some history and related work.

The multiplication operator, defined roughly speaking as the pointwise multiplication by a real-valued measurable function, is a well-studied transformation. This operator received considerable attention over the past several decades. Multiplication operators generalize the notion of operator given by a diagonal matrix. More precisely, one of the results of operator theory is a spectral theorem, which states that every self-adjoint operator on a Hilbert space is unitarily equivalent to a multiplication operator on an L_2 space (see e.g. [12]). There exist several papers devoted to the study of the multiplication operator, on L_p spaces [13, 18], on Lorentz spaces [2], on Orlicz-Lorentz spaces [6], on Weak L_p spaces [9], on Cesàro spaces [15], on variable L_p spaces [7], on Köthe sequence spaces [17], on Lorentz sequence spaces [8] and on bounded variation spaces [3, 10]. For some of the history of the multiplication operator and open problems, see [16]. It is natural to extend the study to mixed norm Lebesgue spaces.

In order to carry on this study, we introduce at the end of this section some previous definitions. In Section 2 we characterize the boundedness of the multiplication operator on mixed norm Lebesgue spaces. In Section 3, we give necessary and sufficient conditions to guarantee the closed range of the multiplication operator. Finally, in Section 4 we introduce a subspace of the mixed norm Lebesgue space and then we establish some results about the compactness of the multiplication operator.

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Denote by $L_0(\mathbb{R}^2)$ the class of all measurable and almost everywhere finite functions f on $\mathbb{R}^2$. Fix indices $p, q \in (0, \infty)$. A function $f \in L_0(\mathbb{R}^2)$ belongs to the mixed norm Lebesgue space $L^q(u)[L^p(v)]$ if

$$\|f\|_{L^q(u)[L^p(v)]} = \left[\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,y)|^p v(x)\, dx\right)^{q/p} u(y)\, dy\right]^{1/q} < \infty.$$

Where u and v are weight functions, i.e., u and v are non-negative locally integrable functions.

$\|\cdot\|_{L^q(u)[L^p(v)]}$ is a norm only when $p \geq 1$ and $q \geq 1$, moreover $L^q(u)[L^p(v)]$ is a Banach space. For details, we refer the reader to [5].

We denote by $m_2(E)$ the Lebesgue measure of a measurable set $E \subset \mathbb{R}^2$.

If $F(X)$ is a function space on a non-empty set X, and $\varphi : X \to \mathbb{R}$ is a function such that $\varphi \cdot f \in F(X)$ whenever $f \in F(X)$, then the transformation $f \mapsto \varphi \cdot f$ is denoted by M_φ. In case $F(X)$ is a topological space, M_φ is called the *multiplication operator induced by* φ.

2. Boundedness of the multiplication operator on $L^q(u)[L^p(v)]$

In the following theorem we characterize the boundedness of M_φ, defined on $L^q(u)[L^p(v)]$

Theorem 2.1. The operator $M_\varphi : L^q(u)[L^p(v)] \to L^q(u)[L^p(v)]$ given by

$$(M_\varphi f)(x,y) = M_\varphi(f(x,y)) = \varphi(x,y) \cdot f(x,y)$$

is bounded if and only if φ is essentially bounded. Moreover,

$$\|M_\varphi\| = \|\varphi\|_\infty .$$

Proof. We prove first the sufficiency. Let φ be an essentially bounded function. Since $|\varphi(x,y)| \leq \|\varphi\|_\infty$ a.e., we have

$$|\varphi(x,y) f(x,y)| \leq \|\varphi\|_\infty |f(x,y)| \text{ a.e.}$$

Raising to p, multiplying by $v(x)$ and integrating, we get

$$\int_{\mathbb{R}} |\varphi(x,y) f(x,y)|^p v(x)\, dx \leq \int_{\mathbb{R}} \left[\|\varphi\|_\infty |f(x,y)|\right]^p v(x)\, dx.$$

Raising to q/p and multiplying by the weight u,

$$\left(\int_{\mathbb{R}} |\varphi(x,y) f(x,y)|^p v(x)\, dx\right)^{q/p} u(y) \leq \left(\int_{\mathbb{R}} \left[\|\varphi\|_\infty |f(x,y)|\right]^p v(x)\, dx\right)^{q/p} u(y).$$

Finally, we integrate and raise to $1/q$, to obtain

$$\left[\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\varphi(x,y) f(x,y)|^p v(x)\, dx\right)^{q/p} u(y)\, dy\right]^{1/q} \leq$$

$$\left[\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left[\|\varphi\|_\infty |f(x,y)|\right]^p v(x)\, dx\right)^{q/p} u(y)\, dy\right]^{1/q}.$$

So, $\|M_\varphi f\|_{L^q(u)[L^p(v)]} \leq \|\varphi\|_\infty \|f\|_{L^q(u)[L^p(v)]}$, i.e.

$$\|M_\varphi\| \leq \|\varphi\|_\infty . \tag{2.1}$$

Then M_φ is bounded.

Conversely, suppose that M_φ is a bounded operator. Suppose also that φ is not essentially bounded. Then, the set $E_n = \left\{(x,y) \in \mathbb{R}^2 : |\varphi(x,y)| > n\right\}$ has a positive measure. Therefore, for any $n \in \mathbb{N}$ and any $(x,y) \in \mathbb{R}^2$, we have

$$|(\varphi \chi_{E_n})(x,y)| \geq n \chi_{E_n}(x,y).$$

Raising to p, multiplying by v and then integrating,

$$\int_{\mathbb{R}} |(\varphi\chi_{E_n})(x,y)|^p v(x)\,dx \geq \int_{\mathbb{R}} [n\chi_{E_n}(x,y)]^p v(x)\,dx.$$

Raising to q/p and then multiplying by u,

$$\left(\int_{\mathbb{R}} |(\varphi\chi_{E_n})(x,y)|^p v(x)\,dx\right)^{q/p} u(y) \geq \left(\int_{\mathbb{R}} [n\chi_{E_n}(x,y)]^p v(x)\,dx\right)^{q/p} u(y).$$

Now, integrating and then raising to $1/q$, we have

$$\left[\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |(\varphi\chi_{E_n})(x,y)|^p v(x)\,dx\right)^{q/p} u(y)\,dy\right]^{1/q} \geq$$
$$\left[\int_{\mathbb{R}} \left(\int_{\mathbb{R}} [n\chi_{E_n}(x,y)]^p v(x)\,dx\right)^{q/p} u(y)\,dy\right]^{1/q}.$$

Hence

$$\|M_\varphi \chi_{E_n}\|_{L^q(u)[L^p(v)]} \geq n\,\|\chi_{E_n}\|_{L^q(u)[L^p(v)]}\,.$$

This contradicts the boundedness of M_φ. So φ must be essentially bounded.

In order to prove that the norm of M_φ is actually $\|\varphi\|_\infty$, for $\varepsilon > 0$, let $E = \{(x,y) \in \mathbb{R}^2 : |\varphi(x,y)| \geq \|\varphi\|_\infty - \varepsilon\}$. Note that $m_2(E) > 0$. Then

$$|\varphi(x,y)\chi_E(x,y)| \geq (\|\varphi\|_\infty - \varepsilon)\,\chi_E(x,y) \quad \forall\ (x,y) \in \mathbb{R}^2.$$

Following the same steps as above, one concludes that

$$\|M_\varphi \chi_E\|_{L^q(u)[L^p(v)]} \geq (\|\varphi\|_\infty - \varepsilon)\,\|\chi_E\|_{L^q(u)[L^p(v)]}\,.$$

Hence

$$\|M_\varphi\| \geq \|\varphi\|_\infty - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\|M_\varphi\| \geq \|\varphi\|_\infty\,. \tag{2.2}$$

From (2.1) and (2.2) we conclude that

$$\|M_\varphi\| = \|\varphi\|_\infty\,.$$ □

3. Closed range of the multiplication operator

Now, we study the closed range of the multiplication operator.

Although we will need M_φ to be an injective operator, this is not always the case. Take $\varphi(x,y) = \chi_{[0,1]\times[0,1]}(x,y)$ and $f(x,y) = \chi_{[2,3]\times[2,3]}(x,y)$. Then,

$$(M_\varphi f)(x,y) = \varphi(x,y)\cdot f(x,y) = \chi_{[0,1]\times[0,1]}(x,y)\cdot\chi_{[2,3]\times[2,3]}(x,y) = 0.$$

Hence, since $\ker(M_\varphi) \neq \{0\}$, M_φ is not one to one.

In order to guarantee the injectivity of M_φ, we need to take into account the support of φ, which is defined as

$$\operatorname{supp}\varphi = \left\{(x,y) \in \mathbb{R}^2 : \varphi(x,y) \neq 0\right\}.$$

Take $S = \operatorname{supp}\varphi$ and define the restricted space $L^q(u)\,[L^p(v)]\,(S)$ as

$$L^q(u)\,[L^p(v)]\,(S) = \{f\chi_S : f \in L^q(u)\,[L^p(v)]\}\,.$$

The following result gives us a relation between the injectivity of M_φ and the restricted space $L^q(u)\,[L^p(v)]\,(S)$.

Proposition 3.1. $M_\varphi : L^q(u)\,[L^p(v)]\,(S) \to L^q(u)\,[L^p(v)]\,(S)$ is an injective operator.

Proof. If $M_\varphi \tilde{f} = 0$ where $\tilde{f} = f\chi_S$, we have $\varphi(x,y)\tilde{f}(x,y) = 0$ for all $(x,y) \in \mathbb{R}^2$, i.e. $\varphi(x,y)f(x,y)\chi_S(x,y) = 0$ for all $(x,y) \in \mathbb{R}^2$ and since $S = \operatorname{supp}\varphi$, then

$$\begin{aligned} \varphi(x,y)f(x,y) &= 0, \quad \forall\ (x,y) \in S \\ f(x,y) &= 0, \quad \forall\ (x,y) \in S \\ f(x,y)\chi_S(x,y) &= 0, \quad \forall\ (x,y) \in \mathbb{R}^2. \end{aligned}$$

Then $\tilde{f}(x,y) = 0$. Hence $\ker(M_\varphi) = \{0\}$ and then we have injectivity of M_φ on the set $L^q(u)\,[L^p(v)]\,(S)$. □

We recall the definition of a *bounded below operator*.

Definition 3.2. An operator $T : X \to Y$ between normed spaces is said to be bounded below if there exists a constant $C > 0$ such that

$$\|Tx\| \geq C\,\|x\|$$

for each $x \in X$.

The following theorem (see e.g. [1]) allows us to obtain some results about the range of M_φ.

Theorem 3.3. Let T be a bounded linear operator, $T : X \to Y$, where X and Y are Banach spaces. Then T is bounded below if and only if T is one-to-one and has closed range.

As an immediate consequence of Proposition 3.1 and Theorem 3.3, we have the following corollary.

Corollary 3.4. The multiplication operator

$$M_\varphi : L^q(u)\,[L^p(v)]\,(S) \to L^q(u)\,[L^p(v)]\,(S)$$

is bounded below if and only if M_φ has closed range.

Now we show the main theorem of this section.

Theorem 3.5. The multiplication operator $M_\varphi : L^q(u)\,[L^p(v)] \to L^q(u)\,[L^p(v)]$ has closed range if and only if there exists $\delta > 0$ such that $|\varphi(x,y)| \geq \delta$ for m_2-almost all $(x,y) \in \operatorname{supp}\varphi$.

Proof. We prove first the converse implication. Write $S = \operatorname{supp}\varphi$. Suppose that there exists $\delta > 0$ for which $|\varphi(x,y)| \geq \delta$ a.e. on $\operatorname{supp}\varphi$. Then

$$|\varphi(x,y)f(x,y)\chi_S(x,y)| \geq \delta\,|f(x,y)\chi_S(x,y)| \quad \text{a.e.}$$

From this we have

$$\left(\int_{\mathbb{R}} |\varphi(x,y)f(x,y)\chi_S(x,y)|^p\,v(x)\,dx\right)^{q/p} \geq \left(\int_{\mathbb{R}} [\delta\,|f(x,y)\chi_S(x,y)|]^p\,v(x)\,dx\right)^{q/p}.$$

And then

$$\left[\int_{\mathbb{R}}\left(\int_{\mathbb{R}} |\varphi(x,y)f(x,y)\chi_S(x,y)|^p\,v(x)\,dx\right)^{q/p} u(y)\,dy\right]^{1/q} \geq \left[\int_{\mathbb{R}}\left(\int_{\mathbb{R}} [\delta\,|f(x,y)\chi_S(x,y)|]^p\,v(x)\,dx\right)^{q/p} u(y)\,dy\right]^{1/q}.$$

Hence

$$\|M_\varphi f\chi_S\|_{L^q(u)[L^p(v)]} \geq \delta\,\|f\chi_S\|_{L^q(u)[L^p(v)]}\,.$$

This means that M_φ is bounded below on $L^q(u)\,[L^p(v)]\,(S)$. Following similar lines to [18, Lemma 2.2], one concludes that $M_\varphi : L^q(u)\,[L^p(v)] \to L^q(u)\,[L^p(v)]$ has closed range.

Now we prove the reverse implication. Suppose that M_φ has closed range on $L^q(u)\,[L^p(v)]$. Again, by [18, Lemma 2.2], there exists $\varepsilon > 0$ such that

$$\left\| M_\varphi \tilde{f} \right\|_{L^q(u)[L^p(v)]} \geq \varepsilon \left\| \tilde{f} \right\|_{L^q(u)[L^p(v)]} \tag{3.1}$$

for all $\tilde{f} \in L^q(u)\,[L^p(v)]\,(S)$. Let $E = \{(x,y) \in S : |\varphi(x,y)| < \varepsilon/2\}$. If $m_2(E) > 0$, we can find a measurable set $F \subset E$ such that $0 < m_2(E) < m_2(F)$ and so $\chi_F \in L^q(u)\,[L^p(v)]\,(S)$. Then we have

$$|\varphi(x,y)\chi_F(x,y)| \leq \frac{\varepsilon}{2}\,|\chi_F(x,y)|\,.$$

Following the same steps as above, one concludes that

$$\left[\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\varphi(x,y)\chi_F(x,y)|^p\, v(x)\, dx\right)^{q/p} u(y)\, dy\right]^{1/q} \leq \left[\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left[\frac{\varepsilon}{2}\,|\chi_F(x,y)|\right]^p v(x)\, dx\right)^{q/p} u(y)\, dy\right]^{1/q}.$$

Thus,

$$\|M_\varphi \chi_F\|_{L^q(u)[L^p(v)]} \leq \frac{\varepsilon}{2}\,\|\chi_F\|_{L^q(u)[L^p(v)]}\,. \tag{3.2}$$

Inequalities (3.1) and (3.2) together lead to a contradiction. Therefore $m_2(E) = 0$. In other words, $|\varphi(x,y)| \geq \varepsilon/2$ for m_2-almost all $(x,y) \in S$. □

4. Compactness of the multiplication operator

Before we continue, we recall the definition of *invariant subspace*.

Definition 4.1. Let $T : X \to X$ be an operator. A subspace V of X is said to be invariant under T (or T-invariant) if $T(V) \subseteq V$.

The next lemma will be useful later.

Lemma 4.2. Let $T : X \to X$ be an operator. If T is compact and V is a closed T-invariant subspace of X, then $T|_V$ is compact.

A proof of the above lemma may be found in [6].

For $\varepsilon > 0$, we define

$$A_\varepsilon(\varphi) = \left\{(x,y) \in \mathbb{R}^2 : |\varphi(x,y)| \geq \varepsilon\right\},$$

and we also define

$$L^q(u)\,[L^p(v)]\,(A_\varepsilon(\varphi)) = \left\{ f\chi_{A_\varepsilon(\varphi)} : f \in L^q(u)\,[L^p(v)]\right\}.$$

Lemma 4.3. Let M_φ be a compact operator. Then $L^q(u)\,[L^p(v)]\,(A_\varepsilon(\varphi))$ is a closed invariant subspace of $L^q(u)\,[L^p(v)]$ under M_φ. Moreover,

$$M_\varphi|_{L^q(u)[L^p(v)](A_\varepsilon(\varphi))}$$

is a compact operator.

Proof. Let $F, G \in L^q(u)\,[L^p(v)]\,(A_\varepsilon(\varphi))$, then $F = f\chi_{A_\varepsilon(\varphi)}$ and $G = g\chi_{A_\varepsilon(\varphi)}$ with $f, g \in L^q(u)\,[L^p(v)]$. So,

$$\begin{aligned}\lambda F + \mu G &= \lambda f \chi_{A_\varepsilon(\varphi)} + \mu g \chi_{A_\varepsilon(\varphi)}\\ &= (\lambda f + \mu g)\chi_{A_\varepsilon(\varphi)}.\end{aligned}$$

Since $\lambda f + \mu g \in L^q(u)\,[L^p(v)]$, the above equation shows that

$$\lambda F + \mu G \in L^q(u)\,[L^p(v)]\,(A_\varepsilon(\varphi))\,.$$

So this is a subspace of $L^q(u)\,[L^p(v)]\,(A_\varepsilon(\varphi))$.

Now, let $h \in M_\varphi\left(L^q(u)\left[L^p(v)\right](A_\varepsilon(\varphi))\right)$. Then there exist F such that F belongs to $L^q(u)\left[L^p(v)\right](A_\varepsilon(\varphi))$ and $M_\varphi F = h$. Since $F = f\chi_{A_\varepsilon(\varphi)}$ for some $f \in L^q(u)\left[L^p(v)\right]$, we have

$$h = M_\varphi F = \varphi F = \varphi\left(f\chi_{A_\varepsilon(\varphi)}\right) = (\varphi f)\chi_{A_\varepsilon(\varphi)}.$$

Since $\varphi f \in L^q(u)\left[L^p(v)\right]$, the above equation shows that $h \in L^q(u)\left[L^p(v)\right](A_\varepsilon(\varphi))$. This proves that $L^q(u)\left[L^p(v)\right](A_\varepsilon(\varphi))$ is M_φ-invariant.

To prove the closedness of $L^q(u)\left[L^p(v)\right](A_\varepsilon(\varphi))$, let $\{F_k\}_{k\in\mathbb{N}}$ be a sequence in $L^q(u)\left[L^p(v)\right](A_\varepsilon(\varphi))$ such that

$$F_k \to F \quad \text{in } L^q(u)\left[L^p(v)\right](A_\varepsilon(\varphi)).$$

We need to show that $F \in L^q(u)\left[L^p(v)\right](A_\varepsilon(\varphi))$. In order to do this, we write

$$F = F\chi_{A_\varepsilon(\varphi)} + F\chi_{A_\varepsilon^{\complement}(\varphi)}.$$

It is enough to prove that $F\chi_{A_\varepsilon^{\complement}(\varphi)} = 0$. For any $\varepsilon > 0$, there exists n_0 such that $\|F - F_{n_0}\|_{L^q(u)[L^p(v)]} < \varepsilon$, but

$$\begin{aligned}
\left\|F\chi_{A_\varepsilon^{\complement}(\varphi)}\right\|_{L^q(u)[L^p(v)]} &= \left\|\left(F - F_{n_0} + F_{n_0}\right)\chi_{A_\varepsilon^{\complement}(\varphi)}\right\|_{L^q(u)[L^p(v)]} \\
&\le \left\|\left(F - F_{n_0}\right)\chi_{A_\varepsilon^{\complement}(\varphi)}\right\|_{L^q(u)[L^p(v)]} + \left\|F_{n_0}\chi_{A_\varepsilon^{\complement}(\varphi)}\right\|_{L^q(u)[L^p(v)]} \\
&= \left\|\left(F - F_{n_0}\right)\chi_{A_\varepsilon^{\complement}(\varphi)}\right\|_{L^q(u)[L^p(v)]} + \left\|f\chi_{A_\varepsilon(\varphi)}\chi_{A_\varepsilon^{\complement}(\varphi)}\right\|_{L^q(u)[L^p(v)]} \\
&= \left\|\left(F - F_{n_0}\right)\chi_{A_\varepsilon^{\complement}(\varphi)}\right\|_{L^q(u)[L^p(v)]} < \varepsilon.
\end{aligned}$$

Hence $\left\|F\chi_{A_\varepsilon^{\complement}(\varphi)}\right\|_{L^q(u)[L^p(v)]} < \varepsilon$. Since ε was arbitrary, we have

$$\left\|F\chi_{A_\varepsilon^{\complement}(\varphi)}\right\|_{L^q(u)[L^p(v)]} = 0.$$

Therefore $F\chi_{A_\varepsilon^{\complement}(\varphi)} = 0$ and $F = F\chi_{A_\varepsilon(\varphi)} \in L^q(u)\left[L^p(v)\right](A_\varepsilon(\varphi))$.

Now, by using Lemma 4.2, we conclude that $M_\varphi|_{L^q(u)[L^p(v)](A_\varepsilon(\varphi))}$ is a compact operator. □

Finally, we have the following theorem.

Theorem 4.4. Let $M_\varphi : L^q(u)\left[L^p(v)\right] \to L^q(u)\left[L^p(v)\right]$ be a bounded linear operator. Then M_φ is compact if and only if $L^q(u)\left[L^p(v)\right](A_\varepsilon(\varphi))$ is finite dimensional for each $\varepsilon > 0$.

Proof. Suppose that M_φ is a compact operator. Note that, for all $(x, y) \in A_\varepsilon(\varphi)$,

$$|\varphi(x, y)| \ge \varepsilon.$$

Then

$$|\varphi(x, y)f(x, y)\chi_{A_\varepsilon}(x, y)| \ge \varepsilon\,|f(x, y)\chi_{A_\varepsilon}(x, y)| \quad \forall\ (x, y) \in \mathbb{R}^2.$$

From this we have

$$\left(\int_{\mathbb{R}} |\varphi(x, y)f(x, y)\chi_{A_\varepsilon}(x, y)|^p\, v(x)\, dx\right)^{q/p} \ge \left(\int_{\mathbb{R}} \left[\varepsilon\,|f(x, y)\chi_{A_\varepsilon}(x, y)|\right]^p v(x)\, dx\right)^{q/p}.$$

Consequently

$$\left[\int_{\mathbb{R}}\left(\int_{\mathbb{R}} |\varphi(x, y)f(x, y)\chi_{A_\varepsilon}(x, y)|^p\, v(x)\, dx\right)^{q/p} u(y)\, dy\right]^{1/q} \ge$$
$$\left[\int_{\mathbb{R}}\left(\int_{\mathbb{R}} \left[\varepsilon\,|f(x, y)\chi_{A_\varepsilon}(x, y)|\right]^p v(x)\, dx\right)^{q/p} u(y)\, dy\right]^{1/q}.$$

From the last inequality we infer that

$$\|M_\varphi f\chi_{A_\varepsilon}\|_{L^q(u)[L^p(v)]} \geq \varepsilon \|f\chi_{A_\varepsilon}\|_{L^q(u)[L^p(v)]}. \tag{4.1}$$

Hence $M_\varphi|_{L^q(u)[L^p(v)](A_\varepsilon(\varphi))}$ has closed range.

Now, if M_φ is a compact, then from Lemma 4.3, $L^q(u)\,[L^p(v)]$ is a closed invariant subspace of M_φ and by Lemma 4.2,

$$M_\varphi|_{L^q(u)[L^p(v)](A_\varepsilon(\varphi))}$$

is a compact operator. Also, $M_\varphi : L^q(u)\,[L^p(v)]\,(A_\varepsilon(\varphi)) \to L^q(u)\,[L^p(v)]\,(A_\varepsilon(\varphi))$ is invertible (in fact, its inverse is $M_\varphi^{-1} = M_{\varphi^{-1}}$). Therefore, $L^q(u)\,[L^p(v)]\,(A_\varepsilon(\varphi))$ is finite dimensional for each $\varepsilon > 0$.

Conversely, suppose that $L^q(u)\,[L^p(v)]\,(A_\varepsilon(\varphi))$ is finite dimensional for any $\varepsilon > 0$. Particularly, $L^q(u)\,[L^p(v)]\left(A_{1/n}(\varphi)\right)$ is finite dimensional for all $n \in \mathbb{N}$.

For each n, we define $\varphi_n : \mathbb{R}^2 \to \mathbb{C}$ as follows

$$\varphi_n(x,y) = \begin{cases} \varphi(x,y), & \text{if } |\varphi(x,y)| \geq \frac{1}{n} \\ 0, & \text{if } |\varphi(x,y)| < \frac{1}{n}. \end{cases}$$

Then we have $|\varphi_n(x,y) - \varphi(x,y)| \leq 1/n$. Following the same steps as above, one concludes that

$$\|M_{\varphi_n} f - M_\varphi f\|_{L^q(u)[L^p(v)]} \leq \frac{1}{n}\|f\|_{L^q(u)[L^p(v)]}.$$

Then M_{φ_n} converges to M_φ uniformly.

Since each one of the spaces $L^q(u)\,[L^p(v)]\left(A_{1/n}(\varphi)\right)$ is finite dimensional, we have that M_{φ_n} is a finite rank operator, which in turn implies that M_{φ_n} is compact. Finally, the uniform convergence implies the compactness of M_φ. □

Remark 4.5. The results obtained in this paper can easily be extended to another types of mixed norm spaces. For example, the mixed norm Lorentz spaces $\Lambda^q(u)\,[\Lambda^p(v)]$, which are the set of all functions $f \in L_0\left(\mathbb{R}^2\right)$ such that

$$\|f\|_{\Lambda^q(u)[\Lambda^p(v)]} := \left(\int_0^\infty \left[\left(\int_0^\infty [f^{*y}(\cdot,t)]^p\, v(t)\, dt\right)^{*x}(s)\right]^{q/p} u(s)\, ds\right)^{1/q}$$

is finite, where $0 < p, q < \infty$, v, w are weights in $\mathbb{R}_+$, and h^* denotes the usual decreasing rearrangement of h.

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