Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 48 (3) (2019), 779 – 784
DOI : 10.15672/HJMS.2018.548

Research Article

L-paracompactness and L_2-paracompactness

Lutfi Kalantan

King Abdulaziz University, Department of Mathematics, P.O.Box 80203, Jeddah 21589, Saudi Arabia

Abstract

A topological space X is called *L-paracompact* if there exist a paracompact space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f \restriction_A : A \longrightarrow f(A)$ is a homeomorphism for each Lindelöf subspace $A \subseteq X$. A topological space X is called *L_2-paracompact* if there exist a Hausdorff paracompact space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f \restriction_A : A \longrightarrow f(A)$ is a homeomorphism for each Lindelöf subspace $A \subseteq X$. We investigate these two properties.

Mathematics Subject Classification (2010). 54C10, 54D20

Keywords. Lindelöf, paracompact, countably normal, C-paracompact, C_2-paracompact, L-paracompact, L_2-paracompact, L-normal, CP, C_2P

The purpose of this paper is to investigate two new properties, L-paracompactness and L_2-paracompactness. Some of their aspects are similar to L-normality [4], and some are distinct. Throughout this paper, we denote an ordered pair by $\langle x, y \rangle$, the set of positive integers by $\mathbb{N}$, and the set of real numbers by $\mathbb{R}$. A T_4 space is a T_1 normal space and a Tychonoff space ($T_{3\frac{1}{2}}$) is a T_1 completely regular space. We do not assume T_2 in the definition of compactness, countable compactness, and paracompactness. We do not assume regularity in the definition of Lindelöfness. For a subset A of a space X, int$A$ and $\overline{A}$ denote the interior and the closure of A, respectively. An ordinal γ is the set of all ordinal α such that $\alpha < \gamma$. The first infinite ordinal is ω_0, the first uncountable ordinal is ω_1, and the successor cardinal of ω_1 is ω_2.

Definition 1. (A. V. Arhangel'skiĭ)
A topological space X is called *C-paracompact* if there exist a paracompact space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f \restriction_A : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$. A topological space X is called *C_2-paracompact* if there exist a Hausdorff paracompact space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f \restriction_A : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$.

C-paracompactness and C_2-paracompactness were studied in [8]. We use the idea of Arhangel'skiĭ's definition above and give the following definition.

Email address: lnkalantan@hotmail.com; lkalantan@kau.edu.sa
Received: 21.08.2017; Accepted: 12.01.2018

Definition 2. A topological space X is called *L-paracompact* if there exist a paracompact space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f \restriction_A : A \longrightarrow f(A)$ is a homeomorphism for each Lindelöf subspace $A \subseteq X$. A topological space X is called *L_2-paracompact* if there exist a Hausdorff paracompact space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f \restriction_A : A \longrightarrow f(A)$ is a homeomorphism for each Lindelöf subspace $A \subseteq X$.

Observe that a function $f : X \longrightarrow Y$ witnessing the C - paracompactness (C_2 - paracompactness) of X need not to be continuous. But it will be if it has the property that for each convergent sequence $x_n \longrightarrow x$ in X we have $f(x_n) \longrightarrow f(x)$ [8]. This happens if X is a Hausdorff sequential space or a k-space. Similarly, a function $f : X \longrightarrow Y$ witnessing the L-paracompactness (L_2-paracompactness) of X need not to be continuous, see Example 11 below. But it will be if X is of countable tightness. Recall that a space X is of *countable tightness* if for each subset A of X and each $x \in X$ with $x \in \overline{A}$ there exists a countable subset $B \subseteq A$ such that $x \in \overline{B}$ [1]. For a set A, we let $[A]^{\leq \omega_0} = \{ B \subseteq A : B$ is countable $\}$.

Theorem 3. *If X is L-paracompact (L_2-paracompact) and of countable tightness and $f : X \longrightarrow Y$ is a witness function of the L-paracompactness (L_2-paracompactness) of X, then f is continuous.*

Proof. Let $A \subseteq X$ be arbitrary. We have

$$f(\overline{A}) = f(\bigcup_{B \in [A]^{\leq \omega_0}} \overline{B}) = \bigcup_{B \in [A]^{\leq \omega_0}} f(\overline{B}) \subseteq \bigcup_{B \in [A]^{\leq \omega_0}} \overline{f(B)} \subseteq \overline{f(A)}.$$

Therefore, f is continuous □

Since any first countable space is Fréchet [1, 1.6.14], any Fréchet space is sequential [1, 1.6.14], and any sequential space is of countable tightness [1, 1.7.13(c)], we conclude that a witness function of the L-paracompactness (L_2-paracompactness) first countable (Fréchet, sequential) space X is continuous. The following corollary is also clear.

Corollary 4. *Any L_2-paracompact space which is of countable tightness must be at least Hausdorff.*

Since any compact space is Lindelöf, then any L-paracompact space is C-paracompact and any L_2-paracompact space is C_2-paracompact. The converse is not true in general. Obviously, no Lindelöf non-paracompact space is L-paracompact. So, no uncountable set X with countable complement topology [11] is L-paracompact, but it is C_2-paracompact, hence C-paracompact, because the only compact subspaces are the finite subspaces, and the countable complement topology is T_1 so compact subspaces are discrete. Hence the discrete topology on X and the identity function will witness C_2-paracompactness.

Any paracompact space is L-paracompact, just by taking $Y = X$ and the identity function. It is clear from the definitions that any L_2-paracompact is L-paracompact. In general, the converse is not true. Assume that X is Lindelöf and L_2-paracompact, then the witness function is a homeomorphism which gives that X is Hausdorff. Thus, any paracompact Lindelöf space which is not Hausdorff is an L-paracompact space that cannot be L_2-paracompact. In particular, any compact space which is not Hausdorff cannot be L_2-paracompact. For examples, the modified Fort space [11], and the overlapping intervals space [11]. There is a case when the L-paracompactness implies L_2-paracompactness given in the next theorem.

Theorem 5. *If X is T_3 separable L-paracompact and of countable tightness, then X is paracompact T_4.*

Proof. Let Y be a paracompact space and $f : X \longrightarrow Y$ be a bijective witness to L-paracompactness of X. Then f is continuous because X is of countable tightness, by Theorem 3. Let D be a countable dense subset of X. We show that f is closed. Let H be any non-empty closed proper subset of X. Suppose that $f(p) = q \in Y \setminus f(H)$; then $p \notin H$. Using regularity, let U and V be disjoint open subsets of X containing p and H, respectively. Then $U \cap (D \cup \{p\})$ is open in the Lindelöf subspace $D \cup \{p\}$ containing p, so $f(U \cap (D \cup \{p\}))$ is open in the subspace $f(D \cup \{p\})$ of Y containing q. Thus $f(U \cap (D \cup \{p\})) = f(U) \cap f(D \cup \{p\}) = W \cap f(D \cup \{p\})$ for some open subset W in Y with $q \in W$. We claim that $W \cap f(H) = \emptyset$. Suppose otherwise, and take $y \in W \cap f(H)$. Let $x \in H$ such that $f(x) = y$. Note that $x \in V$. Since D is dense in X, D is also dense in the open set V. Thus $x \in \overline{V \cap D}$. Now since W is open in Y and f is continuous, $f^{-1}(W)$ is an open set in X; it also contains x. Thus we can choose $d \in f^{-1}(W) \cap V \cap D$. Then $f(d) \in W \cap f(V \cap D) \subseteq W \cap f(D \cup \{p\}) = f(U \cap (D \cup \{p\}))$. So $f(d) \in f(U) \cap f(V)$, a contradiction. Thus $W \cap f(H) = \emptyset$. Note that $q \in W$. As $q \in Y \setminus f(H)$ was arbitrary, $f(H)$ is closed. So f is a homeomorphism and X is paracompact. Since X is also T_2, X is normal. Note that X is also Lindelöf being separable and paracompact. ☐

We conclude from Theorem 5 that the Niemytzki plane [11] and Mrówka space $\Psi(\mathcal{A})$, where $\mathcal{A} \subset [\omega_0]^{\omega_0} = \{ B \subset \omega_0 : B \, \text{is infinite} \}$ is mad [12], are examples of Tychonoff spaces which are not L-paracompact. L-paracompactness is not hereditary, neither is L_2-paracompactness. Take any compactification of the Niemytzki plane. We still do not know if L-paracompactness is hereditary with respect to closed subspaces.

Recall that a *Dowker space* is a T_4 space whose product with I, $I = [0, 1]$ with its usual metric, is not normal. M. E. Rudin used the existence of a Suslin line to obtain a Dowker space which is hereditarily separable and first countable [6]. Using CH, I. Juhász, K. Kunen, and M. E. Rudin constructed a first countable hereditarily separable real compact Dowker space [2]. Weiss constructed a first countable separable locally compact Dowker space whose existence is consistent with MA + ¬ CH [13]. By Theorem 5, such spaces are consistent examples of Dowker spaces are not L-paracompact.

Theorem 6. *L-paracompactness (L_2-paracompactness) is a topological property.*

Proof. Let X be an L-paracompact space and $X \cong Z$. Let Y be a paracompact space and $f : X \longrightarrow Y$ be a bijection such that $f\restriction_A : A \longrightarrow f(A)$ is a homeomorphism for each Lindelöf subspace A of X. Let $g : Z \longrightarrow X$ be a homeomorphism. Then $f \circ g : Z \longrightarrow Y$ satisfies all requirements. ☐

Theorem 7. *L-paracompactness (L_2-paracompactness) is an additive property.*

Proof. Let X_α be an L-paracompact space for each $\alpha \in \Lambda$. We show that their sum $\oplus_{\alpha\in\Lambda} X_\alpha$ is L-paracompact. For each $\alpha \in \Lambda$, pick a paracompact space Y_α and a bijective function $f_\alpha : X_\alpha \longrightarrow Y_\alpha$ such that $f_{\alpha\restriction_{C_\alpha}} : C_\alpha \longrightarrow f_\alpha(C_\alpha)$ is a homeomorphism for each Lindelöf subspace C_α of X_α. Since Y_α is paracompact for each $\alpha \in \Lambda$, then the sum $\oplus_{\alpha\in\Lambda} Y_\alpha$ is paracompact, [1, 5.1.30]. Consider the function sum, see [1, 2.2.E], $\oplus_{\alpha\in\Lambda} f_\alpha : \oplus_{\alpha\in\Lambda} X_\alpha \longrightarrow \oplus_{\alpha\in\Lambda} Y_\alpha$ defined by $\oplus_{\alpha\in\Lambda} f_\alpha(x) = f_\beta(x)$ if $x \in X_\beta, \beta \in \Lambda$. Now, a subspace $C \subseteq \oplus_{\alpha\in\Lambda} X_\alpha$ is Lindelöf if and only if the set $\Lambda_0 = \{\alpha \in \Lambda : C \cap X_\alpha \neq \emptyset\}$ is countable and $C \cap X_\alpha$ is Lindelöf in X_α for each $\alpha \in \Lambda_0$. If $C \subseteq \oplus_{\alpha\in\Lambda} X_\alpha$ is Lindelöf, then $(\oplus_{\alpha\in\Lambda} f_\alpha) \restriction_C$ is a homeomorphism because $f_{\alpha\restriction_{C\cap X_\alpha}}$ is a homeomorphism for each $\alpha \in \Lambda_0$. ☐

Theorem 8. *Every second countable L_2-paracompact space is metrizable.*

Proof. If X is a second countable space, then X is Lindelöf. If X is also L_2-paracompact, then X will be homeomorphic to a T_2 paracompact space Y and, in particular, Y is T_4. Thus X is second countable and regular, hence metrizable [1, 4.2.9]. ☐

Corollary 9. *Every T_2 second countable L-paracompact space is metrizable.*

Recall that a topological space X is called *L-normal* if there exist a normal space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f \restriction_A : A \longrightarrow f(A)$ is a homeomorphism for each Lindelöf subspace $A \subseteq X$ [4]. Since any T_2 paracompact space is normal, it is clear that any L_2-paracompact space is L-normal. In general, L-paracompactness does not imply L-normality. Observe that any finite space which is not discrete is compact, hence paracompact, thus L-paracompact. So, any finite space which is not normal will be an example of an L-paracompact which is neither L_2-paracompact nor L-normal. In general, L-normality does not imply L-paracompactness. Here is an example.

Example 10. Let $X = [0, \infty)$. Define $\mathcal{T} = \{\emptyset, X\} \cup \{[0, x) : x \in \mathbb{R}, 0 < x\}$. Observe that $(X, \mathcal{T})$ is normal because there are no two non-empty closed disjoint subsets. Thus $(X, \mathcal{T})$ is L-normal. Observe that $(X, \mathcal{T})$ is second countable, hence hereditarily Lindelöf. $(X, \mathcal{T})$ cannot be paracompact because $\mathcal{T}$ is coarser than the particular point topology on X [11], where the particular point is 0. That's because any non-empty open set contains 0. Therefore, X is L-normal but not L-paracompact.

Recall that a subset A of a space X is called *closed domain* [1], called also *regularly closed, κ-closed,* if $A = \overline{\text{int}A}$. A space X is called *mildly normal* [10], called also *κ-normal* [9], if for any two disjoint closed domains A and B of X there exist two disjoint open sets U and V of X such that $A \subseteq U$ and $B \subseteq V$, see also [5] and [3]. Any uncountable set X with countable complement topology is mildly normal, because the only closed domains are the empty set and the ground set X, but not L-paracompact. Here is an example of a Tychonoff L_2-paracompact space which is not mildly normal.

Example 11. We modify the Dieudonné Plank [11] to define a new topological space. Let

$$X = ((\omega_2 + 1) \times (\omega_0 + 1)) \setminus \{\langle \omega_2, \omega_0 \rangle\}.$$

Write $X = A \cup B \cup N$, where $A = \{\langle \omega_2, n \rangle : n < \omega_0\}$, $B = \{\langle \alpha, \omega_0 \rangle : \alpha < \omega_2\}$, and $N = \{\langle \alpha, n \rangle : \alpha < \omega_2 \text{ and } n < \omega_0\}$. The topology $\mathcal{T}$ on X is generated by the following neighborhood system: For each $\langle \alpha, n \rangle \in N$, let $\mathcal{B}(\langle \alpha, n \rangle) = \{\{\langle \alpha, n \rangle\}\}$. For each $\langle \omega_2, n \rangle \in A$, let $\mathcal{B}(\langle \omega_2, n \rangle) = \{V_\alpha(n) = (\alpha, \omega_2] \times \{n\} : \alpha < \omega_2\}$. For each $\langle \alpha, \omega_0 \rangle \in B$, let $\mathcal{B}(\langle \alpha, \omega_0 \rangle) = \{V_n(\alpha) = \{\alpha\} \times (n, \omega_0] : n < \omega_0\}$. Then X is Tychonoff non-normal space which is neither locally compact nor locally Lindelöf as any basic open neighborhood of any element in A is not Lindelöf. Now, define $Y = X = A \cup B \cup N$. Generate a topology $\mathcal{T}'$ on Y by the following neighborhood system: Elements of $B \cup N$ have the same local base as in X. For each $\langle \omega_2, n \rangle \in A$, let $\mathcal{B}(\langle \omega_2, n \rangle) = \{\{\langle \omega_2, n \rangle\}\}$. Then Y is T_4 space because it is paracompact. Y and the identity functions gives the L-normality of the modified Dieudonné Plank X is L-normal, see [4]. Since Y is also T_2 paracompact, then X is L_2-paracompact. Observe that the identity function is not continuous on X because it is not continuous at each point in A.

We show that X is not mildly normal. X is not normal because A and B are closed disjoint subsets which cannot be separated by two disjoint open sets. Let $E = \{n < \omega_0 : n \text{ is even}\}$ and $O = \{n < \omega_0 : n \text{ is odd}\}$. Let K and L be subsets of ω_2 such that $K \cap L = \emptyset$, $K \cup L = \omega_2$, and the cofinality of K and L are ω_2; for instance, let K be the set of limit ordinals in ω_2 and L be the set of successor ordinals in ω_2. Then $K \times E$ and $L \times O$ are both open subsets of N. Define $C = \overline{K \times E}$ and $D = \overline{L \times O}$; then C and D are closed domains in X, being closures of open set, and they are disjoint. Note that $C = \overline{K \times E} = (K \times E) \cup (K \times \{\omega_0\}) \cup (\{\omega_2\} \times E)$ and $D = \overline{L \times O} = (L \times O) \cup (L \times \{\omega_0\}) \cup (\{\omega_2\} \times O)$. Let $U \subseteq X$ be any open set such that $C \subseteq U$. For each $n \in E$ there exists an $\alpha_n < \omega_2$ such that $V_{\alpha_n}(n) \subseteq U$. Let $\beta = \sup\{\alpha_n : n \in E\}$; then $\beta < \omega_2$. Since L is cofinal in ω_2, then there exists $\gamma \in L$ such that $\beta < \gamma$ and then any basic open set of $\langle \gamma, \omega_0 \rangle \in D$ will meet U. Thus C and D cannot be separated. Therefor, the modified Dieudonné plank X is L-normal but is not mildly normal.

A space X is called *countably normal* if there exist a normal space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f \restriction_A : A \longrightarrow f(A)$ is a homeomorphism for each countable subspace $A \subseteq X$ [7]. By a similar way, analogous to Definition 1, we can define CP and C_2P properties, these are the best suitable names for us we could choose. A topological space X is called CP if there exist a paracompact space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f \restriction_A : A \longrightarrow f(A)$ is a homeomorphism for each countable subspace $A \subseteq X$. A topological space X is called C_2P if there exist a Hausdorff paracompact space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f \restriction_A : A \longrightarrow f(A)$ is a homeomorphism for each countable subspace $A \subseteq X$.

It is clear that any L-paracompact space is CP and any L_2-paracompact space is C_2P. The converse is not true in general. For example, any uncountable set X with countable complement topology is C_2P because any countable subspace of X is a discrete subspace, hence the discrete topology on X and the identity function will witness the C_2P property. But it is not L-paracompact because it is a Lindelöf non-paracompact space. Most of the aspects of L-paracompactness will go for CP property, for instant Theorem 3 and Theorem 5.

In general, L-paracompactness and L_2-paracompactness are not preserved by a discrete extension. Let us recall the definitions. Let M be a non-empty proper subset of a topological space $(X, \mathcal{T})$. Define a new topology $\mathcal{T}_{(M)}$ on X as follows: $\mathcal{T}_{(M)} = \{U \cup K : U \in \mathcal{T} \text{ and } K \subseteq X \setminus M\}$. $(X, \mathcal{T}_{(M)})$ is called a *discrete extension* of $(X, \mathcal{T})$ and we denote it by X_M, see [11, Examples 70 and 71] and also [1, 5.1.22]. Thus the spaces X and X_M have the same underlying set, but their topologies are in general distinct. The topology of X_M is finer than the topology of X, i.e., $\mathcal{T} \subseteq \mathcal{T}_{(M)}$. The set $X \setminus M$ and all its subsets are open in X_M, so that $X \setminus M$ is an open discrete subspace of X_M. The subspace $M \subseteq X_M$ is closed and its topology coincides with the topology induced on M by the topology of X. The space X_M has the following neighborhood system: For each $x \in X \setminus M$, let $\mathcal{B}(x) = \{\{x\}\}$ and for each $x \in M$, let $\mathcal{B}(x) = \{U \in \mathcal{T} : x \in U\}$. Some topological properties are shared by X and X_M for any non-empty proper subset M for X. For examples, if X is first countable or T_i, where $i \in \{0, 1, 2, 3, 3\frac{1}{2}\}$, then so is X_M for any $\emptyset \neq M \subset X$, see [1, 5.1.22]. Here is an example of an L_2-paracompact space whose a discrete extension of it is Tychonoff but not L-paracompact.

Example 12. It is will-known that $\mathbb{R}$ with the rational sequence topology is a first countable Tychonoff locally compact separable space which is neither normal nor paracompact [11, Example 65]. Thus $\mathbb{R}$ with the rational sequence topology has a one-point compactification. Let $X = \mathbb{R} \cup \{p\}$, where $p \notin \mathbb{R}$, be a one-point compactification of $\mathbb{R}$. Since X is T_2 compact, then it is L_2-paracompact. We prove that the discrete extension $X_{\mathbb{R}}$ is not L_2-paracompact. Observe that in $X_{\mathbb{R}} = \mathbb{R} \cup \{p\}$, the singleton $\{p\}$ is closed-and-open. $X_{\mathbb{R}}$ is first countable and T_3 because $\mathbb{R}$ with the rational sequence topology is, thus $X_{\mathbb{R}}$ is of countable tightness. $X_{\mathbb{R}}$ is also separable because $\mathbb{Q} \cup \{p\}$ is a countable dense subset of $X_{\mathbb{R}}$. Now, $\mathbb{R}$ with the rational sequence topology is not normal. Pick any two closed disjoint subsets A and B of $\mathbb{R}$ that cannot be separated by disjoint open sets in $\mathbb{R}$. Then A and B will be also closed and disjoint in $X_{\mathbb{R}}$ that cannot be separated by disjoint open sets in $X_{\mathbb{R}}$. We conclude that $X_{\mathbb{R}}$ is not normal. By Theorem 5, we conclude that $X_{\mathbb{R}}$ cannot be L-paracompact.

Observe that Example 12 above shows that, in general, CP and C_2P are not preserved by the discrete extension.

References

[1] R. Engelking, *General Topology*, PWN, Warszawa, 1977.
[2] I. Juhász, K. Kunen and M.E. Rudin, *Two More Hereditarily Separable non-Lindelöf spaces*, Cand. J. Math. **28**, 998-1005, 1976.
[3] L. Kalantan, *Results about κ-normality*, Topology Appl. **125**, 47-62, 2002.
[4] L. Kalantan and M. Saeed, *L-Normality*, Topology Proceedings **50**, 141-149, 2017.
[5] L. Kalantan and P. Szeptycki, *κ-normality and products of ordinals*, Topology Appl. **123** (3), 537-545, 2002.
[6] M.E. Rudin, *A Separable Dowker space, Symposia Mathematica*, Instituto Nazionale di Alta Mathematica, 1973.
[7] M.M. Saeed, *Countable Normality*, J. Math. Anal. **9** (1), 116-123, 2018.
[8] M.M. Saeed, L. Kalantan and H. Alzumi, *C - Paracompactness and C_2 - Paracompactness*, Turk. J. Math. **43**, 9-20, 2019.
[9] E.V. Shchepin, *Real Valued Functions and Spaces Close to Normal*, Sib. J. Math. **13**, 1182-1196, 1972.
[10] M.K. Singal and A.R. Singal, *Mildly Normal Spaces*, Kyungpook Math J. **13**, 29-31, 1973.
[11] L. Steen and J.A. Seebach, *Counterexamples in Topology*, Dover Publications, INC. 1995.
[12] E.K. van Douwen, *The Integers and Topology*, in: Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 111-167, 1984.
[13] W. Weiss, *Small Dowker Spaces*, Pacific J. Math. **94**, 485-492, 1981.