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Research Article

A link between topology and soft topology

Marimuthu Kiruthika[1], Periannan Thangavelu*[2]

[1] *Department of Mathematics, Suguna College of Engineering, Coimbatore-641005, India.*
[2] *Professor of Mathematics(Retired), Karunya Deemed University & Aditanar College, TN, India*

Abstract

Muhammad Shabir and Munazza Naz have shown that every soft topology gives a parametrized family of topologies on a set X. In this paper such a link between topology and soft topology is further discussed.

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1. Introduction

The theory of soft sets gives a vital mathematical tool for handling uncertainties and vague concepts. In the year 1999, Molodtsov [15] initiated the study of soft sets. Soft set theory has been applied in several directions. Following this Maji, Biswas, and Roy [13,14] discussed soft set theoretical operations and gave an application of soft set theory to a decision making problem. Recently Shabir and Naz introduced the notion of soft topology [16] and established that every soft topology induces a collection of topologies called the parametrized family of topologies induced by the soft topology. Several mathematicians published papers on applications of soft sets and soft topology [1–3,8,10]. Soft sets and soft topology have applications to data mining, image processing, decision making problems, spatial modeling and neural patterns [4–7, 9, 11–13, 17]. The purpose of this paper is to study a link between a soft topology and the parametrized family of topologies induced by the soft topology. In particular, we give conditions on a given parametrized family of topologies which ensure there exists a soft topology whose induced family of topologies is the given family.

2. Preliminaries

Throughout this paper X denotes the universal set and E denotes the parameter space.

Definition 2.1. A pair (F,E) is called a soft set over X, where $F : E \to 2^X$ is a mapping. We denote (F,E) by $\tilde{F}$ and we write $\tilde{F} = \{(e, F(e)) : e \in E\}$.

According to Shabir and Naz [16], for each subset A of E (F_A, E) is a soft set over the universal set X, where $F_A : A \to 2^X$ is a mapping. However $F_A : A \to 2^X$ can be extended to E by setting $F_A(e) = \phi$ for all $e \in E - A$. This motivates us to fix the parameter space.

*Corresponding Author.
Email addresses: kiruthi.karpagam@gmail.com (M. Kiruthika), ptvelu12@gmail.com (P. Thangavelu)
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In this paper, the definitions and results of Shabir and Naz [16] are taken and the subset A of E is replaced by the fixed parameter space E. Accordingly the following definitions and results are due to Shabir and Naz [16].

Definition 2.2. For any two soft sets $\tilde{F}$ and $\tilde{G}$ over a common universe X, $\tilde{F}$ is a soft subset of $\tilde{G}$ if $F(e) \subset G(e)$ for all e∈E. If $\tilde{F}$ is a soft subset of $\tilde{G}$ then we write $\tilde{F}\tilde{\subseteq}\tilde{G}$

Two soft sets $\tilde{F}$ and $\tilde{G}$ over a common universe X are soft equal if $\tilde{F}\tilde{\subseteq}\tilde{G}$ and $\tilde{G}\tilde{\subseteq}\tilde{F}$. That is $\tilde{F} = \tilde{G}$ if and only if $F(e) = G(e)$ for all $e \in E$

Definition 2.3. A soft set $\tilde{\Phi}$ over X is said to be the NULL soft set if $\tilde{\Phi} = \{(e, \phi) : e \in E\}$.

Definition 2.4. A soft set $\tilde{X}$ over X is said to be the absolute soft set if $\tilde{X} = \{(e, X) : e \in E\}$

Definition 2.5. The union of two soft sets $\tilde{F}$ and $\tilde{G}$ over X is defined as $\tilde{F}\tilde{\cup}\tilde{G} = (F\tilde{\cup}G, E)$ where $(F\tilde{\cup}G)(e) = F(e) \cup G(e)$ for all $e \in E$.

Definition 2.6. The intersection of two soft sets $\tilde{F}$ and $\tilde{G}$ over X is defined as $\tilde{F}\tilde{\cap}\tilde{G} = (F\tilde{\cap}G, E)$ where $(F\tilde{\cap}G)(e) = F(e) \cap G(e)$ for all $e \in E$.

The arbitrary union and the arbitrary intersection of soft sets are defined as follows:

$$\tilde{\cup}\{\tilde{F_\alpha} : \alpha \in \Delta\} = (\tilde{\cup}\{F_\alpha : \alpha \in \Delta\}, E)$$

and

$$\tilde{\cap}\{\tilde{F_\alpha} : \alpha \in \Delta\} = (\tilde{\cap}\{F_\alpha : \alpha \in \Delta\}, E)$$

where $(\tilde{\cup}\{F_\alpha : \alpha \in \Delta\})(e) = \cup\{F_\alpha(e) : \alpha \in \Delta\}$ and $(\tilde{\cap}\{F_\alpha : \alpha \in \Delta\})(e) = \cap\{F_\alpha(e) : \alpha \in \Delta\}$.

Definition 2.7. The complement of a soft set $\tilde{F}$ is denoted by $(\tilde{F})' = (F', E)$ where$F' : E \to 2^X$ is the mapping given by $F'(e) = X - F(e)$ for all $e \in E$.

Definition 2.8. If $\tilde{\tau}$ is a collection of soft sets over X, then $\tilde{\tau}$ is said to be a soft topology on X if

(i) $\tilde{\Phi}, \tilde{X}$ belong to $\tilde{\tau}$,
(ii) arbitrary union of soft sets in $\tilde{\tau}$ belongs to $\tilde{\tau}$,
(iii) the intersection of any two soft sets in $\tilde{\tau}$ belongs to $\tilde{\tau}$.

If $\tilde{\tau}$ is a soft topology over a universal set X with parameter space E, then $(X, \tilde{\tau}, E)$ is called a soft topological space and the members of $\tilde{\tau}$ are called soft open sets over (X, E).

Shabir and Naz introduced a parametrized family of topologies and established that every soft topology induces the parametrized family of topologies as shown in the following lemma.

Lemma 2.9. *Let (X, $\tilde{\tau}$, E) be a soft topological space over X. Then the collection $\tilde{\tau}_e = \{F(e) : \tilde{F} \in \tilde{\tau}\}$ for each $e \in E$, defines a topology on X.*

3. Link

Definition 3.1. Let (X, $\tilde{\tau}$, E) be a soft topological space over X. Then the collection $E(\tilde{\tau}) = \{\tilde{\tau}_e : e \in E\}$ denotes the parameterized family of topologies induced by the soft topology $\tilde{\tau}$.

Proposition 3.2. *Let $(X, \tilde{\tau}, E)$ be a soft topology over X with parameter space E. Then $|E(\tilde{\tau})| \leq |E|$ and $|\tilde{\tau}_e| \leq |\tilde{\tau}|$ for every $e \in E$.*

Proof. Let $\tilde{\tau}$ be a soft topological space over X with parameter space E. Define $\varphi : E \to E(\tilde{\tau})$ by $\varphi(e) = \tilde{\tau}_e$. Clearly φ is onto but it need not be one-to-one. This proves that $|E(\tilde{\tau})| \leq |E|$. Now define $\theta_e : \tilde{\tau} \to \tilde{\tau}_e$ by $\theta_e(\tilde{F}) = F(e)$. θ_e is onto but need not be one-to-one. Therefore $|\tilde{\tau}_e| \leq |\tilde{\tau}|$ □

The above proposition has been illustrated in the following examples.

Example 3.3. Let $X = \{h_1, h_2, h_3\}, E = \{e_1, e_2\}$ and $\tilde{\tau} = \{\tilde{\Phi}, \tilde{X}, \tilde{F_1}, \tilde{F_2}, \tilde{F_3}, \tilde{F_4}, \tilde{F_5}, \tilde{F_6}, \tilde{F_7}, \tilde{F_8}, \tilde{F_9}\}$ where $\tilde{\Phi}, \tilde{X}, \tilde{F_1}, \tilde{F_2}, \tilde{F_3}, \tilde{F_4}, \tilde{F_5}, \tilde{F_6}, \tilde{F_7}, \tilde{F_8}$ and $\tilde{F_9}$ are soft sets over X. The soft sets are defined as follows:

$\tilde{F_1} = \{(e_1, \{h_2\}), (e_2, \{h_1\})\}$
$\tilde{F_2} = \{(e_1, \{h_2, h_3\}), (e_2, \{h_1, h_2\})\}$
$\tilde{F_3} = \{(e_1, \{h_1, h_2\}), (e_2, \{h_1, h_2\})\}$
$\tilde{F_4} = \{(e_1, \{h_1, h_2\}), (e_2, \{h_1, h_3\})\}$
$\tilde{F_5} = \{(e_1, X), (e_2, \{h_1, h_2\})\}$
$\tilde{F_6} = \{(e_1, \{h_2\}), (e_2, \{h_1, h_2\})\}$
$\tilde{F_7} = \{(e_1, \{h_2, h_3\}), (e_2, X)\}$
$\tilde{F_8} = \{(e_1, \{h_1, h_2\}), (e_2, X)\}$
$\tilde{F_9} = \{(e_1, \{h_2\}), (e_2, X)\}$

Then $\tilde{\tau}$ defines a soft topology on X and (X, $\tilde{\tau}$, E) is a soft topological space over X. It can be easily seen that $\tilde{\tau}_{e_1} = \{\phi, X, \{h_2\}, \{h_2, h_3\}, \{h_1, h_2\}\}$ and $\tilde{\tau}_{e_2} = \{\phi, X, \{h_1\}, \{h_1, h_3\}, \{h_1, h_2\}\}$ are topologies on X. Here $e_1 \neq e_2$ and $\tilde{\tau}_{e_1} \neq \tilde{\tau}_{e_2}$. Since $\varphi(e_1) \neq \varphi(e_2)$. φ is one-to-one. Here $|E(\tilde{\tau})| = 2$, $|E| = 2$ and $|E(\tilde{\tau})| = |E|$. Also $\tilde{F_1}(e_1) = \tilde{F_6}(e_1)$ but $\tilde{F_1} \neq \tilde{F_6}$. Since $\theta_{e_1}(\tilde{F_1}) = \theta_{e_1}(\tilde{F_6})$, θ_{e_1} is not one-to-one. Here $|\tilde{\tau}_{e_1}| = 5$ and $|\tilde{\tau}| = 11$. Therefore $|\tilde{\tau}_{e_1}| < |\tilde{\tau}|$. Again since $\theta_{e_2}(\tilde{F_2}) = \theta_{e_2}(\tilde{F_3}) = \{h_1, h_2\}$. θ_{e_2} is not one-to-one. Here $|\tilde{\tau}_{e_2}| = 5 < 11 = |\tilde{\tau}|$.

Example 3.4. Let $X = \{h_1, h_2\}, E = \{e_1, e_2\}$ and $\tilde{\tau} = \{\tilde{\Phi}, \tilde{X}, \tilde{F_1}, \tilde{F_2}, \tilde{F_3}, \tilde{F_4}, \tilde{F_5}, \tilde{F_6}\}$ where $\tilde{\Phi}, \tilde{X}, \tilde{F_1}, \tilde{F_2}, \tilde{F_3}, \tilde{F_4}, \tilde{F_5}, \tilde{F_6}$ are soft sets over X. The soft sets are defined as follows:

$\tilde{F_1} = \{(e_1, \{h_2\}), (e_2, \{h_2\})\}$,
$\tilde{F_2} = \{(e_1, X), (e_2, \{h_2, h_3\})\}$
$\tilde{F_3} = \{(e_1, \{h_2\}), (e_2, X)\}$
$\tilde{F_4} = \{(e_1, \{h_2\}), (e_2, \{h_2, h_3\})\}$
$\tilde{F_5} = \{(e_1, \{h_2, h_3\}), (e_2, X)\}$
$\tilde{F_6} = \{(e_1, \{h_2, h_3\}), (e_2, \{h_2, h_3\})\}$

Then $\tilde{\tau}$ defines a soft topology on X and hence (X,$\tilde{\tau}$,E) is a soft topological space over X. It can be easily seen that $\tilde{\tau}_{e_1} = \{\phi, X, \{h_2\}, \{h_2, h_3\}\}$ and $\tilde{\tau}_{e_2} = \{\phi, X, \{h_2\}, \{h_2, h_3\}\}$ are topologies on X. Here $e_1 \neq e_2$ but $\tilde{\tau}_{e_1} = \tilde{\tau}_{e_2}$. Since $\varphi(e_1) = \varphi(e_2)$, φ is not one-to-one. Here $|E(\tilde{\tau})| = 1$, $|E| = 2$ and $|E(\tilde{\tau})| < |E|$. Also $\tilde{F_1}(e_1) = \tilde{F_4}(e_1)$ but $\tilde{F_1} \neq \tilde{F_4}$. Since $\theta_{e_1}(\tilde{F_1}) = \theta_{e_1}(\tilde{F_4})$, θ_{e_1} is not one-to-one. Here $|\tilde{\tau}_{e_1}| = 4$ and $|\tilde{\tau}| = 8$. Therefore $|\tilde{\tau}_{e_1}| < |\tilde{\tau}|$. Again since $\theta_{e_2}(\tilde{F_4}) = \theta_{e_2}(\tilde{F_6}) = \{h_2, h_3\}$, θ_{e_2} is not one-to-one. Here $|\tilde{\tau}_{e_2}| = 4, |\tilde{\tau}| = 8$. Therefore $|\tilde{\tau}_{e_2}| < |\tilde{\tau}| = 8$.

Example 3.5. Let $X = \{h_1, h_2\}, E = \{e_1, e_2\}$ and $\tilde{\tau} = \{\tilde{\Phi}, \tilde{X}, \tilde{F_1}, \tilde{F_2}\}$ where $\tilde{\Phi}, \tilde{X}, \tilde{F_1}, \tilde{F_2}$ are soft sets over X. The soft sets are defined as follows :
$\tilde{F_1} = \{(e_1, \{h_2, h_3\})\}, (e_2, \{h_2, h_3\})\}$
$\tilde{F_2} = \{(e_1, \{h_2\}), (e_2, \{h_2\})\}$
Then $\tilde{\tau}$ defines a soft topology on X and hence (X,$\tilde{\tau}$,E) is a soft topological space over X. It can be easily seen that here $\tilde{\tau}_{e_1} = \{\phi, X, \{h_2\}, \{h_2, h_3\}\}$ and $\tilde{\tau}_{e_2} = \{\phi, X, \{h_2\}, \{h_2, h_3\}\}$ are topologies on X. Here $e_1 \neq e_2$ and $\tilde{\tau}_{e_1} = \tilde{\tau}_{e_2}$. Since $\varphi(e_1) = \varphi(e_2)$, φ is not one-to-one. Here $|E(\tilde{\tau})| = 1$ and $|E| = 2$. Therefore $|E(\tilde{\tau})| < |E|$. Also $\tilde{F_1}(e_1) \neq \tilde{F_2}(e_1)$ and $\tilde{F_1} \neq \tilde{F_2}$. Since $\theta_{e_1}(\tilde{F_1}) \neq \theta_{e_1}(\tilde{F_2})$, θ_{e_1} is one-to-one and onto. Here $|\tilde{\tau}_{e_1}| = 4$ and $|\tilde{\tau}| = 4$. Therefore $|\tilde{\tau}_{e_1}| = |\tilde{\tau}|$. Again since $\theta_{e_2}(\tilde{F_1}) \neq \theta_{e_2}(\tilde{F_2})$, θ_{e_2} is one-to-one. Here $|\tilde{\tau}_{e_2}| = 4, |\tilde{\tau}| = 4$. Therefore $|\tilde{\tau}_{e_2}| = |\tilde{\tau}| = 4$

Shabir and Naz established that every soft topology induces a parameterized family of topologies and further gave an example (Example 2, page 1790 of [16]) to show that the converse is not true. Then the following question will arise.

Given a collection $\{\tau_e : e \in E\}$ of topologies on X, are there conditions under which there exists a soft topology $\tilde{\tau}$ over X with parameter space E such that $\tau_e = \tilde{\tau}_e$, for all $e \in E$?

The following theorem gives an answer to the above question.

Theorem 3.6. *Let X be a universal set and E be a parameter space. Let $\{\tau_\alpha : \alpha \in E\}$ be a family of topologies on X satisfying the following conditions.*

(i) *There is an index set J such that for each $\alpha \in E, \tau_\alpha = \{G_{\alpha j} : j \in J\}$*

(ii) *If $\Delta \subseteq J$, then $\exists\ r, s \in J$ such that $\cap\{G_{\alpha_j} : j \in \Delta\} = G_{\alpha_r}$ for finite Δ and $\cup\{G_{\alpha_j} : j \in \Delta\} = G_{\alpha_s}$ for each $\alpha \in E$.*

(iii) *There exist $j_0, j_1 \in J$ such that $G_{\alpha j_0} = \phi$ and $G_{\alpha j_1} = X$ for all $\alpha \in E$.*

Then $\tilde{\tau} = \{\tilde{F}_j : j \in J\}$ where $\tilde{F}_j(\alpha) = G_{\alpha j}$ for each $\alpha \in E$ is a soft topology on X with parameter space E satisfying $\tilde{\tau}_\alpha = \tau_\alpha$ for all $\alpha \in E$.

Proof. For each $j \in J$ define $\tilde{F}_j : E \to 2^X$ by $\tilde{F}_j(\alpha) = G_{\alpha j}$ for all $\alpha \in E$. Then $\{\tilde{F}_j : j \in J\}$ is a collection of soft sets over X with parameter space E.

Claim: $\tilde{\tau} = \{\tilde{F}_j : j \in J\}$ is a soft topology on X.

Let Δ be a non-empty subset of J and let $\alpha \in E$.

$$\begin{aligned} (\tilde{\cup}\{\tilde{F}_j : j \in \Delta\})(\alpha) &= \cup\{\tilde{F}_j(\alpha) : j \in \Delta\} \\ &= \cup\{G_{\alpha_j} : j \in \Delta\} \\ &= G_{\alpha_s} \quad \text{for some } s \in J \end{aligned}$$

Since this is true for every $\alpha \in E$, $(\tilde{\cup}\{\tilde{F}_j : j \in \Delta\})(\alpha) = \tilde{F}_s(\alpha)$. Therefore $\tilde{\tau}$ is closed under arbitrary union.

$$\begin{aligned} (\tilde{F}_j \tilde{\cap} \tilde{F}_k)(\alpha) &= F_j(\alpha) \cap F_k(\alpha) \\ &= G_{\alpha j} \cap G_{\alpha k} = G_{\alpha r} \\ &= F_r(\alpha) \end{aligned}$$

That is $\tilde{F}_j \tilde{\cap} \tilde{F}_k = \tilde{F}_r \in \tilde{\tau}$. To prove that $\tilde{\tau}_\alpha = \tau_\alpha$ for all $\alpha \in E$

$$\begin{aligned} \tilde{\tau}_\alpha &= \{\tilde{F}_j(\alpha) : \tilde{F}_j \in \tilde{\tau}\} \\ &= \{G_{\alpha j} : j \in J\} \\ &= \tau_\alpha \quad \text{for all } \alpha \in \text{E}. \end{aligned}$$

□

Remark 3.7. Obtaining the necessary and sufficient conditions for theorem 3.6 is an open problem for researchers in soft topology.

4. Conclusion

In this paper, a link between a soft topology and the parametrized family of topologies induced by the soft topology is identified and characterized.

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