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Research Article

# On $k$-circulant matrices involving geometric sequence

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## Abstract

In this paper we consider a $k$-circulant matrix with geometric sequence, where $k$ is a nonzero complex number. The eigenvalues, the determinant, the Euclidean norm and bounds for the spectral norm of such matrix are investigated. The method for obtaining the inverse of a nonsingular $k$-circulant matrix, was presented in [On $k$-circulant matrices (with geometric sequence), Quaest. Math. 2016]. A generalization of that method is given in this paper, and using it, the inverse of a nonsingular $k$-circulant matrix with geometric sequence is obtained. The Moore-Penrose inverse of a singular $k$-circulant matrix with geometric sequence is determined in a different way than the way using in [On $k$-circulant matrices (with geometric sequence), Quaest. Math. 2016].

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## 1. Introduction

Throughout this paper, $k$ is a nonzero complex number. By $\mathbb{C}^{m\times n}$ we denote the set of all $m\times n$ complex matrices. The symbol $\mathbb{C}_r^{m\times n}$ denotes the set of all $m\times n$ complex matrices having rank equal to $r$. The eigenvalues, the determinant and the inverse of $C\in\mathbb{C}^{n\times n}$ are denoted by $\lambda_j(C), j=\overline{0,n-1}$, $|C|$ and $C^{-1}$, respectively. The symbols $C^*$, $\|C\|_E$ and $\|C\|_2$ stand for the conjugate (or Hermitian) transpose, the Euclidean norm and the spectral norm of $C\in\mathbb{C}^{m\times n}$, respectively. The identity matrix of order $n$ is denoted by $I_n$. By $diag\,[c_{1,1}, c_{2,2}, c_{3,3}, \ldots, c_{n,n}]$ we denote a diagonal matrix of order $n$.

*A $k$-circulant matrix* is a matrix of the form:

$$\begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ kc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ kc_{n-2} & kc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ kc_2 & kc_3 & kc_4 & \cdots & c_0 & c_1 \\ kc_1 & kc_2 & kc_3 & \cdots & kc_{n-1} & c_0 \end{bmatrix}. \tag{1.1}$$

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A matrix of the form (1.1) is completely determined by $k$ and its first row. So, if a matrix $C$ has the form (1.1), then we shall write $C = circ_n\{_k(c_0, c_1, c_2, \ldots, c_{n-1})\}$ and we call it a circulant (a skew circulant) matrix for $k = 1$ ($k = -1$). If C is a circulant matrix, we shall not indicate that $k = 1$ and we shall write $C = circ_n\{(c_0, c_1, c_2, \ldots, c_{n-1})\}$. Necessary and sufficient conditions for a complex square matrix to be a $k$-circulant matrix were presented by R. E. Cline, R. J. Plemmons and G. Worm in the paper [5] (see Lemmas 2 and 3 in [5]). In the paper [8], the authors showed how $C^q$ can be obtained, where $C$ is a matrix of the form (1.1) and $q$ is a positive integer greater than 1.

It is necessary to point out that $k$-circulant matrices (especially, circulant and skew circulant matrices) play important role in many areas (probability, statistics, numerical analysis, signal and image processing, coding theory, engineering model etc.). There are many papers devoted to $k$-circulant matrices whose entries are different types of number sequences. Circulant matrices with geometric sequence were considered by Bueno in [2] and [3]. In [2] ([3]), Bueno obtained the eigenvalues, the determinants, the Euclidean norms and the spectral norms of $circ_n\{(g, gq, gq^2, \ldots, gq^{n-1})\}$ ($circ_n\{(gq^j, gq^{j+1}, gq^{j+2}, \ldots, gq^{j+n-1})\}$), where $g \neq 0$ and $q \neq 0, 1$ (where $j$ is an arbitrary natural number, $g \neq 0$ and $q \neq 0, 1$), and their inverses. Bueno, in [4], investigated the eigenvalues, the determinants, the Euclidean norms and the inverses of $circ_n\{(\frac{F_0}{g}, \frac{F_1}{gq}, \frac{F_2}{gq^2}, \ldots, \frac{F_{n-1}}{gq^{n-1}})\}$, where $F_n$ is the $n^{th}$ Fibonacci number, $g \neq 0$ and $q \neq 0, 1$. The paper [11] is devoted to $k$-circulant matrices with geometric sequence and, in that paper, the author obtained the inverses of such nonsingular matrices and the Moore-Penrose inverses of such singular matrices. Circulant and skew circulant matrices with binomial coefficients were considered in [16] and the spectral norms of such matrices were investigated in that paper. The paper [14] is devoted to circulant and skew circulant matrices whose entries are binomial coefficients combined with either Fibonacci numbers or Lucas numbers. In that paper, the authors investigated the spectral norms of such matrices and obtained identity estimations for these spectral norms. Circulant and skew circulant matrices whose entries are product of binomial coefficients with harmonic numbers were considered in the paper [15]. The spectral norms of such matrices were investigated in that paper, and the explicit identities for these spectral norms were obtained. In [12] and [13], the authors considered circulant matrices with the generalized $r$-Horadam numbers $\{H_{r,n}\}$ (the numbers defined as follows:

$$H_{r,n+2} = f(r)H_{r,n+1} + g(r)H_{r,n}, \quad n \geq 0, \tag{1.2}$$

where $r \in \mathbb{R}^+$, $H_{r,0} = a$, $H_{r,1} = b$, $a, b \in \mathbb{R}$ and $f^2(r) + 4g(r) > 0$), and obtained the spectral norms, the eigenvalues, the determinants and the inverses of such matrices.

In this paper, we shall investigate the eigenvalues, the determinant, the Euclidean norm and bounds for the spectral norm of a $k$-circulant matrix with geometric sequence and extend some results presented in [2]. The method for obtaining the inverse of a nonsingular $k$-circulant matrix, presented in [11], will be generalized and, using it, we shall obtain the inverse of a nonsingular $k$-circulant matrix with geometric sequence. Using the formula for the Moore-Penrose inverse of an arbitrary $k$-circulant matrix, given by Boman in [1], the Moore-Penrose inverse of a singular $k$-circulant matrix with geometric sequence will be determined. Before that, let us recall that *a geometric sequence* is a sequence having the following form:

$$g_0 = g,\ g_1 = gq,\ g_2 = gq^2,\ g_3 = gq^3, \ldots \tag{1.3}$$

where $g \in \mathbb{C} \setminus \{0\}$ and $q \in \mathbb{R} \setminus \{0\}$ i.e. $g_i = gq^i, i \in \mathbb{N}_0$. If $q = 1$, then (1.3) is a constant sequence. If $q = -1$, then (1.3) is an alternating sequence.

Our results will be presented in the next section.

## 2. The main results

First, we investigate the eigenvalues of

$$circ_n\{_k(g, gq, gq^2, \ldots, gq^{n-1})\}, \tag{2.1}$$

where $g \in \mathbb{C}\setminus\{0\}$ and $q \in \mathbb{R}\setminus\{0\}$. Before that, let us point out that the symbols $\omega$ and $\psi$ denote any primitive $n^{th}$ root of unity and any $n^{th}$ root of $k$, respectively. In order to determine the eigenvalues of (2.1), we shall use the following lemma.

**Lemma 2.1** ([5, Lemma 4])**.** *Let $C$ be a matrix of the form (1.1). The eigenvalues of $C$ are:*

$$\lambda_j(C) = \sum_{i=0}^{n-1} c_i(\psi\omega^{-j})^i,\ j = \overline{0,\ n-1}. \tag{2.2}$$

*Moreover, in this case*

$$c_i = \frac{1}{n}\sum_{j=0}^{n-1} \lambda_j(C)(\psi\omega^{-j})^{-i},\ i = \overline{0,\ n-1}\,. \tag{2.3}$$

Now, we can prove the following theorem.

**Theorem 2.2.** *Let $G$ be a matrix of the form (2.1). The eigenvalues of $G$ are given by the following formulae:*

1) *If $q\psi\omega^{-j} = 1$, then*

$$\lambda_j(G) = ng, \tag{2.4}$$

2) *If $q\psi\omega^{-j} \neq 1$, then*

$$\lambda_j(G) = g\frac{1-kq^n}{1-q\psi\omega^{-j}}\,. \tag{2.5}$$

***Proof.*** Using Lemma 2.1, it follows:

1) Suppose that $q\psi\omega^{-j} = 1$. Then,

$$\lambda_j(G) = \sum_{i=0}^{n-1} g_i(\psi\omega^{-j})^i = \sum_{i=0}^{n-1} gq^i(\frac{1}{q})^i = g\sum_{i=0}^{n-1} 1 = ng\,.$$

2) Suppose that $q\psi\omega^{-j} \neq 1$. Then,

$$\lambda_j(G) = \sum_{i=0}^{n-1} g_i(\psi\omega^{-j})^i = g\sum_{i=0}^{n-1}(q\psi\omega^{-j})^i = g\frac{1-kq^n}{1-q\psi\omega^{-j}}\,.$$

□

**Remark 2.3.** If $k=1$, then we obtain the result of Theorem 3.2 in [2].

The determinant of

$$circ_n\{_k(1, q, q^2, \ldots, q^{n-1})\}, \tag{2.6}$$

where $q\in\mathbb{R}\setminus\{0\}$, is given by the following theorem.

**Theorem 2.4.** *Let $Q$ be a matrix of the form (2.6). The determinant of $Q$ is:*

$$|\,Q\,| = (1-kq^n)^{n-1}. \tag{2.7}$$

***Proof.*** Applying the properties of the determinant of a matrix we obtain:

$$|\,Q\,| = \begin{vmatrix} 1 & q & q^2 & \ldots & q^{n-2} & q^{n-1} \\ kq^{n-1} & 1 & q & \ldots & q^{n-3} & q^{n-2} \\ kq^{n-2} & kq^{n-1} & 1 & \ldots & q^{n-4} & q^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ kq^2 & kq^3 & kq^4 & \ldots & 1 & q \\ kq & kq^2 & kq^3 & \ldots & kq^{n-1} & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & q & q^2 & \dots & q^{n-2} & q^{n-1} \\ 0 & 1-kq^n & q(1-kq^n) & \dots & q^{n-3}(1-kq^n) & q^{n-2}(1-kq^n) \\ 0 & 0 & 1-kq^n & \dots & q^{n-4}(1-kq^n) & q^{n-3}(1-kq^n) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1-kq^n & q(1-kq^n) \\ 0 & 0 & 0 & \dots & 0 & 1-kq^n \end{vmatrix}$$

Therefore, $|Q| = (1-kq^n)^{n-1}$. □

**Corollary 2.5.** *Let $G$ be a matrix of the form (2.1). The determinant of $G$ is:*

$$|G| = g^n(1-kq^n)^{n-1}. \tag{2.8}$$

***Proof.*** It follows from Theorem 2.4 and the fact: $|\alpha A| = \alpha^n |A|$, which holds for any complex matrix $A$ of order $n$ and any complex number $\alpha$. □

**Remark 2.6.** If $k=1$, then we obtain the result of Theorem 3.1 in [2].

The inverse $C^{-1}$ of a nonsingular $k$-circulant matrix $C$ is always $k$-circulant and it can be obtained using the following lemma.

**Lemma 2.7.** ([11, Lemma 2.2]) *Let $C = circ_n\{_k(c_0, c_1, c_2, \dots, c_{n-1})\}$ be a nonsingular matrix with complex entries. Then, $C^{-1} = circ_n\{_k(c'_0, c'_1, c'_2, \dots, c'_{n-1})\}$, where $(c'_0, c'_1, c'_2, \dots, c'_{n-1})$ is the unique solution of the following system of linear equations:*

$$C \begin{bmatrix} x_0 \\ kx_{n-1} \\ \vdots \\ kx_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{2.9}$$

From the proof of the previous lemma which was given in [11], it follows that the inverse of a nonsingular $k$-circulant matrix can also be obtained by solving the system of linear equations which is different than the system (2.9). Namely, from the proof of Lemma 2.7, it follows that the following lemma is also true.

**Lemma 2.8.** *Let $C = circ_n\{_k(c_0, c_1, c_2, \dots, c_{n-1})\}$ be a nonsingular matrix with complex entries. Then, $C^{-1} = circ_n\{_k(c'_0, c'_1, c'_2, \dots, c'_{n-1})\}$, where $(c'_0, c'_1, c'_2, \dots, c'_{n-1})$ is the unique solution of one of the following systems of linear equations:*

$$C \begin{bmatrix} x_{j-1} \\ \vdots \\ x_1 \\ x_0 \\ kx_{n-1} \\ \vdots \\ kx_j \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow j\,, \\ \\ \\ \\ \end{matrix} \; j = \overline{1, n}. \tag{2.10}$$

For $j = 1$, Lemma 2.8 becomes Lemma 2.7. In the paper [11] the inverse of (2.6) was obtained, provided that $1-kq^n \neq 0$, using Lemma 2.7 (see Theorem 2.2 in [11]). In this paper we shall determine the inverse of (2.6), provided that $1-kq^n \neq 0$, using Lemma 2.8 and, for example, $j = n$.

**Theorem 2.9.** *Let $Q$ be a matrix of the form (2.6). If $1-kq^n \neq 0$, the inverse of $Q$ is:*

$$Q^{-1} = \frac{1}{kq^n - 1} circ_n\{_k(-1, q, 0, \dots, 0)\}. \tag{2.11}$$

***Proof.*** Let $Q^{-1}=circ_n\{_k(q'_0,q'_1,q'_2,\ldots,q'_{n-1})\}$. Based on Lemma 2.8 $(q'_0,q'_1,q'_2,\ldots,q'_{n-1})$ is the unique solution of the following system of linear equations:

$$Q\begin{bmatrix}x_{n-1}\\ \vdots\\ x_1\\ x_0\end{bmatrix}=\begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix}. \tag{2.12}$$

Applying elementary row operations to the augmented matrix we obtain:

$$\overline{Q}=\begin{bmatrix}1 & q & q^2 & \ldots & q^{n-2} & q^{n-1} & 0\\ kq^{n-1} & 1 & q & \ldots & q^{n-3} & q^{n-2} & 0\\ kq^{n-2} & kq^{n-1} & 1 & \ldots & q^{n-4} & q^{n-3} & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ kq^2 & kq^3 & kq^4 & \ldots & 1 & q & 0\\ kq & kq^2 & kq^3 & \ldots & kq^{n-1} & 1 & 1\end{bmatrix}$$
$$\sim\begin{bmatrix}1 & q & q^2 & \ldots & q^{n-2} & q^{n-1} & 0\\ 0 & 1-kq^n & q(1-kq^n) & \ldots & q^{n-3}(1-kq^n) & q^{n-2}(1-kq^n) & 0\\ 0 & 0 & 1-kq^n & \ldots & q^{n-4}(1-kq^n) & q^{n-3}(1-kq^n) & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \ldots & 1-kq^n & q(1-kq^n) & 0\\ 0 & 0 & 0 & \ldots & 0 & 1-kq^n & 1\end{bmatrix}.$$

Therefore, the system (2.12) is equivalent to the following system:

$$\begin{cases}\displaystyle\sum_{i=0}^{n-1} q^i x_{n-(i+1)}=0,\\ \displaystyle\sum_{i=0}^{j} q^{j-i}x_i=0, j=\overline{1,n-2},\\ x_0=-\frac{1}{kq^n-1}.\end{cases} \tag{2.13}$$

The solution of the system (2.13) is:

$$\begin{cases}x_0=-\frac{1}{kq^n-1},\\ x_1=\frac{q}{kq^n-1},\\ x_i=0,\ i=\overline{2,n-1}.\end{cases} \tag{2.14}$$

Since the system (2.13) is equivalent to the system (2.12), it follows that (2.14) is also the solution of the system (2.12). □

The Moore-Penrose inverse $C^\dagger$ (i.e. the unique matrix satisfying the following identities $CC^\dagger C=C$, $C^\dagger CC^\dagger=C^\dagger$, $(CC^\dagger)^*=CC^\dagger$ and $(C^\dagger C)^*=C^\dagger C$) of a singular $k$-circulant matrix $C$ need not be $k$-circulant. Namely, in the paper [5], the authors proved the following statements.

**Theorem 2.10.** ([5, Lemma 5]) *Let $C$ be a singular $k$-circulant matrix. If $C^\dagger$ is $k$'-circulant for some $k$', then $k'=\frac{1}{\overline{k}}$.*

**Theorem 2.11.** ([5, Theorem 3]) *Let $C$ be a singular $k$-circulant matrix. Then $C^\dagger$ is $k$-circulant if and only if $k$ lies on the unit circle.*

The problem of characterizing $C^\dagger$ for an arbitrary $k$-circulant matrix $C$ was solved by Boman in [1]. Before we present the Boman's result, let us mention that $F=[f_{i,j}]$ is the Fourier matrix defined as follows:

$$f_{i,j} = \frac{1}{\sqrt{n}}\,\omega^{(i-1)(j-1)},\ \ i,j=\overline{1,n}, \tag{2.15}$$

and $W=[w_{i,j}]$ is the matrix defined as follows:

$$w_{i,j} = \begin{cases} \psi^{i-1}, & i=j \\ 0, & \text{otherwise} \end{cases}, \qquad i,j=\overline{1,n}, \tag{2.16}$$

i.e. $W = diag\left[1,\psi,\psi^2,\dots,\psi^{n-1}\right]$, where $\omega$ is any primitive $n^{th}$ root of unity and $\psi$ is any $n^{th}$ root of $k$, as we pointed out at the beginning of this section.

**Theorem 2.12** ([1, Theorem 3])**.** *Let $C\in\mathbb{C}_r^{m\times n}$ be a singular $k$-circulant matrix. Without loss of generality assume that in block form*

$$F^*W^{-1}CWF = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix},$$

*where $D$ is a diagonal matrix of order $r$ and $0$ is the zero matrix of appropriate dimensions. Then,*

$$C^\dagger = \overline{W^{-1}}F \begin{bmatrix} (NDM)^{-1} & 0 \\ 0 & 0 \end{bmatrix} F^*\overline{W}, \tag{2.17}$$

*where $M$ and $N$ are the submatrices of order $r$ (at the first $r$ columns and the first $r$ rows) of $(F^*\overline{W}WF)^{-1}$ and $F^*\overline{W}WF$, respectively.*

If $1-kq^n=0$ (i.e. $k=\frac{1}{q^n}$), then (2.6) is a singular matrix. The Moore-Penrose inverse of such matrix will be obtained using Theorem 2.12.

**Theorem 2.13.** *Let $n$ be an arbitrary natural number greater than 1 and*

$$Q = circ_n\{_{\frac{1}{q^n}}(1,q,q^2,\dots,q^{n-1})\}. \tag{2.18}$$

a) *If $q\in\mathbb{R}\setminus\{-1,0,1\}$, then*

$$Q^\dagger = q^{2(n-1)}\left(\frac{1-q^2}{1-q^{2n}}\right)^2 circ_n\{_{q^n}(1,\frac{1}{q},\frac{1}{q^2},\dots,\frac{1}{q^{n-1}})\}. \tag{2.19}$$

b) *If $q=1$, then*

$$Q^\dagger = \frac{1}{n^2}\,circ_n\{(1,1,\dots,1)\}. \tag{2.20}$$

c) *If $q=-1$ and $n$ is an arbitrary odd natural number, then*

$$Q^\dagger = \frac{1}{n^2}\,circ_n\{_{-1}(1,-1,1,\dots,-1,1)\}. \tag{2.21}$$

d) *If $q=-1$ and $n$ is an arbitrary even natural number, then*

$$Q^\dagger = \frac{1}{n^2}\,circ_n\{(1,-1,\dots,1,-1)\}. \tag{2.22}$$

***Proof.*** Let $F$ and $W$ be the matrices defined, respectively, by (2.15) and (2.16).

a) If $q\in\mathbb{R}\setminus\{-1,0,1\}$, then $W = diag\left[1,\frac{1}{q},\frac{1}{q^2},\dots,\frac{1}{q^{n-1}}\right]$. Thus,

$$F^*\overline{W}WF = \frac{q^{2(1-n)}(q^{2n}-1)}{n}\,circ_n\{(\tfrac{1}{q^2-1},\tfrac{1}{q^2-\omega},\tfrac{1}{q^2-\omega^2},\dots,\tfrac{1}{q^2-\omega^{n-1}})\}$$

and

$$F^*W^{-1}QWF = \left[\begin{array}{c|c} n & 0 \\ \hline 0 & 0 \end{array}\right].$$

Therefore,

$$M=\frac{1}{n}\frac{1-q^{2n}}{1-q^2},\quad D=n\quad\text{and}\quad N=\frac{q^{2(1-n)}}{n}\frac{1-q^{2n}}{1-q^2}.$$

Based on Theorem 2.12, it follows

$$\begin{aligned}Q^{\dagger} &= \overline{W^{-1}}F\begin{bmatrix}(NDM)^{-1} & 0\\ 0 & 0\end{bmatrix}F^{*}\overline{W}\\ &= \dots\\ &= q^{2(n-1)}(\frac{1-q^2}{1-q^{2n}})^2\, circ_n\{_{q^n}(1,\frac{1}{q},\frac{1}{q^2},\dots,\frac{1}{q^{n-1}})\},\end{aligned}$$

b) If $q=1$, then $W=I_n$.

c) If $q=-1$ and $n$ is an arbitrary odd natural number, then $W=diag\,[1,-1,1,\dots,-1,1]$.

d) If $q=-1$ and $n$ is an arbitrary even natural number, then $W=diag\,[1,-1,\dots,1,-1]$.

In these cases (b), c) and d)),

$$F^{*}\overline{W}WF=I_n$$

and

$$F^{*}W^{-1}QWF=\left[\begin{array}{c|c} n & 0\\ \hline 0 & 0\end{array}\right].$$

Therefore,

$$M=1,\quad D=n\quad\text{and}\quad N=1\ .$$

Based on Theorem 2.12, it follows

b)

$$\begin{aligned}Q^{\dagger} &= \overline{W^{-1}}F\begin{bmatrix}(NDM)^{-1} & 0\\ 0 & 0\end{bmatrix}F^{*}\overline{W}\\ &= \dots\\ &= \frac{1}{n^2}\, circ_n\{(1,1,\dots,1)\},\end{aligned}$$

c)

$$\begin{aligned}Q^{\dagger} &= \overline{W^{-1}}F\begin{bmatrix}(NDM)^{-1} & 0\\ 0 & 0\end{bmatrix}F^{*}\overline{W}\\ &= \dots\\ &= \frac{1}{n^2}\, circ_n\{_{-1}(1,-1,1,\dots,-1,1)\},\end{aligned}$$

d)

$$\begin{aligned}Q^{\dagger} &= \overline{W^{-1}}F\begin{bmatrix}(NDM)^{-1} & 0\\ 0 & 0\end{bmatrix}F^{*}\overline{W}\\ &= \dots\\ &= \frac{1}{n^2}\, circ_n\{(1,-1,\dots,1,-1)\}.\end{aligned}$$

□

The Euclidean norm of (2.1) is given by the following theorem. Let us recall that the Euclidean norm of $C=[c_{i,j}]\in\mathbb{C}^{n\times n}$ is $\|C\|_E=\sqrt{\sum_{i,j=1}^{n}|c_{i,j}|^2}$. We shall use the following formula.

For all $x$,

$$\sum_{i=1}^{n-1} ix^i=\frac{x-nx^n+(n-1)x^{n+1}}{(1-x)^2}\,. \tag{2.23}$$

**Theorem 2.14.** *Let $G$ be a matrix of the form (2.1). The Euclidean norm of $G$ is given by the following formulae:*

$$\|G\|_E = \begin{cases} |g|\sqrt{n\frac{1-q^{2n}}{1-q^2} + (|k|^2-1)\frac{q^2-nq^{2n}+(n-1)q^{2(n+1)}}{(1-q^2)^2}}, & q\in\mathbb{R}\setminus\{-1,0,1\} \\ |g|\sqrt{n^2+(|k|^2-1)\frac{(n-1)n}{2}}, & q=-1 \text{ or } q=1 \end{cases}. \tag{2.24}$$

***Proof.*** From the definition of the Euclidean norm of a matrix and (2.23), it follows:

1) If $q\in\mathbb{R}\setminus\{-1,0,1\}$, then

$$\begin{aligned}(\|G\|_E)^2 &= \sum_{i,j=1}^{n} |g_{i,j}|^2 = n|g_0|^2 + \left[(n-1)+|k|^2\right]|g_1|^2 + \cdots + \left[1+(n-1)|k|^2\right]|g_{n-1}|^2 \\ &= \sum_{i=0}^{n-1}(n-i)|g_i|^2 + |k|^2\sum_{i=1}^{n-1} i|g_i|^2 = n\sum_{i=0}^{n-1}|g_i|^2 + (|k|^2-1)\sum_{i=1}^{n-1} i|g_i|^2 \\ &= n|g|^2\sum_{i=0}^{n-1} q^{2i} + (|k|^2-1)|g|^2\sum_{i=1}^{n-1} iq^{2i} \\ &= n|g|^2\frac{1-q^{2n}}{1-q^2} + (|k|^2-1)|g|^2\frac{q^2-nq^{2n}+(n-1)q^{2(n+1)}}{(1-q^2)^2} \\ &= |g|^2\left[n\frac{1-q^{2n}}{1-q^2} + (|k|^2-1)\frac{q^2-nq^{2n}+(n-1)q^{2(n+1)}}{(1-q^2)^2}\right],\end{aligned}$$

2) If $q=-1$ or $q=1$. then

$$\begin{aligned}(\|G\|_E)^2 &= \sum_{i,j=1}^{n} |g_{i,j}|^2 = \cdots = n\sum_{i=0}^{n-1}|g_i|^2 + (|k|^2-1)\sum_{i=1}^{n-1} i|g_i|^2 \\ &= n|g|^2\sum_{i=0}^{n-1} 1 + (|k|^2-1)|g|^2\sum_{i=1}^{n-1} i = |g|^2\left[n^2+(|k|^2-1)\frac{(n-1)n}{2}\right].\end{aligned}$$

Therefore,

$$\|G\|_E = \begin{cases} |g|\sqrt{n\frac{1-q^{2n}}{1-q^2} + (|k|^2-1)\frac{q^2-nq^{2n}+(n-1)q^{2(n+1)}}{(1-q^2)^2}}, & q\in\mathbb{R}\setminus\{-1,0,1\} \\ |g|\sqrt{n^2+(|k|^2-1)\frac{(n-1)n}{2}}, & q=-1 \text{ or } q=1 \end{cases}.$$

□

**Remark 2.15.** If $k=1$, then we obtain the result of Theorem 3.3 in [2]. Let notice that in [2] the author considered a matrix of the form:

$$circ_n\{(g, gq, gq^2, \ldots, gq^{n-1})\}, \tag{2.25}$$

where $g\neq 0$ and $q\neq 0,1$, and obtained, by Theorem 3.3 in [2], the Euclidean norm of such matrix. But the result of Theorem 3.3 in [2] is not applicable if $q=-1$. Hence, the Euclidean norm of

$$circ_n\{(g, -g, g, \ldots, g(-1)^{n-1})\} \tag{2.26}$$

is missing in Theorem 3.3 in [2] and it can be obtained from the previous theorem (if $k=1$ and $q=-1$).

The following theorems are devoted to the spectral norm of (2.1). Before we continue, let us recall that the spectral norm of $C\in\mathbb{C}^{n\times n}$ is $\|C\|_2 = \sqrt{\max_{1\le i\le n} \lambda_i(C^*C)}$.

**Theorem 2.16.** ([9, Theorem 3.1 and Remark 3.1]) *Let $n$ be an arbitrary even natural number and let $G$ be a real matrix of the form (2.26). The spectral norm of $G$ is:*

$$\|G\|_2 = n|g|. \tag{2.27}$$

Next, we determine the upper and lower bounds for the spectral norm of (2.1). In order to obtain the upper and lower bounds for the spectral norm of (2.1), we shall use the following inequalities (see Theorem 1 and Table 1 in [17]):

$$\frac{\|C\|_E}{\sqrt{n}} \leq \|C\|_2 \leq \|C\|_E, \tag{2.28}$$

which hold for any complex matrix $C$ of order $n$, and the following lemma.

**Lemma 2.17.** ([7]) *Let $A=[a_{i,j}]$ and $B=[b_{i,j}]$ be $m\times n$ matrices. Then,*

$$\|A\circ B\|_2 \leq r_1(A)\cdot c_1(B), \tag{2.29}$$

*where $A\circ B=[a_{i,j}b_{i,j}]$ is the Hadamard product (or the Schur product) of matrices $A$ and $B$ (see [6] and [10]),*

$$r_1(A) = \max_{1\leq i\leq m}\sqrt{\sum_{j=1}^{n} \mid a_{i,j}\mid^2} \quad \textit{and} \quad c_1(B) = \max_{1\leq j\leq n}\sqrt{\sum_{i=1}^{m} \mid b_{i,j}\mid^2}\,.$$

Now, we can obtain the upper and lower bounds for the spectral norm of (2.1).

**Theorem 2.18.** *Let $G$ be a matrix of the form (2.1).*

*I) If $q\in\mathbb{R}\setminus\{-1,0,1\}$ and*

*1) $|k|\geq 1$, then*

$$|g|\sqrt{\frac{1-q^{2n}}{1-q^2}} \leq \|G\|_2 \leq |g|\sqrt{(1+(n-1)|k|^2)\left(\frac{1-q^{2n}}{1-q^2}\right)}, \tag{2.30}$$

*2) $|k|<1$, then*

$$|kg|\sqrt{\frac{1-q^{2n}}{1-q^2}} \leq \|G\|_2 \leq |g|\sqrt{n\frac{1-q^{2n}}{1-q^2}}\,. \tag{2.31}$$

*II) If $q=-1$ or $q=1$ and*

*1) $|k|\geq 1$, then*

$$|g|\sqrt{n} \leq \|G\|_2 \leq |g|\sqrt{n\,(1+(n-1)|k|^2)}\,, \tag{2.32}$$

*2) $|k|<1$, then*

$$|kg|\sqrt{n} \leq \|G\|_2 \leq n|g|\,. \tag{2.33}$$

***Proof.*** From the definition of the Euclidean norm of a matrix, it follows:

I) Suppose that $q\in\mathbb{R}\setminus\{-1,0,1\}$ and

1) $|k|\geq 1$. Then,

$$\|G\|_E^2 \geq \sum_{i=0}^{n-1}(n-i)|g_i|^2 + \sum_{i=1}^{n-1} i|g_i|^2 = n\sum_{i=0}^{n-1}|g_i|^2 = n|g|^2\sum_{i=0}^{n-1}q^{2i} = n|g|^2\frac{1-q^{2n}}{1-q^2}.$$

Therefore,

$$\frac{\|G\|_E}{\sqrt{n}} \geq |g|\sqrt{\frac{1-q^{2n}}{1-q^2}}\,.$$

We conclude from (2.28) that

$$\|G\|_2 \geq |g|\sqrt{\frac{1-q^{2n}}{1-q^2}}\,.$$

Now, we shall obtain the upper bound for the spectral norm of $G$. Let $P$ and $Q$ be the following matrices:

$$P = \begin{bmatrix} g_0 & g_0 & g_0 & \cdots & g_0 \\ kg_0 & g_0 & g_0 & \cdots & g_0 \\ kg_0 & kg_0 & g_0 & \cdots & g_0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ kg_0 & kg_0 & kg_0 & \cdots & g_0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 1 & q & q^2 & \cdots & q^{n-1} \\ q^{n-1} & 1 & q & \cdots & q^{n-2} \\ q^{n-2} & q^{n-1} & 1 & \cdots & q^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q & q^2 & q^3 & \cdots & 1 \end{bmatrix}. \tag{2.34}$$

Then,

$$\begin{aligned} r_1(P) &= \max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n} \mid p_{i,j}\mid^2} = \sqrt{|g_0|^2 + \sum_{j=1}^{n-1}|kg_0|^2} = \sqrt{|g|^2+(n-1)|kg|^2} \\ &= |g|\sqrt{1+(n-1)|k|^2} \end{aligned}$$

and

$$c_1(Q) = \max_{1\leq j\leq n}\sqrt{\sum_{i=1}^{n} \mid q_{i,j}\mid^2} = \sqrt{\sum_{i=0}^{n-1}|q|^{2i}} = \sqrt{\tfrac{1-q^{2n}}{1-q^2}}\,.$$

Since $G = P \circ Q$, based on Lemma 2.17, we can write

$$\|G\|_2 \leq r_1(P)\cdot c_1(Q) = |g|\sqrt{(1+(n-1)|k|^2)\left(\frac{1-q^{2n}}{1-q^2}\right)}\,.$$

2) $|k| < 1$. Then,

$$\begin{aligned} \|G\|_E^2 &\geq \sum_{i=0}^{n-1}(n-i)|k|^2|g_i|^2 + \sum_{i=1}^{n-1} i|k|^2|g_i|^2 = n|k|^2\sum_{i=0}^{n-1}|g_i|^2 \\ &= n|kg|^2\sum_{i=0}^{n-1}q^{2i} = n|kg|^2\frac{1-q^{2n}}{1-q^2}. \end{aligned}$$

Therefore,

$$\frac{\|G\|_E}{\sqrt{n}} \geq |kg|\sqrt{\frac{1-q^{2n}}{1-q^2}}\,.$$

We conclude from (2.28) that

$$\|G\|_2 \geq |kg|\sqrt{\frac{1-q^{2n}}{1-q^2}}\,.$$

Now, we shall obtain the upper bound for the spectral norm of $G$. Let $R$ and $S$ be the following matrices:

$$R = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ k & 1 & 1 & \cdots & 1 \\ k & k & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k & k & k & \cdots & 1 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} g_0 & g_1 & g_2 & \cdots & g_{n-1} \\ g_{n-1} & g_0 & g_1 & \cdots & g_{n-2} \\ g_{n-2} & g_{n-1} & g_0 & \cdots & g_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_1 & g_2 & g_3 & \cdots & g_0 \end{bmatrix}. \tag{2.35}$$

Then,

$$r_1(R) = \max_{1 \le i \le n} \sqrt{\sum_{j=1}^{n} \mid r_{i,j} \mid^2} = \sqrt{n}$$

and

$$c_1(S) = \max_{1 \le j \le n} \sqrt{\sum_{i=1}^{n} \mid s_{i,j} \mid^2} = \sqrt{\sum_{i=0}^{n-1} |g_i|^2} = \sqrt{|g|^2 \sum_{i=0}^{n-1} q^{2i}} = |g|\sqrt{\frac{1-q^{2n}}{1-q^2}} \,.$$

Since $G = R \circ S$, based on Lemma 2.17, we can write

$$\|G\|_2 \le r_1(R) \cdot c_1(S) = |g|\sqrt{n\frac{1-q^{2n}}{1-q^2}} \,.$$

II) Suppose that $q = -1$ or $q = 1$ and

1) $|k| \ge 1$. Then,

$$\|G\|_E^2 \ge \sum_{i=0}^{n-1} (n-i)|g_i|^2 + \sum_{i=1}^{n-1} i|g_i|^2 = n\sum_{i=0}^{n-1} |g_i|^2 = n|g|^2 \sum_{i=0}^{n-1} 1 = n^2|g|^2.$$

Therefore,

$$\frac{\|G\|_E}{\sqrt{n}} \ge |g|\sqrt{n} \,.$$

We conclude from (2.28) that

$$\|G\|_2 \ge |g|\sqrt{n} \,.$$

Now, we shall obtain the upper bound for the spectral norm of $G$. Let $P$ and $Q$ be the matrices as in (2.34).

Then,

$$\begin{aligned} r_1(P) &= \max_{1 \le i \le n} \sqrt{\sum_{j=1}^{n} \mid p_{i,j} \mid^2} = \sqrt{|g_0|^2 + \sum_{j=1}^{n-1} |kg_0|^2} = \sqrt{|g|^2 + (n-1)|kg|^2} \\ &= |g|\sqrt{1 + (n-1)|k|^2} \end{aligned}$$

and

$$c_1(Q) = \max_{1 \le j \le n} \sqrt{\sum_{i=1}^{n} \mid q_{i,j} \mid^2} = \sqrt{\sum_{i=0}^{n-1} |q|^{2i}} = \sqrt{n} \,.$$

Since $G = P \circ Q$, based on Lemma 2.17, we can write

$$\|G\|_2 \le r_1(P) \cdot c_1(Q) = |g|\sqrt{n\,(1 + (n-1)|k|^2)} \,.$$

2) $|k| < 1$. Then,

$$\|G\|_E^2 \ \ge \ \sum_{i=0}^{n-1} (n-i)|k|^2|g_i|^2 + \sum_{i=1}^{n-1} i|k|^2|g_i|^2 = n|k|^2 \sum_{i=0}^{n-1} |g_i|^2 = n|kg|^2 \sum_{i=0}^{n-1} 1 = n^2|kg|^2.$$

Therefore,

$$\frac{\|G\|_E}{\sqrt{n}} \ge |kg|\sqrt{n}.$$

We conclude from (2.28) that

$$\|G\|_2 \ge |kg|\sqrt{n}.$$

Now, we shall obtain the upper bound for the spectral norm of $G$. Let $R$ and $S$ be the matrices as in (2.35).

Then,

$$r_1(R) = \max_{1\le i\le n} \sqrt{\sum_{j=1}^{n} \mid r_{i,j}\mid^2} = \sqrt{n}$$

and

$$c_1(S) = \max_{1\le j\le n} \sqrt{\sum_{i=1}^{n} \mid s_{i,j}\mid^2} = \sqrt{\sum_{i=0}^{n-1} |g_i|^2} = \sqrt{|g|^2 \sum_{i=0}^{n-1} 1} = |g|\sqrt{n}\ .$$

Since $G = R \circ S$, based on Lemma 2.17, we can write

$$\|G\|_2 \le r_1(R)\cdot c_1(S) = n|g|\ .$$

□

## 3. Conclusion

In this paper we determined the eigenvalues, the determinant, the Euclidean norm and bounds for the spectral norm of a $k$-circulant matrix with geometric sequence, where $k$ is a nonzero complex number, and extended some results obtained in [2]. A generalization of the method for obtaining the inverse of a nonsingular $k$-circulant matrix, presented in [11], was given and, using it, the inverse of a nonsingular $k$-circulant matrix with geometric sequence was obtained. Using the formula in [1], the Moore-Penrose of a singular $k$-circulant matrix with geometric sequence was also determined.

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