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Determination of the temperature distribution in a rectangular cooling fin using the finite element method

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ABSTRACT

This paper involves the use of the Galerkin finite element method to determine the temperature distribution in a rectangular cooling fin. The governing equation is a one-dimensional second order differential equation. The result shows that the temperature at the tip of the rectangular cooling fin which was 100°C and begins to drop as it proceeds to the other end of the rectangular cooling fin which is 61.5518°C at 0.1m. The result obtained from the finite element solutions when compared with the analytical solution, shows that the accuracy was very high with the highest percentage error of 0.000432875. It can be stated that the finite element solution is an accurate method for determining the temperature distribution in a rectangular cooling fin.

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1. Introduction

Rectangular fins are used to remove heat from the surface of a body by conduction along the fins and convection from the surface of the fins into the surroundings. Fins are the most effective instrument for increasing the rate of heat transfer. As we know, they increase the area of heat transfer and cause an increase in the transferred heat amount. A complete review on this topic is presented by Kraus et al. [1]. Fins are widely used in many industrial applications such as air conditioning, refrigeration, automobile, chemical processing equipment and electrical chips.

Although there are various types of the fins, but the rectangular fin is widely used among them, probably, due to simplicity of its design and its easy manufacturing process. For ordinary fins problem, the thermal conductivity assumes to be constant, but when temperature difference between the tip and base of the fin is large, the effect of the temperature on thermal conductivity must be considered. Also, it is very realistic that to consider the heat generation in the fin (due to electric current or etc.) as a function of temperature. Domairi and Fazeli [2] used the least squares method for predicting the performance of a longitudinal fin with temperature-dependent internal heat generation and thermal conductivity and they compared their results by Homotopy Perturbation Method (HPM), Variational Iteration Method (VIM) and double series regular perturbation method and found that the least squares method was a simpler method.

Razani and Ahmadi [3] considered circular fins with an arbitrary heat source distribution and a nonlinear temperature-dependent thermal conductivity and obtained the results for the optimum fin design. Unal [4] conducted an analytical study of a rectangular and longitudinal fin with temperature-dependent internal heat generation and temperature-dependent heat transfer coefficient. Another study about this issue (convective fin with both temperature dependent thermal conductivity and internal heat generation) was performed by Shouman [5]. Kundu [6] had solved a problem about thermal analysis and optimization of longitudinal and pin fins of uniform thickness subject to fully wet, partially wet and fully dry surface conditions. Domairry and Fazeli [7] solved the nonlinear straight fin differential equation by the Homotopy Analysis Method (HAM) to evaluate the temperature distribution and fin efficiency. Also, temperature distribution for annual fins with temperature-dependent thermal conductivity was studied by Ganji et al. [8] using HPM. The effects of temperature-dependent thermal conductivity of a moving fin with considering the radiation losses have been studied by Aziz and Kaani [9]. Furthermore, Bouaziz and Aziz [10] introduced a double optimal linearization method (DOLM) to get a simple and accurate solution for the temperature distribution in a straight rectangular convective–radiative fin with temperature-dependent thermal conductivity. Inc [11] used HAM to obtain the efficiency of straight fin with temperature dependent thermal conductivity.

The concept of Differential Transformation Method (DTM) was firstly introduced by Ghafoori et al. [12] which was used to solve both linear and nonlinear initial value problems in electric circuit analysis. This method can be applied directly for linear and nonlinear differential equations without requiring linearization, discretization or perturbation and this is the main benefit of this method. Ghafoori, et al. [13] used the DTM for solving the nonlinear oscillation equation. Abdel-Halim Hassan [14] applied DTM for different systems of differential equations and he discussed the convergence of this method in several examples of linear and nonlinear systems of differential equations. Recently, analytical methods were used for solving the heat transfer through the porous fins with different geometries [15-19]. It is obvious that a number of researchers seem not to have analyzed the temperature distribution in a rectangular cooling fin. But none has attempted to use the finite element method. Hence, this paper using the finite element analysis tends to fill this gap.

2. Methodology

Consider a rectangular cooling fin. The governing equation (i.e. balance of energy) is

$$-\frac{d^2T}{dx^2}+\frac{\beta}{ka}(T-T_\infty)=0 \tag{1}$$

where T is the temperature, k is the thermal conductivity, β is the film coefficient, a is the thickness and T_∞ is the temperature of the surrounding fluid (i.e., ambient temperature) [20].
Eq. 1 was derived by approximating the true physical situation. Therefore, the assumptions are:

i. The temperature is a function of the x direction alone

ii. No heat is lost from the end or from the edges

iii. The heat flux at the surface is given by $q_x = h(T-T_a)$, where h is a constant and T depends on x [21].

The boundary conditions of the problem are

$$T(0) = T_w \qquad \text{(wall temperature)} \tag{2}$$

$$\left(kA\frac{dT}{dx}\right)\bigg|_{x=L}=0 \tag{3}$$

2.1. Solution

The equations can be recast in the residual form as:

$$-\frac{d^2T}{dx^2}+\frac{\beta T}{ka}-\frac{\beta T_\infty}{ka}=0 \tag{4}$$

In the analysis involving Finite Element method, the governing equation can only be solved if it is in order one. But the governing equation for the temperature distribution in a rectangular cooling fin is in order two, so, the need to weaken the governing equation to order one. Therefore, the weak form of eq. 4 is as shown in eq. 5.

$$\int_{x_A}^{x_B}\frac{\partial w}{\partial x}\frac{\partial T}{\partial x}dx+\frac{\beta}{ka}\int_{x_A}^{x_B}wTdx-\frac{\beta T_\infty}{ka}\int_{x_A}^{x_B}wdx- \\ -wQ_A-wQ_B=0 \tag{5}$$

This is followed by the introduction of the interpolation functions to unable us derive the finite element model. The weak form in eq. 5 requires that the approximation chosen for T should be at least quadratic in x so that there are no terms in eq. 5 that are identically zero. Since the primary variable is simply the function itself, the Lagrange family of interpolation functions is admissible. We proposed that T is the approximation over a typical finite element domain by the expression:

$$T(x)=\sum_{j=1}^{n}T_j\psi_j^e(x) \quad and \quad w=\psi_i^e(x) \tag{6}$$

In Galerkin's weighted residual method, the weighting functions are chosen to be identical to the trial functions as shown in eq. 6.
Substituting eq. 6 into eq. 5 and simplifying, eq. 5 reduces to eq. 7.

$$\sum_{j=1}^{n}T_j(x)\int_{x_A}^{x_A+h}\frac{\partial\psi_i}{\partial x}\frac{\partial\psi_j}{\partial x}dx+\sum_{j=1}^{n}T_j(x)\frac{\beta}{ka}\int_{x_A}^{x_A+h}\psi_i\psi_j dx- \\ -\frac{\beta T_\infty}{ka}\int_{x_A}^{x_A+h}\psi_i dx-wQ_A-wQ_B=0 \tag{7}$$

$$\text{where } x_B=x_A+h \tag{8}$$

Eq. 7 is the developed finite element model. This is then used to generate the elemental matrices.
Eq. 7 can be written in the condensed form as:

$$\left[K_{ij}^e\right]\left\{T_j^e\right\}+\frac{\beta}{ka}\left\{M_{ij}^e\right\}\left\{T_j^e\right\}=\frac{\beta T_\infty}{ka}\left\{f_i^e\right\}+\left\{Q_i^e\right\} \tag{9}$$

$$\text{where } K_{ij}^e=\int_{x_A}^{x_A+h_e}\frac{\partial\psi_i^e(x)}{\partial x}\frac{\partial\psi_j^e(x)}{\partial x}dx \tag{10}$$

$$M_{ij}^{e}=\int_{x_A}^{x_A+h_e}\psi_i^e(x)\psi_j^e(x)\,dx \tag{11}$$

$$f_i^e=\int_{x_A}^{x_A+h_e}\psi_i\,dx \tag{12}$$

Hence, the one-dimensional Lagrange quadratic interpolation function for Equation becomes

$$\psi_1(x)=\left(1-\frac{x}{h_e}\right)\left(1-\frac{2x}{h_e}\right) \tag{13}$$

$$\psi_2(x)=\frac{4x}{h_e}\left(1-\frac{x}{h_e}\right) \tag{14}$$

$$\psi_3(x)=-\frac{x}{h_e}\left(1-\frac{2x}{h_e}\right) \tag{15}$$

where h_e = Elemental length of the rectangular cooling fin

To evaluate the K_{ij}, f_i and M_{ij} matrices, we substitute eq. 13-15 accordingly into eq. 10, 11 and 12 respectively, we have;

$$K^e=\frac{1}{3h_e^3}\begin{bmatrix} 7h_e^2-24h_e\xi_A+48\xi_A^2 & -8\left(h_e^2-3h_e\xi_A+12\xi_A^2\right) & h_e^2+48\xi_A^2 \\ -8\left(h_e^2-3h_e\xi_A+12\xi^2\right) & 16\left(h_e^2+12\xi_A^2\right) & -8\left(h_e^2+3h_e\xi_A+12\xi_A^2\right) \\ h_e^2+48\xi_A^2 & -8\left(h_e^2+3h_e\xi_A+12\xi_A^2\right) & 7h_e^2+24h_e\xi_A+48\xi_e^2 \end{bmatrix} \tag{16}$$

$$M^e=\frac{1}{30h^3}\begin{bmatrix} 4h^4-30h^3\xi_A+90h^2\xi_A^2-120h\xi_A^3+120\xi_A^4 & 2h^4-60h^2\xi_A^2+120h\xi_A^3-240\xi_A^4 & -h^4+30h^2\xi_A^2+120\xi_A^4 \\ 2h^4-60h^2\xi_A^2+120h\xi_A^3-240\xi_A^4 & 16h^4+480\xi_A^4 & 2h^4-60h^2\xi_A^2-120h\xi_A^3-240\xi_A^4 \\ -h^4+30h^2\xi_A^2+120\xi_A^4 & 2h^4-60h^2\xi_A^2-120h\xi_A^3-240\xi_A^4 & 4h^4+30h^3\xi_A+90h^2\xi_A^2+120h\xi_A^3+120\xi_A^4 \end{bmatrix} \tag{17}$$

$$f^e=\begin{Bmatrix} \dfrac{h_e}{6}-x_A+\dfrac{2x_A^2}{h_e} \\ \dfrac{2\left(h_e^2-6x_A^2\right)}{3h_e} \\ \dfrac{h_e}{6}+x_A+\dfrac{2x_A^2}{h_e} \end{Bmatrix} \tag{18}$$

Assembly of the Matrix using four elements

The assembled K^e matrix is given as:

$$\left[K^e\right]=\frac{1}{3h_e}\begin{bmatrix} 7 & -8 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -8 & 16 & -8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -8 & 38 & -80 & 49 & 0 & 0 & 0 & 0 \\ 0 & 0 & -80 & 208 & -128 & 0 & 0 & 0 & 0 \\ 0 & 0 & 49 & -128 & 230 & -344 & 193 & 0 & 0 \\ 0 & 0 & 0 & 0 & -344 & 784 & -440 & 0 & 0 \\ 0 & 0 & 0 & 0 & 193 & -440 & 614 & -800 & 433 \\ 0 & 0 & 0 & 0 & 0 & 0 & -800 & 1744 & -944 \\ 0 & 0 & 0 & 0 & 0 & 0 & 433 & -944 & 511 \end{bmatrix} \tag{19}$$

The assembled M^e matrix is given as:

$$M^e=\frac{h}{30}\begin{bmatrix} 4 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 16 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 68 & -178 & 149 & 0 & 0 & 0 & 0 \\ 0 & 0 & -178 & 496 & -418 & 0 & 0 & 0 & 0 \\ 0 & 0 & 149 & -418 & 1628 & -3118 & 2039 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3118 & 7696 & -5038 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2039 & -5038 & 10508 & -16738 & 9989 \\ 0 & 0 & 0 & 0 & 0 & 0 & -16738 & 38896 & -23218 \\ 0 & 0 & 0 & 0 & 0 & 0 & 9989 & -23218 & 138964 \end{bmatrix} \tag{20}$$

The assembled f^e matrix is given as:

$$f^e = \frac{h_e}{6}\begin{Bmatrix} 1 \\ 4 \\ 2 \\ 4 \\ 2 \\ 4 \\ 2 \\ 4 \\ 1 \end{Bmatrix} \tag{21}$$

Eq. 19, 20 and 21 are substituted into eq. 9 to obtain the temperature distribution in the rectangular cooling fin. But the assembled matrix cannot be solved directly. But with the introduction of either the boundary condition or initial conditions or a combination of both the initial and boundary conditions, the nodal values of the parameter (Temperature) can be determined.

3. Results

In other to solve this problem analytically, we recast eq. 1, 2 and 3 by introducing the following non-dimensional quantities;

$$\theta = \frac{T - T_\infty}{T_w - T_\infty}; \qquad \xi = \frac{x}{L}; \qquad N = \sqrt{\left(\frac{\beta L^2}{ka}\right)} \tag{22}$$

Therefore, eq. 1, 2 and 3 becomes respectively

$$-\frac{d^2\theta}{d\xi^2} + N^2\theta = 0 \tag{23}$$

$$\theta(0) = 1 \tag{24}$$

$$\left.\left(\frac{d\theta}{d\xi}\right)\right|_{\xi=1} = 0 \tag{25}$$

The analytical solution of the problem is given in eq. 26

$$\theta = \cosh N\xi - (\tanh N)\sinh N\xi \tag{26}$$

The data used in analysing this problem are given thus:

$\beta = 35W/m^2\,{}^0C,\ k = 170W/m\,{}^0C,\ T_0 = 100^0C,$

$T_\infty = 20^0C,\ L = 100mm,\ t = 1mm$

4. Discussion

The Finite Element solutions obtained in this problem can be used to determine the temperature distribution in a rectangular cooling fin. This is as a result of substituting the appropriate values of the domains and boundary conditions into the formulated coefficient matrix equations. The results are represented in the Table 1 for both the Finite Element Solution and the analytical solution.

Table 1. *Comparison between FEM solution and analytical solution.*

Length (m)	*T ⁰C (FEM Solution)*	*T ⁰C (Analytical Solution)*	*% ERROR*
0.0000	100.0000	100.0000	0.0000
0.0125	89.8610	89.8614	4.3288E-04
0.0250	81.6539	81.6537	-3.1399E-04
0.0375	75.1120	75.1120	9.4111E-05
0.0500	70.0259	70.0255	-5.5540E-04
0.0625	66.2302	66.2301	-1.5269E-04
0.0750	63.6037	63.6032	-7.0786E-04
0.0875	62.0604	62.0602	-2.8332E-04
0.1000	61.5519	61.5514	-7.6003E-04

The graph of the finite element solution and the analytical solution converges as shown in Figure 1 which shows a decline in temperature as the length of the fin increases.

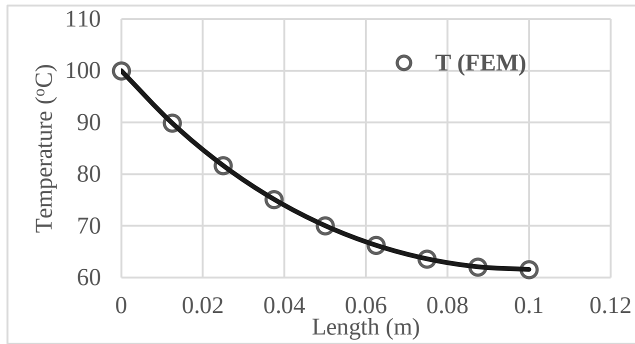

Figure 1. *A graph of Temperature and Length of fin for FEM and analytical solution.*

Figure 1 shows that the temperature at the tip of the rectangular cooling fin is *100ºC* and begins to drop as it proceeds to the other end of the rectangular cooling fin which is *61.5518ºC* at *0.1m*. As a result of the decrease in temperature from one end of the cooling fin to the other, more heat is released into the surrounding air thereby raising the ambient temperature (T_∞). This increase in the ambient temperature (T_∞) has a direct relationship with the temperature in the rectangular cooling fin. This means that the higher the ambient temperature (T_∞), the higher the temperature in the rectangular cooling fin. This is shown in Figure 2.

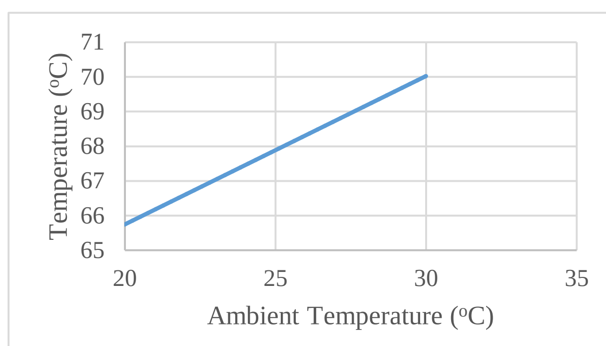

Figure 2. *A graph of fin temperature against ambient temperature.*

A mathematical model was developed keeping other parameters constant to show the relationship between the fin temperature and the ambient temperature. The model developed is as shown in eq. 27.

$$T = 0.4282T_{\infty} + 57.18 \qquad (27)$$

It was observed that the model fits in well with a coefficient of determination (R^2) of 100%. This shows that the ambient temperature was able to account for 100% of the variation in the temperature of the rectangular cooling fin.

5. Conclusion

In this paper, the Galerkin finite element method has been used to determine the temperature distribution in a rectangular cooling fin. The results shows that finite element method is a more reliable and accurate method for determining the temperature distribution in a rectangular cooling fin successfully.

References

[1]. Kraus A.D., Aziz A., Welty J.R. Extended Surface Heat Transfer John Wiley, New York, (2002).

[2]. Aziz A., Bouaziz M.N. “A least squares method for a longitudinal fin with temperature dependent internal heat generation and thermal conductivity”. Energy Convers Manag, 52, (2011), 2876-2882.

[3]. Razani A., Ahmadi G. “On optimization of circular fins with heat generation”. J Frankl Inst, 303 (2), (1977), 211-218.

[4]. Unal H.C. “Temperature distributions in fins with uniform and non-uniform heat generation and non-uniform heat transfer coefficient”. International Journal of Heat and Mass Transfer, 30 (7), (1987), 1465-1477.

[5]. Shouman AR. “Nonlinear heat transfer and temperature distribution through fins and electric elements of arbitrary geometry with temperature dependent properties and heat generation”. NASA technical note, TN D-4257, (1968).

[6]. Kundu B. “Performance and optimum design analysis of longitudinal and pin fins with simultaneous heat and mass transfer: unified and comparative investigations”. Appl Therm Eng, 27, (2007), 976-987.

[7]. Domairry G., Fazeli M. “Homotopy analysis method to determine the fin efficiency of convective straight fins with temperature-dependent thermal conductivity”. Commun Nonlinear Sci Numer Simul, 14, (2009), 489-499.

[8]. Ganji D.D., Ganji Z.Z., Ganji H.D. “Determination of temperature distribution for annual fins with temperature-dependent thermal conductivity by HPM”. Therm Sci, 15, (2011), 111-115.

[9]. Aziz A., Khani F. “Convection–radiation from a continuously moving fin of a variable thermal conductivity”. J Frankl Inst, 348, (2011), 640-651.

[10]. Bouaziz M.N., Aziz A. “Simple and accurate solution for convective–radiative fin with temperature dependent thermal conductivity using double optimal linearization”. Energy Convers Manag, 51, (2010), 2776-2782.

[11]. Mustafa Inc, “Application of homotopy analysis method for fin efficiency of convective straight fins with temperature-dependent thermal conductivity”. Math Comput Simul, 79, (2008), 189-200.

[12]. Zhou J.K. “Differential Transformation Method and its Application for Electrical Circuits”. Hauzhang University press, Wuhan, China, (1986).

[13]. Ghafoori S., Motevalli M., Nejad M.G., Shakeri F., Ganji D.D., Jalaal M. “Efficiency of differential transformation method for nonlinear oscillation: comparison with HPM and VIM”. Curr Appl Phys, 11, (2011), 965-971.

[14]. Abdel-Halim Hassan I.H. “Application to differential transformation method for solving systems of differential equations”. Appl Math Model, 32, (2008), 2552-2559.

[15]. Hatami M., Ganji D.D. “Thermal behavior of longitudinal convective–radiative porous fins with different section shapes and ceramic materials (SiC and Si3N4)”. Ceram Int, 40, (2014), 6765-6775.

[16]. Hatami M., Ganji D.D. “Thermal and flow analysis of microchannel heat sink (MCHS) cooled by Cu–water nanofluid using porous media approach and least square method”. Energy Convers Manag, 78, (2014), 347-358.

[17]. Hatami M., Ganji D.D. “Investigation of refrigeration efficiency for fully wet circular porous fins with variable sections by combined heat and mass transfer analysis”. International Journal of Refrigeration, 40, (2014), 140-151.

[18]. Hatami M., Ganji D.D. “Thermal performance of circular convective-radiative porous fins with different section shapes and materials”. Energy Convers Manag, 76, (2013), 185-193.

[19]. Hatami M., Hasanpour A., Ganji D.D. “Heat transfer study through porous fins (Si3N4 and Al) with temperature-dependent heat generation”. Energy Convers Manag, 74, (2013), 9-16.

[20]. Reddy J.N. An Introduction to the Finite Element Method, Second Edition, McGraw-Hill, Texas, (1993).

[21]. Bird R.B., Stewart W.E., Lightfoot E.N. “Transport Phenomena” Second Edition, John Wiley & Sons, Inc, New York, (2002), 310.