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# Two-dimensional parabolic problem with a rapidly oscillating free term

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**ABSTRACT**

In this paper, it is aimed to construct regularized asymptotics of the solution of a two-dimensional partial differential equation of parabolic type with a small parameter for all spatial derivatives and a rapidly oscillating free term.

The case when the first derivative of the phase of the free term at the initial point vanishes is considered. The two-dimensionality of the equation leads to the existence of a two-dimensional boundary layer. The presence in the free term as a rapidly oscillating factor leads to the inclusion in the asymptotic of the boundary layer with a rapidly oscillating nature of change. Vanishing of the derived phase of the free term leads to the asymptotic of a new type of boundary layer function. A complete asymptotic solution of the problem is constructed by the method of regularization of singularly perturbed problems developed by S.A. Lomov and adapted the authors for singularly perturbed parabolic equations.

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## 1. Introduction

Singularly perturbed problems with rapidly oscillating free terms were studied in [1-3]. In [5], the solution was found using the regularization method for singularly perturbed problems. The method used in [5] was also exploited for differential equations of parabolic type with a small parameter, where fast-oscillating functions are free members, were studied in [2-3]. The one-dimensional parabolic equation, when the scalar equation contains a free term consisting of a finite sum of rapidly oscillating functions is studied in [4]. In the current paper, a two-dimensional parabolic equation are studied.

The asymptotic of the scalar equation contains a rapidly oscillating, power, parabolic boundary layer function and their product [5], while the asymptotic solution of a multidimensional equation additionally contains a multidimensional boundary layer function.

## 2. Asymptotic construction

### *2.1. Statement of the Problem*

In this paper, the following problem is studied:

$$L_\varepsilon u(x,t,\varepsilon) \equiv \partial_t u - \varepsilon^2 \Delta_a u - b(x,t)u = f(x,t)exp\left(\frac{i\theta(t)}{\varepsilon}\right), (x,t)\epsilon E, \quad (1)$$
$$u|_{t=0} = 0, u|_{\partial_\Omega=0} = 0,$$

where $\varepsilon > 0$ – is a small parameter, $x = (x_1, x_2)$, $\Omega = (0 < x_1 < 1)\, x\, (0 < x_2 < 1)$, $E = (0 < t \leq T)\, x\, \Omega$, $\Delta_a \equiv \sum_{l=1}^{2} a_l(x_l)\partial_{x_l}^2$.

The problem is solved under the following assumptions:

1. $\forall\, x_l \in [0,1]$ the function $a_l(x_l) \in C^\infty[0,1], l = 1,2;$
2. $b(x,t), f(x,t) \in C^\infty[E];$
3. $\theta'(0) = 0.$

*2.2. Regularization of the Problem*

Following the method of regularization of singularly perturbed problems [5-6], along with the independent variables $(x,t)$ we introduce regularizing variables:

$$\mu = \frac{t}{\varepsilon}, \xi_l = \frac{(-1)^{l-1}}{\sqrt{\varepsilon^3}}\int_{l-1}^{x_1}\frac{ds}{\sqrt{a_1(s)}}, \eta_l = \frac{\varphi_l(x_1)}{\varepsilon^2}, \xi_{l+2} = \frac{(-1)^{l-1}}{\sqrt{\varepsilon^3}}\int_{l-1}^{x_2}\frac{ds}{\sqrt{a_2(s)}}, \eta_{l+2} = \frac{\varphi_{l+2}(x_2)}{\varepsilon^3}$$
$$\sigma = \int_0^t e^{\frac{i[\theta(s)-\theta(0)]}{\varepsilon}}ds, \tau_2 = \frac{i[\theta(t)-\theta(0)]}{\varepsilon}, \tau_1 = \frac{t}{\varepsilon^2},\ \varphi_l(x_r) = (-1)^{l-1}\int_{l-1}^{x_r}\frac{ds}{\sqrt{a_r(s)}} \quad (2)$$

Instead of the desired function $u(x,t,\varepsilon)$ we study the extended function:

$$\tilde{u}(M,\varepsilon),\quad M = (x,t,\tau,\xi,\eta),\ \ \chi = (\tau,\xi,\eta), \tau = (\tau_1,\tau_2), \xi = (\xi_1,\xi_2,\xi_3,\xi_4),$$
$$\eta = (\eta_1,\eta_2,\eta_3,\eta_4), \psi(x,t,\varepsilon) = \left(\frac{t}{\varepsilon^2},\frac{t}{\varepsilon},\frac{i[\theta(t)-\theta(0)]}{\varepsilon},\frac{\varphi(x)}{\varepsilon},\frac{\varphi(x)}{\varepsilon^2}\right),$$
$$\varphi(x) = (\varphi_1(x_1),\varphi_2(x_1),\varphi_3(x_2),\varphi_4(x_2))$$

such that its restriction by regularizing variables coincides with the desired solution:

$$\tilde{u}(M,\varepsilon)|_{\mu=\psi(x,t,\varepsilon)} \equiv u(x,t,\varepsilon), \quad (3)$$

Taking into account (2) and (3), we find the derivatives:

$$\partial_t u \equiv \left(\partial_t\tilde{u} + \frac{1}{\varepsilon}\partial_\mu\tilde{u} + \frac{1}{\varepsilon^2}\partial_{\tau_1}\tilde{u} + \frac{i\theta'(t)}{\varepsilon}\partial_{\tau_2}\tilde{u} + exp(\tau_2)\,\partial_\sigma\tilde{u}\right)|_{\chi=\psi(x,t,\varepsilon)},$$
$$\partial_{x_r}u \equiv \left(\partial_{x_r}\tilde{u} + \sum_{l=2r-1}^{2r}\left[\frac{\varphi_l'(x_r)}{\sqrt{\varepsilon^3}}\partial_{\xi_l}\tilde{u} + \frac{\varphi_l'(x_r)}{\varepsilon^2}\partial_{\zeta_l}\tilde{u}\right]\right)|_{\chi=\psi(x,t,\varepsilon)},$$
$$\partial_{x_r}^2 u \equiv \left(\partial_{x_r}^2\tilde{u} + \sum_{l=2r-1}^{2r}\left[\frac{\varphi_l'^2(x_r)}{\varepsilon^3}\partial_{\xi_l}^2\tilde{u} + \frac{\varphi_l'^2(x_r)}{\varepsilon^4}\partial_{\zeta_l}^2\tilde{u}\right]\right.$$
$$+\sum_{l=2r-1}^{2r}\left[\frac{2\varphi_l'(x_r)}{\sqrt{\varepsilon^3}}\partial_{x_r\xi_l}^2\tilde{u} + \frac{\varphi_l''(x_r)}{\sqrt{\varepsilon^3}}\partial_{\xi_l}\tilde{u}\right.$$
$$\left.\left.+\frac{1}{\varepsilon^2}\left(\varphi_l'(x_r)\partial_{x_r\eta_l}^2\tilde{u} + \varphi_l''(x_r)\partial_{\eta_l}\tilde{u}\right)\right]\right)|_{\chi=\psi(x,t,\varepsilon)}, \quad (4)$$

Below it is shown that the solution of the iterative problems does not contain terms depending on $(\xi_1,\xi_2), (\xi_3,\xi_4), (\zeta_1,\zeta_2), (\zeta_3,\zeta_4), (\xi_l,\zeta_k)$, $l,k = 1,2$. Therefore, to simplify the expression, the mixed derivatives of these variables are omitted. Based on (1), (3), (4), for extended function $\tilde{u}(M,\varepsilon)$ set the problem:

$$\tilde{L}_\varepsilon\tilde{u} \equiv \frac{1}{\varepsilon^2}T_0\tilde{u} + \frac{1}{\varepsilon}i\theta'(t)\partial_{\tau_2}\tilde{u} + \frac{1}{\varepsilon}T_1\tilde{u} + D_\sigma\tilde{u} - L_\eta\tilde{u} - \sqrt{\varepsilon}L_\xi\tilde{u} - \varepsilon^2\varDelta_a\tilde{u} = f(x,t)exp\left(\tau_2 + \frac{i\theta(0)}{\varepsilon}\right), \quad (5)$$

$$\tilde{u}|_{t=\tau_1=\tau_2=0} = 0,\qquad \tilde{u}|_{x_l=r-1,\xi_k=\eta_k=0} = 0,\qquad r = 1,2,\qquad l = 1,2,\qquad k = \overline{1,4}.$$

$$T_0 \equiv \partial_{\tau_1} - \varDelta_\eta,\qquad T_1 \equiv \partial_t - \varDelta_\xi,\ D_\sigma \equiv D_t + ex\,p(\tau_2)\,\partial_\sigma, D_t \equiv \partial_\mu - b(x,t), L_\eta \equiv \sum_{r=1}^{2}\sum_{l=2r-1}^{2r}a_r(x_r)D_{x,\eta}^{r,l},$$

$$D_{x,\xi}^{r,l} \equiv \left[2\varphi_l'(x_r)\partial_{x_r\xi_l}^2 + \varphi_l''(x_r)\partial_{\eta_l}\right], \Delta_\eta \equiv \sum_{k=1}^{4} \partial_{\eta_k}^2,\ E_1 = E\ x\,(0,\infty)^{10}.$$

The problem (5) is regular in ε as ε → 0.

$$(\tilde{L}_\varepsilon \tilde{u})|_{\chi=\psi(x,t,\varepsilon)} \equiv L_\varepsilon u(x,t,\varepsilon). \tag{6}$$

*2.3. Solution of Iterative Problems*

The solution of problem (5) is determined in the form of a series:

$$\tilde{u}(M,\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^{\frac{i}{2}} u_i(M), \tag{7}$$

For the coefficients of this series, the following iterative problems are obtained:

$$\begin{aligned}
&T_0 u_v(M) = 0, v = 0,1, T_0 u_q = -i\theta'(t)\partial_{\tau_2} u_{q-2} - T_1 u_{q-2}, q = 2,3.\\
&T_0 u_4 = f(x,t) exp\left(\tau_2 + \frac{i\theta(0)}{\varepsilon}\right) - T_1 u_2 - D_\sigma u_0 + L_\eta u_0,\\
&T_0 u_i = -i\theta'(t)\partial_{\tau_2} u_{i-2} - T_1 u_{i-2} - D_\sigma u_{i-4} + L_\eta u_{i-4} + L_\xi u_{i-5} + \Delta_a u_{i-8},\\
&u_i|_{t=\tau=0} = 0, u_i|_{x_l=r-1,\xi_k=\eta_k=0} = 0, l,r = 1,2.\ k = \overline{1,4},
\end{aligned} \tag{8}$$

We introduce a class of functions in which the iterative problems are solved:

$$\begin{aligned}
&U_0 = \{V_0(N) = [c(x,t) + F_1(N) + F_2(N)]ex\,p(\tau_2), F_1(N) \in U_4, F_2(N) \in U_5, c(x,t) \in C^\infty(\bar{E})\},\\
&U_1 = \{V_1(M): V_1(M) = v(x,t) + F_1(M) + F_2(M), F_1(M) \in U_4, F_2(M) \in U_5, v(x,t) \in C^\infty(\bar{E})\},\\
&U_2 = \{V_2(M): V_2(M) = [z(x,t) + F_1(M) + F_2(M)]\sigma, F_1(M) \in U_4, F_2(M) \in U_5, z(x,t) \in C^\infty(\bar{E})\},\\
&U_4 = \left\{V_4(M): V_1(M) = \sum_{l=1}^{4} Y^l(N_l), \quad |Y^l(N_l)| < cexp\left(-\frac{\eta_l^2}{8\tau_1}\right)\right\},\\
&U_5 = \left\{V_5(M): V_2(M)\right.\\
&\qquad = \left.\sum_{r,l=1}^{4} Y^{r+2,l}(N_{r+2,l}), \quad |Y^{r+2,l}(N_{r+2,l})| < cexp\left(-\frac{|\eta^{r,l}|^2}{8\tau_1}\right), |\eta^{r,l}| = \sqrt{\eta_r^2 + \eta_l^2}\right\},
\end{aligned}$$

From these spaces, a new space is constructed:

$$U = U_0 \oplus U_1 \oplus U_2.$$

The element $u(M)\epsilon\text{U}$ has the form:

$$\begin{aligned}
u(M) &= v(x,t) + c(x,t)ex\,p(\tau_2) + z(x,t)\sigma\\
&+ \left[\sum_{l=1}^{4} Y^l(N_l) + \sum_{r,l=1}^{2} Y^{r+2,l}(N_{r+2,l})\right] ex\,p(\tau_2) + \sum_{l=1}^{4} w^l(x,t) erfc\left(\frac{\xi_l}{2\sqrt{\mu}}\right)\\
&+ \sum_{l,r=1}^{2} w^{r+2,l}(M_{r+2,l}) + \left[\sum_{l=1}^{4} q^l(x,t) erfc\left(\frac{\xi_l}{2\sqrt{\mu}}\right) + \sum_{l,r=1}^{2} z^{r+2,l}(M_{r+2,l})\right]\sigma,\\
N_l &= (x,t,\tau_1,\eta_l), N_{r+2,l} = (x,t,\tau_1,\eta_l,\eta_{r+2}), M_l = (x,t,\mu,\xi_l), M_{r+2,l} = (x,t,\mu,\xi_l,\xi_{r+2}).
\end{aligned} \tag{9}$$

To satisfy this function to the boundary conditions:

$$
\begin{aligned}
& v(x,0) = -c(x,0), Y^l(N_l)|_{t=\tau_1=0} = 0, Y^{r+2,l}(N_{r+2,l})|_{t=\tau_1=0} = 0, w^l|_{t=0} = \bar{w}^l(x),\\
& q^l|_{t=0} = \bar{q}^l(x), w^{r+2,l}(M_{r+2,l})|_{t=\mu=0} = 0, z^{r+2,l}(M_{r+2,l})|_{t=\mu=0} = 0,\\
& w^l(x,t)|_{x_1=l-1} = -v(l-1,x_2,t), q^l(x,t)|_{x_1=l-1} = -z(l-1,x_2,t),\\
& \\
& Y^l|_{x_1=l-1,\eta_l=0} = -c(l-1,x_2,t), Y^{r+2,l}|_{x_1=l-1,\eta_l=0} = -Y^{r+2,l}(N_{r+2,l})|_{x_1=l-1},\\
& w^{r+2,l}|_{x_1=l-1,\xi_l=0} = -w^{r+2}(l-1,x_2,t)erfc\left(\frac{\xi_{r+2}}{2\sqrt{t}}\right), z^{r+2,l}|_{x_1=l-1,\xi_l=0}\\
& \qquad\qquad = -q^{r+2}(l-1,x_2,t)erfc\left(\frac{\xi_{r+2}}{2\sqrt{t}}\right),\\
& w^l(x,t)|_{x_r=l-1} = -v(x,t)|_{x_r=l-1}, q^l(x,t)|_{x_r=l-1} = -z(x,t)|_{x_r=l-1},\\
& Y^{r+2}|_{x_2=l-1,\eta_{r+2}=0} = -c(x_1,l-1,t), Y^{r+2,l}|_{x_2=l-1,\eta_{r+2}=0} = -Y^{l}|_{x_2=l-1},\\
& w^{r+2,l}|_{x_2=l-1,\xi_{r+2}=0} = -w^l|_{x_2=l-1}erfc\left(\frac{\xi_l}{2\sqrt{t}}\right),\\
& z^{r+2,l}|_{x_2=l-1,\xi_{r+2}=0} = -q^l|_{x_2=l-1}erfc\left(\frac{\xi_l}{2\sqrt{t}}\right), l,r = 1,2.
\end{aligned}
\tag{10}
$$

The action of the operators $T_0, T_1, L_\eta, L_\xi$ on function $u(M) \in U$ is computed

$$
\begin{aligned}
T_1 u(M) &= \sum_{r,l=1}^{2} \{\partial_\mu w^{r+2,l} - \Delta_\xi w^{r+2,l} + \sigma[\partial_\mu z^{r+2,l} - \Delta_\xi z^{r+2,l}]\},\\
L_\eta u &= \sum_{r=1}^{2}\sum_{l=2r-1}^{2r} D_{x,\eta}^{r,l}\, Y^l(N_l) + \sum_{v=1}^{2}\sum_{r,l=1}^{2} D_{x,\eta}^{v,l}\, Y^{r+2,l}(N_{r+2,l}),\\
L_\xi u &= \sum_{r=1}^{2}\sum_{l=2r-1}^{2r} D_{x,\xi}^{r,l}\, w^l(x,t)erfc\left(\frac{\xi_l}{2\sqrt{\mu}}\right) + \sum_{v=1}^{2}\sum_{r,l=1}^{2} D_{x,\xi}^{v,l}\, w^{r+2,l}(M_{r+2,l})\\
&\quad + \sigma\left[\sum_{r=1}^{2}\sum_{l=2r-1}^{2r} D_{x,\xi}^{r,l}\, q^l(x,t)erfc\left(\frac{\xi_l}{2\sqrt{\mu}}\right) + \sum_{v=1}^{2}\sum_{r,l=1}^{2} D_{x,\xi}^{v,l}\, z^{r+2,l}(M_{r+2,l})\right],\\
D_\sigma u(M) &= D_t v(x,t) + \sum_{l=1}^{4} D_t\, w^l(x,t)erfc\left(\frac{\xi_l}{2\sqrt{\mu}}\right)\\
&\quad + \sum_{r,l=1}^{2} D_t w^{r+2,l}(M_{r+2,l}) + \left[D_t c(x,t) + \sum_{l=1}^{4} D_t\, Y^l(N_l) + \sum_{r,l=1}^{2} D_t Y^{r+2,l}(N_{r+2,l})\right] exp(\tau_2)\\
&\quad + \sigma\left[D_t z(x,t) + \sum_{l=1}^{4} D_t\, q^l(x,t)erfc\left(\frac{\xi_l}{2\sqrt{\mu}}\right) + \sum_{r,l=1}^{2} D_t z^{r+2,l}(M_{r+2,l})\right]\\
&\quad + \left[z(x,t) + \sum_{l=1}^{4} q^l\,(x,t)erfc\left(\frac{\xi_l}{2\sqrt{\mu}}\right) + \sum_{r,l=1}^{2} z^{r+2,l}(M_{r+2,l})\right] exp(\tau_2).
\end{aligned}
$$

The iterative equations (8) are written in the form:

$$T_0 u(M) = H(M) \tag{11}$$

**Theorem 1.** Let $H(M) \in U_4 \oplus U_5$ and condition 1) is satisfied. Then the equation (11) is solvable in U, if the equations are solvable:

$$T_0 Y^l(N_l) = H_1(N_l), l = \overline{1,4}, T_0 Y^{r+2,l}(N_{r+2,l}) = H_2(N_{r+2,l}), r, l = 1,2.$$

**Theorem 2.** Let $H_1(N_l) \in U_4$. Then the problem:

$$\partial_{\tau_1} Y^l(N_l) = \Delta_\eta Y^l(N_l) + H_1(N_l), Y^l(N_l)|_{\tau_1=0} = 0, Y^l(N_l)|_{\eta_l=0} = d^l(x,t), l = \overline{1,4} \tag{12}$$

has a solution $Y^l(N_l) \in U_4$.

**Theorem 3.** Let be $H_2(N_{r+2,l}) \in U_5, Y^l(N_l) \in U_4$, then the problem $\partial_{\tau_1} Y^{r+2,l}(N_{r+2,l}) = \Delta_\eta Y^{r+2,l}(N_{r+2,l}) + H_2(N_{r+2,l}), Y^{r+2,l}(N_{r+2,l})|_{\eta_l=0} = -Y^{r+2}(N_{r+2}), Y^{r+2,l}(N_{r+2,l})|_{\eta_{r+2}=0} = -Y^l(N_l),\ r, l = 1,2$ has a solution $Y^{r+2,l}(N_{r+2,l}) \in U_5$.

The proof of these theorems is given in [7].

The equation (8) is homogeneous under $v = 0, 1$. By Theorem 1, it has a solution representable in the form $u_0(M) \in U$ if functions $Y^l(N_l)$ and $Y^{r+2,l}(N_{r+2,l})$ are solutions of the following equations:

$$T_0 Y_v^l(N_l) = 0, T_0 Y_v^{r+2,l}(N_{r+2,l}) = 0.$$

Based on the boundary conditions from (10), the solution is written as:

$$Y_v^l(N_l) = d_v^l(x,t) erfc\left(\frac{\eta_l}{2\sqrt{\tau_1}}\right), l = 1,2,3,4,$$
$$Y_v^{r+2,l}(N_{r+2,l}) = -\int_0^{\tau_1}\int_0^{\infty} Y_v^l(*)\left[\frac{\partial}{\partial_\xi} G(N_l, \xi, \eta, \tau_1 - \tau)\right]|_{\xi=0} d\eta d\tau -$$
$$\int_0^t \int_0^\infty Y_v^{r+2}(*)\left[\frac{\partial}{\partial_\eta} G(N_{r+2,l}, \xi, \eta, \tau_1 - \tau)\right]|_{\eta=0} d\xi d\tau,$$

where $d^l(x,t)$ is an arbitrary function such as:

$$d_v^p(x,t)|_{t=0} = -\bar{d}_v^p(x), d_v^l(x,t)|_{x_1=l-1} = -c_v(l-1, x_2, t),$$
$$G(\eta_l, \eta_{r+2,l}, \xi, \eta, \tau_1) = \frac{1}{4\pi\tau_1}\left\{exp\left(-\frac{(\eta_l-\xi)^2}{4\tau_1}\right) - exp\left(-\frac{(\eta_l+\xi)^2}{4\tau_1}\right)\right\}\left\{exp\left(-\frac{(\eta_{r+2}-\eta)^2}{4\tau_1}\right) - exp\left(-\frac{(\eta_{r+2}+\eta)^2}{4\tau_1}\right)\right\}, \tag{13}$$

The function $d_v^l(x,t)$ is equal to zero under $t = \tau_1 = 0$, that is, $d_0^l(x,t)|_{t=0} = -\bar{d}_0^l(x)$. Thus, $-\bar{d}_0^l(x)$ can be taken as an arbitrary function and its values under $x_1 = l-1$ is determined from the second relation.
According to Theorem 2 and Theorem 3, the functions found by the equation (13) satisfy the estimates:

$$|Y_v^l(N_l)| < c\,exp\left(-\frac{\eta_l^2}{8\tau_1}\right), |Y_v^{r+2,l}(N_{r+2,l})| < c\,exp\left(-\frac{\eta_{r+2}^2+\eta_l^2}{8\tau_1}\right), r, l = 1,2. \tag{14}$$

Free member of equations (8) under $v = 2,3$ has a form:

$$F_{v-2}(M) \equiv T_1 u_{v-2}(M) + i\theta'(t)\partial_\sigma u_{v-2}(M)$$
$$= i\theta'(t)\left[c_{v-2}(x,t) + \sum_{l=1}^{4} Y_{v-2}^{l}(N_l) + \sum_{r,l=1}^{2} Y_{v-2}^{r+2,l}(N_{r+2,l})\right] exp(\tau_2)$$
$$+ \sum_{l,r=1}^{2}\{\partial_\mu w_{v-2}^{r+2,l} - \Delta_\xi w_{v-2}^{r+2,l} + \sigma[\partial_\mu z_{v-2}^{r+2,l} - \Delta_\xi z_{v-2}^{r+2,l}]\}$$

So, that the equation (8), under $v = 2,3$ has a solution in $U$, we set:

$$c_{v-2}(x,t) = 0, T_1 w_{v-2}^{r+2,l} = 0, T_1 z_{v-2}^{r+2,l} = 0.$$

Solutions of last equations under the boundary conditions from (10) has a form (12) for which estimates of the form (15) is satisfied. For $i=4$, the equation (8) has a free term:

$$F_4(M) = -i\theta'(t)\partial_{\tau_2} - T_1 u_2 + f(x,t)exp\left(\frac{i\theta(0)}{\varepsilon}\right) - D_\sigma u_0 + L_\eta u_0$$
$$= -i\theta'(t)\left[c_2(x,t) + \sum_{l=1}^{4} Y_2^l(N_l) + \sum_{r,l=1}^{2} Y_2^{r+2,l}(N_{r+2,l})\right] exp(\tau_2)$$
$$- \sum_{l,r=1}^{2}[T_0 w_2^{r+2,l}(M_{r+2,l}) + \sigma T_0 z_2^{r+2,l}] - D_t v_0(x,t)$$
$$- \sum_{l=1}^{4} D_t w_0^l(x,t) erfc\left(\frac{\xi_l}{2\sqrt{\mu}}\right)$$
$$- \sum_{l,r=1}^{2} D_t w_0^{r+2,l}(x,t) - exp(\tau_2)\left[\partial_t c_0(x,t) + \sum_{l=1}^{4}\partial_t Y_0^l + \sum_{l,r=1}^{2} D_t Y_0^{r+2,l}\right]$$
$$- \sigma\left[D_t z_0(x,t) + \sum_{l=1}^{4} D_t q_0^l(x,t) erfc\left(\frac{\xi_l}{2\sqrt{\mu}}\right) + \sum_{r,l=1}^{2} D_t z_0^{r+2,l}(M_{r+2,l})\right]$$
$$- \left[z_0(x,t) + \sum_{l=1}^{4} q_0^l(x,t) erfc\left(\frac{\xi_l}{2\sqrt{\mu}}\right) + \sum_{r,l=1}^{2} z_0^{r+2,l}(M_{r+2,l})\right] exp(\tau_2)$$
$$+ \sum_{r=1}^{2}\sum_{l=2r-1}^{2r} D_{x,\xi}^{r,l} w_0^p(x,t) erfc\left(\frac{\xi_l}{2\sqrt{\mu}}\right) + \sum_{v=1}^{2}\sum_{r,l=1}^{2} D_{x,\eta}^{v,l} Y_0^{r+2,l}(N_{r+2,l}).$$

Provided that $F_4(M) \in U_4 \oplus U_5$ with regard to $c_v(x,t) = 0, v = 0,1$ we set:

$$-i\theta'(t)c_2(x,t) + f(x,t)exp\left(\frac{i\theta(0)}{\varepsilon}\right) - z_0(x,t) = 0, \tag{15}$$
$$D_t v_0(x,t) = 0, D_t z_0(x,t) = 0, D_t Y_0^l(N_l), T_0 w_2^{r+2,l} = 0, T_0 z_2^{r+2,l} = 0,$$
$$D_t w_0^l = 0, D_t w_0^{r+2,l} = 0,\ D_t Y_0^{r+2,l} = 0, D_t q_0^l\,(x,t) = 0, D_t z_0^{r+2,l}\,(x,t) = 0, D_{x,\xi}^{r,l} w_0^l(x,t),$$
$$= 0,\ D_{x,\eta}^{v,l} Y_0^{r+2,l} = 0,\ D_{x,\eta}^{r,l} Y_0^l = 0,$$

then

$$F_4(M) = -i\theta'(t)\left[\sum_{l=1}^{4} Y_2^l(N_l) + \sum_{r,l=1}^{2} Y_2^{r+2,l}(N_{r+2,l})\right] exp(\tau_2)$$
$$- \left[\sum_{l=1}^{4} q_0^l(x,t) erfc\left(\frac{\eta_l}{2\sqrt{\tau_2}}\right) + \sum_{r,l=1}^{2} z_0^{r+2,l}(N_{r+2,l})\right] exp(\tau_2).$$

In the last equation, the transition from the variables $\frac{\xi_l}{2\sqrt{\mu}}$ to the variables $\frac{\eta_l}{2\sqrt{\tau_2}}$ is occured.
Substitute the value $Y_0^l(N_l) = d_0^l(x,t)erfc\left(\frac{\eta_l}{2\sqrt{\tau_1}}\right)$ into the equation $D_t Y_0^l(N_l) = 0$, with respect to $d_0^l(x,t)$ we get the equation $D_t d_0^l(x,t) = 0$, which is solved under an arbitrary initial condition $d_0^l(x,t)|_{t=0} = \bar{d}_0^l(x)$ . This arbitrary function provides the condition $L_\eta\, Y_0^l = 0$, therefore $D_{x,\eta}\, Y_0^l = 0$. Initial condition for this equation is determined from the relation:

$$d_0^l(x,t)|_{x_1=l-1} = -c_0(l-1, x_2, t), d_0^{l+2}(x,t)|_{x_2=l-1} = -c_0(x_1, l-1, t),$$

which is due to (10) and (13). The function $Y_0^{r+2,l}(N_{r+2,l})$ expresses through $Y_0^l(N_l)$ therefore provided that

$$D_t Y_0^{r+2,l} = 0,\ D_{x,\eta}^{v,l} Y_0^{r+2,l} = 0.$$

The same is true for functions: $w_0^{r+2,l}(M_{r+2,l})$, $z_0^{r+2,l}(M_{r+2,l})$. In other words, the following relations hold: $D_t w_0^{r+2,l} = 0, D_t z_0^{r+2,l} = 0, D_{x,\xi}^{v,l} w_0^{r+2,l} = 0,\ D_{x,\xi}^{v,l} z_0^{r+2,l} = 0$.

Solutions of equations with respect to $w_0^{r+2,l}$, $z_0^{r+2,l}$ under appropriate boundary conditions from (10) are representable as (13) and they are expressed through $w_2^l(x,t)$, $q_2^l(x,t)$. The first equation (16) is solvable if $z_0(x,t)|_{t=0} = f(x,0)exp\left(\frac{i\theta(0)}{\varepsilon}\right)$. This ratio is used by the initial condition for the equation $D_t z_0(x,t) = 0$. The remaining equations from (16) are solvable under the initial conditions from (10).

Thus, the main term of the asymptotics is uniquely determined. As can be seen from the representation (9) and the estimates (15), we note that the asymptotics of the solution has a complex structure. In addition to regular members, it contains various boundary layer functions. The boundary layer functions have rapidly oscillating exponential and power type of change of:

$$c(x,t)ex\,p(\tau_2)\,, \sigma = \int_0^t e^{\frac{i[\theta(s)-\theta(0)]}{\varepsilon}} ds.$$

Parabolic boundary layer functions have an estimate:

$$|Y^l(N_l)| < cexp\left(-\frac{\eta_l^2}{8\tau_1}\right), \left|w^l(x,t)erfc\left(\frac{\xi_l}{2\sqrt{\mu}}\right)\right| < cexp\left(-\frac{\xi_l^2}{8\mu}\right).$$

Multidimensional and angular parabolic boundary layer functions have an estimate:

$$\left|Y^{r+2,l}(N_{r+2,l})\right| < cexp\left(-\frac{\eta_{r+2}^2+\eta_l^2}{8\tau_1}\right),$$
$$\left|w^{r+2,l}(M_{r+2,l})\right| < cexp\left(-\frac{\xi_{r+2}^2+\xi_l^2}{8\mu}\right).$$

In addition, the asymptotic contains the product of the above-mentioned boundary layer functions.
Repeating the above process, the partial sum is constructed:

$$\tilde{u}_{\varepsilon n}(M) = \sum_{i=0}^{n} \varepsilon^{\frac{i}{2}} u_i(M) \qquad (16)$$

### 3. Assessment of remainder

Substituting the function $\tilde{u}(M,\varepsilon) = u_{\varepsilon n}(M) + \varepsilon^{n+\frac{1}{2}} R_{\varepsilon n}(M)$ into problem (5), and taking into account the iterative problems of (8), (10), the following problem is obtained for the remainder term:

$$R_{\varepsilon n}(M)\colon \tilde{L}_\varepsilon R_{\varepsilon n}(M) = g_n(M,\varepsilon),\\ R_{\varepsilon n}(M)|_{t=0} = R_{\varepsilon n}(M)|_{x_l=r-1,\xi_r=0,\eta_k} = 0,\\ r = 1,2;\ k = \overline{1,4}, \tag{17}$$

where

$$g_n(M,\varepsilon) = -i\theta'(t)\partial_{\tau_2}u_{n-1} - \varepsilon^{\frac{1}{2}}i\theta'(t)\partial_{\tau_2}u_n(M)$$
$$-T_1u_{n-1}(M) - \varepsilon^{\frac{1}{2}}T_1u_n(M)\left(D_\sigma - L_\eta\right)\sum_{k=0}^{3}\varepsilon^{\frac{k}{2}}u_{n-3+k}(M) + L_\eta\sum_{k=0}^{5}\varepsilon^{\frac{k}{2}}u_{n-5+k}(M) + \Delta_a\sum_{k=0}^{7}\varepsilon^{\frac{k}{2}}u_{n-7+k}(M).$$

We put in both parts (18) $\chi = \psi(x,t,\varepsilon)$ considering (6), with respect to:

$$L_\varepsilon R_{\varepsilon n}(x,t,\varepsilon) = g_{\varepsilon n}(x,t,\varepsilon), R_{\varepsilon n}|_{t=0} = 0, R_{\varepsilon n}|_{\partial\Omega=0}.$$

By virtue of the above constructions, the function is $|g_{\varepsilon n}(x,t,\varepsilon)| < c, \forall(x,t) \in \overline{E}$,
Therefore, applying the maximum principle, an estimate is established:

$$|R_{\varepsilon n}(x,t,\varepsilon)| < c.$$

Thus, the following has been proved:

**Theorem 5.** Suppose that the conditions 1) -3) are satisfied. Then, using the method, presented above, for solving $u(x,t,\varepsilon)$ of the problem (1) a regularized series (7) can be constructed such that $\forall\ n = 0,1,2,\ldots$ and for small enough $\varepsilon > 0$ inequality holds.

$$|u(x,t,\varepsilon) - u_{\varepsilon n}(x,t,\varepsilon)| = |R_{\varepsilon n}\ (x,t,\varepsilon)| < c\varepsilon^{n+\frac{1}{2}},$$

where $c$ is independent of $\varepsilon$.

## 4. Conclusion

It is shown that the asymptotic solution of the problem has a complex structure. In addition to regular members, it contains various boundary layer functions. The boundary layer functions have rapidly oscillating exponential and power type of change of:

$$c(x,t)ex\,p(\tau_2)\,, \sigma = \int_0^t e^{\frac{i[\theta(s)-\theta(0)]}{\varepsilon}}ds.$$

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