International Electronic Journal of Geometry
Volume 1 No. 1 pp. 12-14 (2008) ©IEJG

POSITIVITY OF c_3 FOR SPANNED REFLEXIVE SHEAVES

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(Communicated by H. Hilmi HACISALIHOGLU)

Abstract. Let X be an n-dimensional projective manifold, $n \geq 3$, and F a sheaf on X spanned outside a codimension ≥ 3 subset and with depth at least $n-1$. Here we prove that $c_3(F|Y) \geq 0$ for a sufficiently general 3-fold $Y \subseteq X$ and that if $c_3(F|Y) = 0$, then F is locally free.

Let X be a smooth and connected projective variety. Set $n := \dim(X)$. Let $A^i(X)$, $i \geq 0$, denote the i-th homology or cohomology group of X either for rational equivalence or for algebraic equivalence or for numerical equivalence [2]. Hence $A^0(X) \cong A^n(X) \cong \mathbb{Z}$ and $A^i(X) = 0$ for all $i > n$. Let F be any coherent sheaf on X. The smoothness of X implies that F has a finite resolution by locally free coherent sheaves on X. As in [5], §2, this is sufficient to define the Chern classes $c_i(F)$, $0 \leq i \leq n$, which behave for short exact sequences of coherent sheaves as do the classical Chern classes for exact sequences of vector bundles. Fix integers i, c such that $1 \leq i \leq n-1$, $\alpha \in A^i(X)$ and a connected i-dimensional smooth subvariety $S \subset X$. Since $A^i(S) \cong \mathbb{Z}$ we may identify $\alpha|S$ with an integer. Notice that in the identification of $A^i(S)$ with $\mathbb{Z}$, there is a unique positive generator of $A^i(S)$. Hence we may do the identification in such a way that $\alpha|S$ is a well-defined integer, not just an integer up to its sign. Hence it make sense to say that $\alpha|S$ is non-negative or that it is positive or that it is at least c. Now assume $F \neq 0$. We recall that depth$(F) \leq n$, that depth$(F) = n$ if and only if F is locally free, that depth$(F) > 0$ if F is torsion free and that depth$(F) \geq 2$ if and only if F is reflexive, i.e. if and only if the natural map $F \to F^{**}$ is an isomorphism ([5], Prop. 1.3).

Here we prove the following result.

Theorem 0.1. *Assume $n \geq 3$. Let F be a rank r torsion free sheaf on X such that depth$(F) \geq n-1$ outside a codimension 4 subscheme of Y. Assume that the evaluation map $H^0(X, F) \otimes \mathcal{O}_X \to F$ is surjective outside a closed subset B of X with codimension at least 3. Fix a very ample linear system V on X. Let Y be a general intersection of $n-3$ members of V. Then $c_3(F|Y) \geq 0$. If $c_3(F|Y) = 0$, then $F|Y$ is locally free and F is locally free outside an algebraic subset of X with codimension at least 4.*

2000 *Mathematics Subject Classification.* 14H60.

Key words and phrases. Chern classes; reflexive sheaves; depth; spanned sheaf.
The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

Proof. We will first do the case $n = 3$.

(a) Assume $n = 3$ and write $Y := X$. If $r = 1$, then F is a line bundle, because Y is assumed to be locally factorial ([5], Prop. 1.9). Hence we may assume $r \geq 2$. The proof of [5], Prop. 2.6, works taking Y instead of $\mathbf{P}^3$ and shows that $c_3(G) \geq 0$ for every rank 2 reflexive sheaf G with equality if and only if G is locally free. Hence we may assume $r \geq 3$. We will adapt the proof of [1], Th. 5.3. Let B the set of non-locally free points of F. Hence B is finite. Since F is reflexive and Y is smooth, $\det(F)$ is a line bundle ([5]). Since F is spanned outside finitely many points, there is an exact sequence

$$0 \to \mathcal{O}_Y^{r-1} \to F \to \mathcal{I}_C \otimes \det(L) \to 0 \qquad (0.1)$$

in which C is a reduced pure one-dimensional closed subscheme of Y which is locally complete intersection outside finitely many points ([6], §2). Appling the functor $Hom(\ -\ , \mathcal{O}_Y)$ to (0.1) we get the following exact sequence:

$$0 \to \det(F)^* \to F^* \to \mathcal{O}_Y^{r-1} \to Ext^1(\mathcal{I}_Y \otimes \det(F), \mathcal{O}_Y) \to Ext^1(F, \mathcal{O}_Y) \to 0 \qquad (0.2)$$

We have $Ext^1(\mathcal{I}_Y \otimes \det(F), \mathcal{O}_Y) \cong \omega_C \otimes (\omega_Y \otimes \det(F)|C)$ ([3], III.7.5). The sheaf $Ext^1(F, \mathcal{O}_Y)$ is supported by the finite set B. Hence $Ext^1(\mathcal{I}_Y \otimes \det(F), \mathcal{O}_Y)$ is generically spanned. Since C is reduced, the $\mathcal{O}_C$-sheaf $\omega_C \otimes (\omega_Y \otimes \det(F)|C)$ as rank 1 on each ireducible component of C. Hence $\omega_C \otimes (\omega_Y \otimes \det(F)|C)$ is genericallly spanned by one section. Such a section and the isomorphism $Ext^1(\mathcal{I}_Y \otimes \det(F), \mathcal{O}_Y) \cong \omega_C \otimes (\omega_Y \otimes \det(F)|C)$ gives a non-trivial extension

$$0 \to \mathcal{O}_Y \to A \to \mathcal{I}_C \otimes \det(F) \to 0 \qquad (0.3)$$

with A reflexive. Since A has rank 2 we saw at the beginning of the proof that $c_3(A) \geq 0$ and that $c_3(A) = 0$ if and only if A is locally free. The extension (0.3) gives $c_3(A) = c_3(\mathcal{I}_C \otimes \det(F))$. The extension (0.1) gives $c_3(F) = c_3(\mathcal{I}_C \otimes \det(F))$. Hence $c_3(F) \geq 0$ and $c_3(F) = 0$ if and only if A is locally free. In this case C is a locally complete intersection and $\omega_C \equiv \det(F) \otimes \omega_Y$ ([5], Th. 4.1). As in [4] these two properties of C imply that F is locally free.

(b) Now assume $n > 3$. Bertini's theorem gives that Y is smooth, connected and 3-dimensional ([3], II.8.18 and III.7.9.1). Bertini's theorem shows that F has depth at least $n-1$ at each point of Y. By [7], Cor. 1.18, for each $P \in Y$ the $n-3$ local equations generating $\mathcal{I}_{Y,P}$ forms a regular sequence for the $\mathcal{O}_{X,P}$-module F_P. Hence for any $P \in Y$ the restriction to Y of a finite free resolution of F_P as an $\mathcal{O}_{X,P}$-module, is a finite $\mathcal{O}_{Y,P}$-resolution. Hence the reduction modulo $\mathcal{I}_{Y,P}$ of $F|Y$ has no torsion and $\operatorname{depth}(F|Y) \geq 2$. Hence $F|Y$ is reflexive. Bertini's theorem gives $\dim(B \cap Y) \leq 0$. Apply part (a) to $F|Y$. □

Remark 0.1. Fix a coherent sheaf F on X and $L \in \operatorname{Pic}(X)$. Set $r := \operatorname{rank}(F)$. As in [5], Lemma 2.1, we have $c_i(F \otimes L) = \sum_{j=0}^{i} c_j(F) \cdot L^{\times(i-j)}$ where $L^{\times(i-j)}$ denote the cup product. This formula was the key to find spanned reflexives sheaves on $\mathbf{P}^n$, $n \geq 4$, with $c_4(F) < 0$ and similarly for c_i with $5 \leq i \leq n$ ([1], Example 1.1).

We work over an algebraically closed field $\mathbb{K}$.

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