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A NEW CONNECTION IN A RIEMANNIAN MANIFOLD

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Abstract. In a Riemannian manifold, the existence of a new connection is proved. In particular cases, this connection reduces to several symmetric, semi-symmetric and quarter-symmetric connections; even some of them are not introduced so far. We also find formula for curvature tensor of this new connection.

1. Introduction

Let $\tilde{\nabla}$ be a linear connection in an n-dimensional differentiable manifold M. The torsion tensor $\tilde{T}$ and the curvature tensor $\tilde{R}$ of $\tilde{\nabla}$ are given respectively by

$$\tilde{T}(X,Y) \equiv \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y],$$
$$\tilde{R}(X,Y)Z \equiv \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z.$$

The connection $\tilde{\nabla}$ is symmetric if its torsion tensor $\tilde{T}$ vanishes, otherwise it is non-symmetric. The connection $\tilde{\nabla}$ is a metric connection if there is a Riemannian metric g in M such that $\tilde{\nabla} g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection. In 1973, B. G. Schmidt [14] proved that if the holonomy group of $\tilde{\nabla}$ is a subgroup of the orthogonal group $\mathcal{O}(n)$, then $\tilde{\nabla}$ is the Levi-Civita connection of a Riemannian metric.

In 1930, H. A. Hayden [7] introduced a metric connection $\tilde{\nabla}$ with a non-zero torsion on a Riemannian manifold. Such a connection is called a Hayden connection. On the other hand, in a Riemannian manifold given a 1-form ω, the *Weyl connection* $\tilde{\nabla}$ constructed with ω and its associated vector B (G. B. Folland 1970, [5]) is a symmetric non-metric connection. In fact, the Riemannian metric of the manifold is recurrent with respect to the Weyl connection with the recurrence 1-form ω, that is, $\tilde{\nabla} g = \omega \otimes g$. Another symmetric non-metric connection is projectively related to the Levi-Civita connection (cf. K. Yano [19], D. Smaranda [17]).

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In 1924, A. Friedmann and J. A. Schouten ([4], [15]) introduced the idea of a semi-symmetric linear connection in a differentiable manifold. A linear connection is said to be a semi-symmetric connection if its torsion tensor $\tilde{T}$ is of the form

$$\tilde{T}(X,Y) = u(Y)X - u(X)Y, \tag{1.1}$$

where u is a 1-form. A Hayden connection with the torsion tensor of the form (1.1) is a semi-symmetric metric connection, which appeared in a study of E. Pak [11]. In 1970, K. Yano [20] considered a semi-symmetric metric connection and studied some of its properties. He proved that a Riemannian manifold is conformally flat if and only if it admits a semi-symmetric metric connection whose curvature tensor vanishes identically. He also proved that a Riemannian manifold is of constant curvature if and only if it admits a semi-symmetric metric connection for which the manifold is a group manifold, where a group manifold [3] is a differentiable manifold admitting a linear connection $\tilde{\nabla}$ such that its curvature tensor $\tilde{R}$ vanishes and its torsion tensor $\tilde{T}$ is covariantly constant with respect to $\tilde{\nabla}$. In [18], L. Tamássy and T. Q. Binh proved that if in a Riemannian manifold of dimension ≥ 4, $\tilde{\nabla}$ is a metric linear connection of nonvanishing constant curvature for which

$$\tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y = 0,$$

then $\tilde{\nabla}$ is the Levi-Civita connection. Some different kind of semi-symmetric non-metric connections are studied in [1], [2], [8] and [16].

In 1975, S. Golab [6] defined and studied quarter-symmetric linear connections in differentiable manifolds. A linear connection is said to be a quarter-symmetric connection if its torsion tensor $\tilde{T}$ is of the form

$$\tilde{T}(X,Y) = u(Y)\varphi X - u(X)\varphi Y, \tag{1.2}$$

where u is a 1-form and φ is a tensor of type $(1,1)$. In 1980, R. S. Mishra and S. N. Pandey [9] studied quarter-symmetric metric connections and, in particular, Ricci quarter-symmetric metric connections. Note that a quarter-symmetric metric connection is a Hayden connection with the torsion tensor of the form (1.2). Studies of various types of quarter-symmetric metric connections and their properties include [9], [12], [13] and [21] among others.

Now we ask the following question. Can there be a unified theory of connections which unifies the concepts of various metric connections such as a semi-symmetric metric connection and a quarter-symmetric metric connection and various non-metric connections such as the Weyl connection and different kind of semi-symmetric non-metric connections? Surprisingly, we answer to this question in affirmative, and in this paper we prove the existence of a new connection, which unifies all these previously known connections and some other connections not introduced so far. We also find formula for curvature tensor of this new connection. In the last we list a number of connections as particular cases.

2. **A new connection**

In this section we prove the existence of a new connection in the following

Theorem 2.1. *Let M be an n-dimensional Riemannian manifold equipped with the Levi-Civita connection ∇ of its Riemannian metric g. Let f_1, f_2 be functions* *in M, u, u_1, u_2 are 1-forms in M and φ is a $(1,1)$ tensor field in M. Let*

$$u(X) \equiv g(U,X), \quad u_1(X) \equiv g(U_1,X), \quad u_2(X) \equiv g(U_2,X), \tag{2.1}$$

$$g(\varphi X,Y) \equiv \Phi(X,Y) = \Phi_1(X,Y) + \Phi_2(X,Y), \tag{2.2}$$

where Φ_1 and Φ_2 are symmetric and skew-symmetric parts of the $(0,2)$ tensor Φ such that

$$\Phi_1(X,Y) \equiv g(\varphi_1 X,Y), \quad \Phi_2(X,Y) \equiv g(\varphi_2 X,Y). \tag{2.3}$$

Then there exists a unique connection $\tilde{\nabla}$ in M given by

$$\begin{aligned} \tilde{\nabla}_X Y &= \nabla_X Y + u(Y)\varphi_1 X - u(X)\varphi_2 Y - g(\varphi_1 X,Y)U \\ &\quad - f_1\{u_1(X)Y + u_1(Y)X - g(X,Y)U_1\} \\ &\quad - f_2\, g(X,Y)U_2, \end{aligned} \tag{2.4}$$

which satisfies

$$\tilde{T}(X,Y) = u(Y)\varphi X - u(X)\varphi Y, \tag{2.5}$$

and

$$\begin{aligned} \left(\tilde{\nabla}_X g\right)(Y,Z) &= 2 f_1 u_1(X) g(Y,Z) \\ &\quad + f_2\left\{ u_2(Y) g(X,Z) + u_2(Z) g(X,Y)\right\}, \end{aligned} \tag{2.6}$$

where $\tilde{T}$ is the torsion tensor of $\tilde{\nabla}$.

Proof. Let $\tilde{\nabla}$ be a linear connection in M given by

$$\tilde{\nabla}_X Y = \nabla_X Y + H(X,Y). \tag{2.7}$$

Now, we shall determine the tensor field H such that $\tilde{\nabla}$ satisfies (2.5) and (2.6). From (2.7) we have

$$\tilde{T}(X,Y) = H(X,Y) - H(Y,X). \tag{2.8}$$

Denote

$$G(X,Y,Z) \equiv \left(\tilde{\nabla}_X g\right)(Y,Z). \tag{2.9}$$

From (2.7) and (2.9), we have

$$g(H(X,Y),Z) + g(H(X,Z),Y) = -G(X,Y,Z). \tag{2.10}$$

From (2.8), (2.10), (2.9) and (2.6), we have

$$\begin{aligned} & g\left(\tilde{T}(X,Y),Z\right) + g\left(\tilde{T}(Z,X),Y\right) + g\left(\tilde{T}(Z,Y),X\right) \\ = \;& g(H(X,Y),Z) - g(H(Y,X),Z) + g(H(Z,X),Y) \\ & - g(H(X,Z),Y) + g(H(Z,Y),X) - g(H(Y,Z),X) \\ = \;& g(H(X,Y),Z) - g(H(X,Z),Y) + G(Y,X,Z) - G(Z,X,Y) \\ = \;& 2g(H(X,Y),Z) + G(X,Y,Z) + G(Y,X,Z) - G(Z,X,Y) \\ = \;& 2g(H(X,Y),Z) \\ & + \{2f_1u_1(X)g(Y,Z) + f_2u_2(Y)g(X,Z) + f_2u_2(Z)g(X,Y)\} \\ & + \{2f_1u_1(Y)g(X,Z) + f_2u_2(X)g(Y,Z) + f_2u_2(Z)g(X,Y)\} \\ & - \{2f_1u_1(Z)g(X,Y) + f_2u_2(X)g(Y,Z) + f_2u_2(Y)g(X,Z)\}, \end{aligned}$$

which in view of (2.1) implies that

$$H(X,Y) = \frac{1}{2}\{\tilde{T}(X,Y) + \tilde{T}'(X,Y) + \tilde{T}'(Y,X)\} - f_1\{u_1(X)Y + u_1(Y)X - f_1 g(X,Y)U_1\} - f_2\, g(X,Y)U_2, \tag{2.11}$$

where

$$g(\tilde{T}'(X,Y),Z) = g\left(\tilde{T}(Z,X),Y\right). \tag{2.12}$$

Using (2.5), (2.1), (2.2) and (2.3) in (2.12), we get

$$\begin{aligned} g(\tilde{T}'(X,Y),Z) &= g(u(X)\varphi Z - u(Z)\varphi X, Y) \\ &= u(X)\Phi(Z,Y) - g(U,Z)\Phi(X,Y) \\ &= u(X)g(\varphi_1 Z,Y) + u(X)g(\varphi_2 Z,Y) \\ &\quad - g(U,Z)\Phi(X,Y), \end{aligned}$$

which implies that

$$\tilde{T}'(X,Y) = u(X)\varphi_1 Y - u(X)\varphi_2 Y - \Phi(X,Y)U. \tag{2.13}$$

In view of (2.5), (2.11) and (2.13), we get

$$H(X,Y) = u(Y)\varphi_1 X - u(X)\varphi_2 Y - g(\varphi_1 X,Y)U - f_1\{u_1(X)Y + u_1(Y)X - g(X,Y)U_1\} - f_2\, g(X,Y)U_2 \tag{2.14}$$

and hence, $\tilde{\nabla}$ is given by (2.4).

Conversely, a connection given by (2.4) satisfies the conditions (2.5) and (2.6). □

3. **Curvature tensor**

The curvature tensor $\tilde{R}$ of the connection $\tilde{\nabla}$ is given by

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y\tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z.$$

For a function h, in view of (2.4) we have

$$\begin{aligned} \tilde{\nabla}_X(hY) &= (Xh)Y + h\nabla_X Y \\ &\quad + hu(Y)\varphi_1 X - hu(X)\varphi_2 Y - hg(\varphi_1 X,Y)U \\ &\quad - f_1\{hu_1(X)Y + hu_1(Y)X - hg(X,Y)U_1\} \\ &\quad - f_2 h\, g(X,Y)U_2. \end{aligned} \tag{3.1}$$

Also from (2.4) it follows that

$$\begin{aligned} \tilde{\nabla}_X\tilde{\nabla}_Y Z &= \tilde{\nabla}_X\nabla_Y Z + \tilde{\nabla}_X(u(Z)\varphi_1 Y) - \tilde{\nabla}_X(u(Y)\varphi_2 Z) \\ &\quad - \tilde{\nabla}_X(g(\varphi_1 Y,Z)U) - \tilde{\nabla}_X(f_1 u_1(Y)Z) \\ &\quad - \tilde{\nabla}_X(f_1 u_1(Z)Y) + \tilde{\nabla}_X(f_1 g(Y,Z)U_1) \\ &\quad - \tilde{\nabla}_X(f_2\, g(Y,Z)U_2). \end{aligned} \tag{3.2}$$

We shall need the following definitions. Let η be a 1-form and ξ be its associated vector field such that

$$\eta(X) = g(\xi,X).$$

We define

$$\beta(\eta, X, Y) = (\nabla_X \eta)Y + u(X)\eta(\varphi_2 Y) - \eta(\varphi_1 X)u(Y) + \eta(U)g(\varphi_1 X, Y), \tag{3.3}$$

$$B(\eta, X) = \nabla_X \xi - u(X)\varphi_2 \xi - \eta(\varphi_1 X)U + \eta(U)\varphi_1 X, \tag{3.4}$$

such that

$$\beta(\eta, X, Y) = g(B(\eta, X), Y). \tag{3.5}$$

Let

$$\alpha(\eta, X, Y) = \beta(\eta, X, Y) - \frac{1}{2}\eta(U)g(\varphi_1 X, Y). \tag{3.6}$$

Then

$$\alpha(\eta, X, Y) = g(A(\eta, X), Y), \tag{3.7}$$

where

$$A(\eta, X) = B(\eta, X) - \frac{1}{2}\eta(U)\varphi_1 X. \tag{3.8}$$

Explicitly, we have

$$\alpha(\eta, X, Y) = (\nabla_X \eta)Y + u(X)\eta(\varphi_2 Y) - \eta(\varphi_1 X)u(Y) + \frac{1}{2}\eta(U)g(\varphi_1 X, Y), \tag{3.9}$$

$$A(\eta, X) = \nabla_X \xi - u(X)\varphi_2 \xi - \eta(\varphi_1 X)U + \frac{1}{2}\eta(U)\varphi_1 X, \tag{3.10}$$

We also define

$$\mu(X, Y) = (\nabla_X \varphi_1)Y \; - \; u(X)\varphi_2 \varphi_1 Y, \tag{3.11}$$

$$R_0(X, Y)Z = g(Y, Z)X - g(X, Z)Y. \tag{3.12}$$

In view of (2.4), (3.1) and (3.2) we obtain the following formula for curvature $\tilde{R}$ of the connection $\tilde{\nabla}$

$$
\begin{aligned}
\tilde{R}(X,Y)Z \;=\;& R(X,Y)Z - 2du(X,Y)\varphi_2 Z \\
& - \alpha(u,Y,Z)\varphi_1 X + \alpha(u,X,Z)\varphi_1 Y \\
& - g(\varphi_1 Y,Z)A(u,X) + g(\varphi_1 X,Z)A(u,Y) \\
& - R_0(U,\mu(X,Y)-\mu(Y,X))Z \\
& + u(X)(\nabla_Y\varphi_2)Z - u(Y)(\nabla_X\varphi_2)Z \\
& - f_1\{2du_1(X,Y)Z - \beta(u_1,Y,Z)X + \beta(u_1,X,Z)Y \\
& \qquad - g(Y,Z)B(u_1,X) + g(X,Z)B(u_1,Y) \\
& \qquad + u(Y)R_0(\varphi X,U_1)Z - u(X)R_0(\varphi Y,U_1)Z\} \\
& + f_2\{g(\varphi Y,Z)u(X)U_2 - g(\varphi X,Z)u(Y)U_2 \\
& \qquad - g(Y,Z)B(u_2,X) + g(X,Z)B(u_2,Y)\} \\
& - (f_1)^2\{g(Y,Z)R_0(X,U_1)U_1 - g(X,Z)R_0(Y,U_1)U_1 \\
& \qquad - u_1(Z)R_0(X,Y)U_1\} \\
& + (f_2)^2\{g(Y,Z)u_2(X)U_2 - g(X,Z)u_2(Y)U_2\} \\
& + f_1 f_2\{g(Y,Z)(R_0(X,U_2)U_1 - u_2(X)U_1) \\
& \qquad - g(X,Z)(R_0(Y,U_2)U_1 - u_2(Y)U_1)\} \\
& - (Xf_1)\{u_1(Y)Z + u_1(Z)Y - g(Y,Z)U_1\} \\
& + (Yf_1)\{u_1(X)Z + u_1(Z)X - g(X,Z)U_1\} \\
& - (Xf_2)g(Y,Z)U_2 + (Yf_2)g(X,Z)U_2,
\end{aligned}
\tag{3.13}
$$

where (3.9), (3.10), (3.3), (3.4), (3.11) and (3.12) are used.

4. **Particular cases**

In this section, we list the following seventeen particular cases.

4.1. **Quarter-symmetric metric connections.**

(1) $f_1 = 0 = f_2$. Then we obtain a **quarter-symmetric metric connection** $\tilde{\nabla}$ given by (K. Yano and T. Imai [22], eq. (3.3))

$$\tilde{\nabla}_X Y = \nabla_X Y + u(Y)\varphi_1 X - u(X)\varphi_2 Y - g(\varphi_1 X, Y)U.$$

(2) $f_1 = 0 = f_2$, $\varphi_2 = 0$. In this case $\varphi = \varphi_1$ and $g(\varphi X, Y) = g(X, \varphi Y)$. Then we obtain a quarter-symmetric metric connection $\tilde{\nabla}$ given by (R. S. Mishra and S. N. Pandey [9], eq. (1.6))

$$\tilde{\nabla}_X Y = \nabla_X Y + u(Y)\varphi X - g(\varphi X, Y)U.$$

In particular, if $f_1 = 0 = f_2$, $\varphi = Q$, where Q is the Ricci operator, then we obtain the *Ricci quarter-symmetric metric connection* $\tilde{\nabla}$ given by (R. S. Mishra and S. N. Pandey [9], eq. (2.2))

$$\tilde{\nabla}_X Y = \nabla_X Y + u(Y)QX - S(X,Y)U,$$

where S is the Ricci tensor. The torsion of this connection satisfies

$$\tilde{T}(X,Y) = u(Y)QX - u(X)QY.$$

(3) $f_1 = 0 = f_2$, $\varphi_1 = 0$. Then we obtain a quarter-symmetric metric connection $\tilde{\nabla}$ given by (K. Yano and T. Imai [22], eq. (3.6))

$$\tilde{\nabla}_X Y = \nabla_X Y \; - \; u(X)\,\varphi Y.$$

This connection was also introduced by R. S. Mishra and S. N. Pandey in an almost Hermitian manifold. (cf. [9], eq. (3.1)). In [10], R. H. Ojha and S. Prasad defined this type of connection in an almost contact metric manifold and called it a *semi-symmetric metric S-connection.*

4.2. **Quarter-symmetric non-metric connections.**

(4) $f_1 \neq 0$, $f_2 = 0$, $\varphi_2 = 0$. Then we obtain a *quarter-symmetric recurrent-metric connection* $\tilde{\nabla}$ given by

$$\begin{aligned}\tilde{\nabla}_X Y \;\; = \;\; & \nabla_X Y \; + \; u(Y)\,\varphi X \; - \; g(\varphi X, Y)\,U \\ & - f_1 \{u_1(X)\,Y \; + \; u_1(Y)\,X \; - \; g(X,Y)\,U_1\}.\end{aligned}$$

This connection satisfies $\tilde{\nabla} g = 2 f_1 u_1 \otimes g$.

(5) $f_1 = 1$, $f_2 = 0$, $\varphi_2 = 0$, $u_1 = u$. Then we obtain a *special quarter-symmetric recurrent-metric connection* $\tilde{\nabla}$ given by

$$\begin{aligned}\tilde{\nabla}_X Y \;\; = \;\; & \nabla_X Y \; + \; u(Y)\,\varphi X \; - \; g(\varphi X, Y)\,U \\ & - u(X)\,Y \; - \; u(Y)\,X \; + \; g(X,Y)\,U.\end{aligned}$$

This connection satisfies $\tilde{\nabla} g = 2u \otimes g$.

(6) $f_1 \neq 0$, $f_2 = 0$, $\varphi_1 = 0$. Then we obtain a *quarter-symmetric recurrent-metric connection* $\tilde{\nabla}$ given by

$$\begin{aligned}\tilde{\nabla}_X Y \;\; = \;\; & \nabla_X Y \; - \; u(X)\,\varphi Y \\ & - f_1 \{u_1(X)\,Y \; + \; u_1(Y)\,X \; - \; g(X,Y)\,U_1\}.\end{aligned}$$

This connection satisfies $\tilde{\nabla} g = f_1 u_1 \otimes g$.

(7) $f_1 = 1$, $f_2 = 0$, $\varphi_1 = 0$, $u_1 = u$. Then we obtain a *special quarter-symmetric recurrent-metric connection* $\tilde{\nabla}$ given by

$$\tilde{\nabla}_X Y = \nabla_X Y \; - \; u(X)\,\varphi Y \; - \; u(X)\,Y \; - \; u(Y)\,X \; + \; g(X,Y)\,U.$$

This connection satisfies $\tilde{\nabla} g = 2u \otimes g$.

(8) $f_1 = 0$, $f_2 \neq 0$, $\varphi_2 = 0$. Then we obtain a *quarter-symmetric non-metric connection* $\tilde{\nabla}$ given by

$$\tilde{\nabla}_X Y = \nabla_X Y \; + \; u(Y)\,\varphi X \; - \; g(\varphi X, Y)\,U \; - \; f_2\, g(X,Y)\,U_2.$$

This connection satisfies

$$\left(\tilde{\nabla}_X g\right)(Y,Z) = f_2 \{u_2(Y)\,g(X,Z) \; + \; u_2(Z)\,g(X,Y)\}.$$

(9) $f_1 = 0$, $f_2 \neq 0$, $\varphi_2 = 0$, $u_2 = u$. Then we obtain a *quarter-symmetric non-metric connection* $\tilde{\nabla}$ given by

$$\tilde{\nabla}_X Y = \nabla_X Y \; + \; u(Y)\,\varphi X \; - \; g(\varphi X, Y)\,U \; - \; f_2\, g(X,Y)\,U.$$

This connection satisfies

$$\left(\tilde{\nabla}_X g\right)(Y,Z) = f_2 \{u(Y)\,g(X,Z) \; + \; u(Z)\,g(X,Y)\}.$$

(10) $f_1 = 0,\ f_2 \neq 0,\ \varphi_1 = 0$. Then we obtain a *quarter-symmetric non-metric connection* $\tilde{\nabla}$ given by

$$\tilde{\nabla}_X Y = \nabla_X Y \ - \ u(X)\,\varphi Y \ - \ f_2\, g(X,Y)\, U_2.$$

This connection satisfies

$$\left(\tilde{\nabla}_X g\right)(Y,Z) = f_2 \left\{u_2(Y)\, g(X,Z) \ + \ u_2(Z)\, g(X,Y)\right\}.$$

(11) $f_1 = 0,\ f_2 \neq 0,\ \varphi_1 = 0,\ u_2 = u$. Then we obtain a *quarter-symmetric non-metric connection* $\tilde{\nabla}$ given by

$$\tilde{\nabla}_X Y = \nabla_X Y \ - \ u(X)\,\varphi Y \ - \ f_2\, g(X,Y)\, U.$$

This connection satisfies

$$\left(\tilde{\nabla}_X g\right)(Y,Z) = f_2 \left\{u(Y)\, g(X,Z) \ + \ u(Z)\, g(X,Y)\right\}.$$

4.3. **Semi-symmetric metric connection.**

(12) $f_1 = 0 = f_2$, $\varphi = Id$. Then we obtain a *semi-symmetric metric connection* $\tilde{\nabla}$ given by (K. Yano, 1970 [20])

$$\tilde{\nabla}_X Y = \nabla_X Y \ + \ u(Y)\, X \ - \ g(X,Y)\, U.$$

4.4. **Semi-symmetric non-metric connections.**

(13) $f_1 \neq 0$, $f_2 = 0$, $\varphi = Id$. Then we obtain a *semi-symmetric recurrent-metric connection* $\tilde{\nabla}$ given by

$$\begin{aligned}\tilde{\nabla}_X Y \ &= \ \nabla_X Y \ + \ u(Y)\, X \ - \ g(X,Y)\, U \\ &\quad - \ f_1 \left\{u_1(X)\, Y \ + \ u_1(Y)\, X \ - \ g(X,Y)\, U_1\right\}.\end{aligned}$$

This connection satisfies $\tilde{\nabla} g = f_1 u_1 \otimes g$. In particular, if $f_1 = 1,\ f_2 = 0$, $\varphi = Id$, then we obtain a *semi-symmetric recurrent-metric connection* $\tilde{\nabla}$ given by (O. C. Andonie, D. Smaranda 1977, [2]; Y. Liang 1994, [8])

$$\begin{aligned}\tilde{\nabla}_X Y \ &= \ \nabla_X Y \ + \ u(Y)\, X \ - \ g(X,Y)\, U \\ &\quad - \ u_1(X)\, Y \ - \ u_1(Y)\, X \ + \ g(X,Y)\, U_1,\end{aligned}$$

This connection satisfies $\tilde{\nabla} g = 2u_1 \otimes g$. In particular, if $f_1 = 1$, $f_2 = 0$, $\varphi = Id$, $u_1 = u$, then we obtain a *semi-symmetric recurrent-metric connection* $\tilde{\nabla}$ given by

$$\tilde{\nabla}_X Y = \nabla_X Y \ - \ u(X)\, Y.$$

This connection satisfies $\tilde{\nabla} g = 2u \otimes g$.

(14) $f_1 = 0,\ f_2 \neq 0,\ \varphi = Id$. Then we obtain a *semi-symmetric non-metric connection* $\tilde{\nabla}$ given by

$$\tilde{\nabla}_X Y = \nabla_X Y \ + \ u(Y)\, X \ - \ g(X,Y)\, U \ - \ f_2\, g(X,Y)\, U_2.$$

This connection satisfies

$$\left(\tilde{\nabla}_X g\right)(Y,Z) = f_2 \left\{u_2(Y)\, g(X,Z) \ + \ u_2(Z)\, g(X,Y)\right\}.$$

In particular, if $f_1 = 0$, $f_2 = \ -1$, $\varphi = Id$, then we obtain a *semi-symmetric non-metric connection* $\tilde{\nabla}$ given by (J. Sengupta, U. C. De, T. Q. Binh 2000, [16])

$$\tilde{\nabla}_X Y = \nabla_X Y \ + \ u(Y)\, X \ - \ g(X,Y)\, U \ + \ g(X,Y)\, U_2.$$

This connection satisfies

$$\left(\tilde{\nabla}_X g\right)(Y,Z) = -\,u_2(Y)\,g(X,Z) \;-\; u_2(Z)\,g(X,Y).$$

In particular, if $f_1 = 0$, $f_2 = -1$, $\varphi = Id$, $u_2 = u$, then we obtain a *semi-symmetric non-metric connection* $\tilde{\nabla}$ given by (N. S. Agashe, M. R. Chafle 1992, [1])

$$\tilde{\nabla}_X Y = \nabla_X Y \;+\; u(Y)\,X.$$

This connection satisfies

$$\left(\tilde{\nabla}_X g\right)(Y,Z) = -\,u(Y)\,g(X,Z) \;-\; u(Z)\,g(X,Y).$$

4.5. **Symmetric connections.**

(15) $u = 0$. Then we obtain a *symmetric non-metric connection* $\tilde{\nabla}$ given by

$$\begin{aligned}\tilde{\nabla}_X Y \;&=\; \nabla_X Y \;-\; f_1\{u_1(X)\,Y \;+\; u_1(Y)\,X \;-\; g(X,Y)\,U_1\} \\ &\quad -\; f_2\,g(X,Y)\,U_2.\end{aligned}$$

This connection satisfies

$$\begin{aligned}\left(\tilde{\nabla}_X g\right)(Y,Z) \;&=\; 2f_1 u_1(X)\,g(Y,Z) \\ &\quad + f_2\{u_2(Y)\,g(X,Z) \;+\; u_2(Z)\,g(X,Y)\}.\end{aligned}$$

(16) $u = 0$, $f_1 = 1/2$, $f_2 = 0$, $u_1 = \omega$, $U_1 = B$. Then we obtain a *Weyl connection* constructed with ω and B (G. B. Folland 1970, [5]) and given by

$$\tilde{\nabla}_X Y = \nabla_X Y \;-\; \frac{1}{2}\{\omega(X)\,Y \;+\; \omega(Y)\,X \;-\; g(X,Y)\,B\}.$$

This connection is a symmetric recurrent-metric connection. The Riemannian metric g is recurrent with respect to the connection $\tilde{\nabla}$ with the recurrence 1-form ω, that is, $\tilde{\nabla} g = \omega \otimes g$.

(17) $u = 0$, $f_1 = -1$, $f_2 = -1$, $u_1 = u_2 = \omega$. In this case, we obtain a *symmetric non-metric connection* $\tilde{\nabla}$ given by (K. Yano, [19] eq. (4.5) p. 17; (see also D. Smaranda 1983, [17]))

$$\tilde{\nabla}_X Y = \nabla_X Y \;+\; \omega(X)\,Y \;+\; \omega(Y)\,X.$$

This connection satisfies

$$(\tilde{\nabla}_X g)(Y,Z) = -\,2\omega(X)\,g(Y,Z) \;-\; \omega(Y)\,g(X,Z) \;-\; \omega(Z)\,g(X,Y).$$

This connection is projectively related to the Levi-Civita connection ∇. Then from (3.13) we get

$$\begin{aligned}\tilde{R}(X,Y)\,Z \;&=\; R(X,Y)\,Z + s(X,Z)\,Y - s(Y,Z)\,X \\ &\quad + \{s(X,Y) - s(Y,X)\}\,Z,\end{aligned}$$

where

$$s(X,Y) = (\nabla_X \omega)\,Y - \omega(X)\,\omega(Y) = \left(\tilde{\nabla}_X \omega\right) Y.$$

Note that

$$s(X,Y) - s(Y,X) = 2d\omega(X,Y),$$

that is, s is symmetric if and only if the 1-form ω is closed.

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