

CANONICAL FORMS OF MATRICES DETERMINING ANALYTICAL MANIFOLDS

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ABSTRACT. In this paper many classes of sets of matrices with entries in F ($F \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$) are introduced. Each class with the corresponding topology determines real analytical, complex or symplectic manifold for $F = \mathbb{R}$, $F = \mathbb{C}$ or $F = \mathbb{H}$ respectively. Any such family is called to be a set of canonical forms of matrices. The construction of such canonical forms of matrices is determined inductively. First basic canonical forms are studied, and then two operations for obtaining new canonical forms by using the old canonical forms are considered. All such manifolds have the property that each of them can be decomposed into cells which are Cartesian products of F ($F \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$).

1. Introduction

In the recent paper [6] are introduced two different classes of canonical forms of matrices over a field F . It easily can be generalized for the quaternions \mathbb{H} . Note that in [6] and also in this paper the term *matrix in canonical form* does not mean any reduction of a given matrix in a special form, but only means that the corresponding matrix belongs to a given family or set of matrices. The term *canonical form* comes from the example at the end of this section. According to the corresponding topology of the classes of canonical forms of matrices (cfm) in this paper are obtained real analytical (for $F = \mathbb{R}$), complex (for $F = \mathbb{C}$) and symplectic (for $F = \mathbb{H}$) manifolds. In all cases are obtained manifolds such that in special cases are obtained the Grassmann manifolds. In the present paper will be described inductively a large class of cfm yielding to analytical manifolds.

Each set of given cfm consists of $n \times m$ matrices with the following properties.

1°. The first property tells about some restrictions concerning the matrices in the given canonical form (cf), about the first zero coordinates of each vector row. This property depends on the choice of the cf.

2°. The second property is fixed for all cf and it states that any two different vector rows are orthogonal.

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3°. The third property is also fixed for all cf and states that any vector row in cf must have norm 1 and the first nonzero coordinate is positive real number.

Thus in order to define a class of cf it is sufficient to specify the property 1°. Note that alternatively the vector rows also can be considered as vectors from RP^{m-1} , CP^{m-1} or HP^{m-1} and then the property 3° should be omitted. Note also that $m \geq n$ according to 2°. Indeed, if $m < n$, then that set of cfm is empty set.

We finish the introduction by the basic example concerning the Grassmann manifolds.

Example 1.1. Let us consider the set of $n \times m$ matrices ($n \leq m$) such that

1°. If $a_1, \dots, a_n \in F^m$ are vector rows, then

$$0 \leq t(a_1) < t(a_2) < \dots < t(a_n) < m,$$

where $t(a_i)$ ($i = 1, \dots, n$), denotes the number of the first zero coordinates of the vector a_i . This set of matrices together with the fixed properties 2° and 3° for cf determines the Grassmann manifold $G_{n,m}(F)$ with the known topology, consisting of all n -dimensional subspaces of F^m generated by the vector rows.

2. Construction of different classes of cfm and the corresponding analytical manifolds

First we determine *basic canonical forms* as unit $n \times n$ matrices for arbitrary n .

Further we introduce two operations over the cfm, such that the result is a new canonical form. After introducing the topology on the new cfm we obtain a new analytical manifold (real, complex and symplectic).

i) *Inner sum.*

Let C_1, C_2, \dots, C_p be p matrices in cf, not necessary in cf of the same type. Assume that C_i is an $n_i \times m$ matrix ($1 \leq i \leq p$) and let C be the following $n \times m$ block matrix

$$C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_p \end{bmatrix},$$

where $n = n_1 + \dots + n_p$. If any two different vector rows of C are orthogonal, we say that C is in cf called *inner sum* of the cfm C_1, \dots, C_p . Thus the inner sum of given p matrices does not always exist, while the set of matrices of whole cfm is nonempty for sufficiently large m . The term "inner" comes from the orthogonality condition and we can say only "sum" of the cfm. The new canonical form will be denoted symbolically by $C_1 + \dots + C_p$. Note that if we neglect the orthogonality condition we obtain the Cartesian product of cfm C_1, \dots, C_p .

Now we introduce the topology on the new cfm as follows. Let us denote by $\{C_1\}, \dots, \{C_p\}$ the sets of all matrices in given cf, and assume that the corresponding topologies are known. Then the set of all matrices in the new cf $\{C_1 + \dots + C_p\}$ is a subset of the Cartesian topology space $\{C_1\} \times \dots \times \{C_p\}$. Thus the topology of the new cf $\{C_1 + \dots + C_p\}$ we define to be the relative topology with respect to the Cartesian topological space $\{C_1\} \times \dots \times \{C_p\}$.

If $\tau : \{1, \dots, p\} \rightarrow \{1, \dots, p\}$ is a permutation, then obviously the cfm $C_1 + \dots + C_p$ and $C_{\tau(1)} + \dots + C_{\tau(p)}$ determine homeomorphic spaces.

ii) *Spreading of a cfm onto a space Σ .*

Let $\{C\}$ be a set of all quadratic $n \times n$ matrices from a given canonical form. Let $k > n$ and Σ be a subspace of F^k such that $n \leq \dim \Sigma \leq k$. Specially, Σ can be the total space F^k . *Spreading* $S_\Sigma(C)$ over Σ is defined to be the set of matrices CX where X is $n \times k$ matrix in canonical form of the Grassmann manifold $G_{n,k}$ (see example in section 1) such that the row vectors of X belong to Σ , i.e. $X \in G_{n,\Sigma}$. Applying this spreading for each matrix C from the given cfm we obtain the total set of matrices of the spread canonical form. Notice that $C_1X_1 = C_2X_2$ implies $C_1 = C_2$ and $X_1 = X_2$ (see [6]).

The topology on the set of matrices in the spread cf we define inductively. The topology of the set $\{C\}$ of quadratic $n \times n$ matrices must be the topology of inner sum. So assume that it is obtained as a sum $C_1 + \dots + C_p$. Then the topology of the set of the spread cfm is defined to be the topology of the inner sum of the spread cfm of C_1, \dots, C_p on Σ separately. Hence it is sufficient to determine the topology of each spread cfm $\{C_i\}$ over Σ . Continuing this process of decreasing the number n , we should finally to determine the topology of spread cfm of the unit matrix (i.e. basic matrix). But, the spreading of the unit matrix is the Grassmann manifold, whose topology is well known.

Assuming that M is an analytical manifold, then the topology of the spreading $S_\Sigma(C)$ is such that it is an analytical manifold which is bundle over $G_{n,\Sigma}$ with projection $\pi : S_\Sigma(C) \rightarrow G_{n,\Sigma}$ defined by: $\pi(CX)$ is the subspace of Σ generated by the vector rows of CX , i.e. of X , and the fiber is the topology space M induced via the set of matrices $\{C\}$ in initial cf. Note that a spreading of a cf can be done only on a set of square matrices in cf. Thus if $\{C\}$ is a set of $n \times m$ matrices in cf, then we can sum with one or more submatrices in cf (example in section 1) determining the Grassmann manifolds in order to obtain cf on $m \times m$ matrices. Specially we can sum with $m - n$ vector rows or we can sum with a set of $(m - n) \times m$ matrices from the Grassmann canonical form.

Now having in mind the topology of any cfm, we are able to prove that each canonical form of $n \times m$ matrices $\{C\}$ yields to an analytical manifold. The coordinates of any such analytical manifold will be constructed in such a way that for any $n \times n$ nonsingular submatrix will be constructed a coordinate neighborhood, like for the standard coordinate neighborhoods for the Grassmann manifolds.

First note that any basic cf of unit $n \times n$ vectors determines a 0-dimensional manifold, i.e. a point. The first operation must be spreading onto F^k and hence in the first step we obtain the Grassmann manifolds $G_{n,k}$ which are analytical (real, complex and symplectic) manifolds.

Further, suppose that a new cfm is obtained via an inner sum of matrices C_1, \dots, C_p in the corresponding cf. If these p cfm determine analytical manifolds, then the set of new cfm is also an analytical manifold. Indeed, let $G_{n_1, \dots, n_p, m}(F)$ be the manifold consisting of $(p + 1)$ -tuples $(\Pi_1, \Pi_2, \dots, \Pi_p, \Pi_{p+1})$ of orthogonal subspaces of F^m where $\dim \Pi_i = n_i$ for $i = 1, 2, \dots, p$ and $\dim \Pi_{p+1} = m - n_1 - \dots - n_p$. Note that this is a flag manifold of included subspaces $V_1 = \Pi_1, V_2 = \Pi_1 + \Pi_2, V_3 = \Pi_1 + \Pi_2 + \Pi_3, \dots$. Thus $G_{n_1, \dots, n_p, m}(F)$ is an analytical manifold. Let us denote by M_i the analytical manifold which is the fiber of the projection of the matrices $\{C_i\}$ on the Grassmann manifold $G_{n_i, m}$. Then the new cf of sum of cfm determines a space which is a bundle with base $G_{n_1, \dots, n_p, m}(F)$ and fiber $M_1 \times \dots \times M_p$. The

projection π is given by

$$\pi\left(\begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_p \end{bmatrix}\right) = (\Pi_1, \Pi_2, \dots, \Pi_p, \Pi_{p+1}),$$

where Π_1 is generated by the vector rows of C_1 , Π_2 is generated by the vector rows of C_2, \dots and Π_{p+1} is the orthogonal complement of $\Pi_1 + \dots + \Pi_p$ into F^m . Thus this space of sum of cfm is also an analytical manifold. The coordinate neighborhoods of the new cf can be constructed as follows. Let C' be any nonsingular $n \times n$ submatrix of C . Then there exist p submatrices: C'_1 submatrix of C_1 of order $n_1 \times n_1, \dots, C'_p$ submatrix of C_p of order $n_p \times n_p$, where $n = n_1 + \dots + n_p$, such that

1. these matrices are submatrices of C' ,
2. the columns of C'_1, \dots, C'_p are distinct,
3. the matrices C'_1, \dots, C'_p are non-singular, and
4. by deleting the rows and columns of the submatrices C'_1, C'_2, \dots, C'_i , the rest $(n_{i+1} + \dots + n_p) \times (n_{i+1} + \dots + n_p)$ submatrix of C' is nonsingular ($i = 1, 2, \dots, p-1$).

Note that such a choice of submatrices C'_1, C'_2, \dots, C'_p is possible because by generalization of the Laplace decomposition of the determinants it holds

$$\det C' = \sum \pm \det D'_1 \cdot \det D'_2 \cdots \det D'_p$$

where D'_j is $n_j \times n_j$ submatrix of C' and submatrix of C_j and such that the columns of D'_1, \dots, D'_p are distinct.

In the next step we choose the coordinate neighborhoods as follows. All $n(m-n)$ elements which do not belong to C' may be changed to be close to the corresponding elements of C . The same choice is for the elements which are simultaneously in the same row as C'_i and in the same column as C'_j for $i > j$, i.e. they may be chosen to be close to the corresponding elements of C' . Also according to the inductive assumption the elements of C'_1, C'_2, \dots, C'_p can be chosen in the corresponding coordinate neighborhoods which they induce respectively on $\{C_1\}, \{C_2\}, \dots, \{C_p\}$. Finally according to the properties 1. - 4. we note that the elements which belong simultaneously in the same row as C'_i and in the same column as C'_j for $i < j$ can uniquely be determined such that the row vectors of C are orthogonal. Hence we showed that the chosen matrix C can be covered by a coordinate neighborhood of $C_1 + \dots + C_p$. The Jacobi matrices for the described coordinate neighborhoods are analytical functions, because of the inductive assumptions for C_1, \dots, C_p and the analytical solutions of linear algebraic systems.

Next we should show how we can associate a coordinate neighborhood for any nonsingular $n \times n$ submatrix C' of spread $n \times k$ matrix $S_\Sigma(C)$. Without loss of generality we can suppose that C is a square $n \times n$ matrix in former cf and by inductive assumption it can be covered with coordinate neighborhoods with analytical elements of the Jacobi matrices. Since $\{S_\Sigma(C)\}$ is a bundle with base $G_{n,k}$ and according to the standard construction for the coordinates induced by any nonsingular $n \times n$ submatrix and the fact that the fiber is an analytical manifold by the inductive assumption, we obtain the required covering.

Note that inductively it follows that all the manifolds obtained via this method of cf are compact. Finally we can resume the previous results in the following two theorems.

Theorem 2.1. *The set of all matrices in a chosen canonical form with the introduced topology is a compact analytical (real, complex or symplectic) manifold.*

Theorem 2.2. *Any manifold determined via a cf of $n \times m$ matrices is a bundle over the base manifold $G_{n,m}(F)$ or $G_{n_1,n_2,\dots,n_p,m}(F)$.*

3. Some examples

In this section will be considered some examples of cfm.

Example 3.1. Assume that $n_1 = 1, \dots, n_p = 1$ ($n = p$) and let $s_1, \dots, s_n \in \{0, \dots, m-1\}$ be given numbers. Then each of the cf of $1 \times m$ matrices belongs to the space FP^{m-1} . For any i , let Σ_i be the subspace generated by $e_{s_i+1}, e_{s_i+2}, \dots, e_m$. Then the inner sum of cf can be described as a set of n orthogonal projective vectors, such that the first starts with at least s_1 zeros, the second starts with at least s_2 zeros and so on. The dimension of this manifold is

$$mn - \frac{n(n+1)}{2} - s_1 - s_2 - \dots - s_n.$$

Let us consider the special case $m = n$. Then there exists a permutation τ such that $s_{\tau(i)} < i$, because in opposite case the corresponding matrix would not be orthogonal. Hence without loss of generality we assume that $s_i < i$, ($i = 1, 2, \dots, n$). Hence there are $n!$ such manifolds. Some of them are homeomorphic. Note that specially, if $s_1 = s_2 = \dots = s_n = 0$, we obtain the full flag manifold.

Example 3.2. Now let us consider the following example. Suppose that m_1, \dots, m_p are fixed positive integers such that $m = m_1 + \dots + m_p$. Let

Σ_1 be the subspace generated by e_1, \dots, e_{m_1} ,

Σ_2 be the subspace generated by $e_{m_1+1}, \dots, e_{m_1+m_2}$,

Σ_3 be the subspace generated by $e_{m_1+m_2+1}, \dots, e_{m_1+m_2+m_3}$,

\dots

Σ_p be the subspace generated by $e_{m_1+\dots+m_{p-1}+1}, \dots, e_m$.

Then the cf which is an inner sum has the following form

$$\begin{bmatrix} C_1 & 0 & 0 & \dots & 0 \\ 0 & C_2 & 0 & \dots & 0 \\ 0 & 0 & C_3 & \dots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & 0 & 0 & \dots & C_p \end{bmatrix}$$

as a block matrix of type $(n_1 + n_2 + \dots + n_p) \times (m_1 + m_2 + \dots + m_p)$. Obviously the induced analytical manifold is the Cartesian product $M_1 \times M_2 \times \dots \times M_p$ where M_i is the analytical manifold induced by the i -th cf of type C_i on $n_i \times m_i$ matrix ($n_i \leq m_i$).

Example 3.3. Let C be an inner sum of C_1, \dots, C_p where C_i is an $n_i \times m_i$ matrix, where $n_1 + n_2 + \dots + n_p = n$ and $m_1 + \dots + m_p = m = n$. Since $n_i \leq m_i$, it must

be $n_i = m_i$ for $i = 1, \dots, p$, i.e.

$$C = \begin{bmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ \vdots & & & \\ \vdots & & & \\ 0 & 0 & \cdots & C_p \end{bmatrix}$$

where C_i is $n_i \times n_i$ matrix in cf. Since $\det C_i \neq 0$ for $i = 1, 2, \dots, p$, one can verify that the spread $n \times m'$ matrix C' is in cf if and only if C' decomposes into block matrices

$$C' = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1p} \\ C_{21} & C_{22} & \cdots & C_{2p} \\ \vdots & & & \\ \vdots & & & \\ C_{p1} & C_{p2} & \cdots & C_{pp} \end{bmatrix}$$

where C_{ij} is an $n_i \times m'_j$ matrix, where m'_1, \dots, m'_p are not fixed but $m'_1 + \dots + m'_p = m'$, such that

- i) $C_{ij} = 0$ for $i > j$,
- ii) $\text{rank} C_{ii} = n_i$,
- iii) the row vectors of C' are orthogonal,
- iv) the first non-zero coordinate of each vector is a positive real number.

In special case if $\{C_1\}, \dots, \{C_p\}$ are the cf of full flag manifolds, we obtain the manifold described in [6] section 3.

Example 3.4. Let be given p positive integers n_1, n_2, \dots, n_p and let $n_1 + n_2 + \dots + n_p = n \leq m$. We consider a set of linearly independent vectors

$$\mathbf{a}_{11}, \mathbf{a}_{12}, \dots, \mathbf{a}_{1n_1}, \mathbf{a}_{21}, \mathbf{a}_{22}, \dots, \mathbf{a}_{2n_2}, \dots, \mathbf{a}_{p1}, \mathbf{a}_{p2}, \dots, \mathbf{a}_{pn_p},$$

of F^m and we denote the matrix with these n row-vectors by A . The matrix A is in a cf if

- i) $t_1 < t_2 < \dots < t_p$, where $t_i = \min\{t(\mathbf{a}_{i1}), t(\mathbf{a}_{i2}), \dots, t(\mathbf{a}_{in_i})\}$ and $t(\mathbf{a})$ denotes the number of the first zero coordinates of \mathbf{a} ,
- ii) each two different vectors of these n vectors are orthogonal,
- iii) each vector row has norm 1 and the first non-zero coordinate is positive real number.

It is not obvious that this cf belongs to the cf of matrices introduced inductively in section 2. One can prove that this set of matrices can be considered as a spreading of $n \times n$ matrices with the same property i), ii) and iii). Thus we should consider the case $m = n$. It is clear that the set of $n \times n$ matrices in cf is an inner sum of n_1 vectors in F^n and set of canonical forms of $(n_2 + \dots + n_p) \times n$ matrices with parameters n_2, n_3, \dots, n_p , projected on the space generated by the vectors e_2, e_3, \dots, e_n . Hence by induction of p we obtain that the considered cfm is included in the family of manifolds obtained in section 2.

Let us consider the following special case for $F = \mathbb{R}$, $p = 2$, $n_1 = 2$, $n_2 = 1$ and $m = 3$. Then the manifold of canonical vectors consists of the following cells

$$C_1 = \begin{bmatrix} x & * & * \\ y & * & * \\ 0 & * & * \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}, \quad C_3 = \begin{bmatrix} 0 & * & * \\ 1 & 0 & 0 \\ 0 & * & * \end{bmatrix},$$

where $x, y > 0$. The cell C_1 is homeomorphic to $\mathbb{R} \times S^1$ because for fixed ratio $\lambda = x/y \in \mathbb{R}^+$ it is homeomorphic to S^1 . The cells C_2 and C_3 are homeomorphic to S^1 . Thus the Euler characteristic of the manifold is $\chi = 0$. It can be described such that each point consists of two orthogonal lines p and q through the coordinate origin in \mathbb{R}^3 such that q lies in the yz -plane. The third line which is orthogonal to p and q is uniquely determined by p and q . This manifold is homeomorphic to the Klein's bottle. Note that if we consider the complex and quaternionic cases, then we obtain complex and symplectic manifolds - analogs of the Klein's bottle.

4. Decomposition into cells

In this section we show the existence of cell decomposition which is analogous to the Schubert's cell decomposition of the Grassmann manifolds.

By the construction of cf described in section 2 we obtain a large class of compact analytical manifolds, three manifolds for each cfm: real, complex and symplectic. All these manifolds have the following property.

Theorem 4.1. *All the manifolds obtained via cfm are such that they can be decomposed into disjoint cells of type F^i .*

Proof. Note that the base cf determine 0-dimensional manifolds and each of them is a point, i.e. F^0 .

First let us prove that if the set of cfm $\{C\}$ satisfies the property of the Theorem 4.1, then $\{S_\Sigma(C)\}$ satisfies that property also. Since $\{C\}$ is a set of quadratic matrices, $\{S_\Sigma(C)\}$ consists of all matrices of type CX where X is matrix of the Grassmann manifolds, and the representation is unique. Hence we obtain that the cells of $\{CX\}$ are products of the cells of $\{C\}$ and the cells of $\{X\}$. The cells of $\{C\}$ are of type F^i because of the inductive assumption and the cells of the Grassmann manifolds $G_{n,\Sigma}$ are also of that type and in this case the proof is finished. Indeed, $G_{n,\Sigma}$ can be decomposed into C_r^n cells of type F^i , where $r = \dim \Sigma$.

Suppose that the manifolds determined by the cf $\{C_i\}$ satisfy the property in Theorem 4.1. Then we will show that the manifold induced by the sum $C_1 + \dots + C_p$ also satisfies that property. This reduces to the special case when C_1, \dots, C_p are spreadings over the corresponding Grassmann manifolds with bases M_i which are quadratic matrices. Indeed this manifold is a bundle over the base $G_{n_1, n_2, \dots, n_p, m}(F)$ and the fiber $M_1 \times M_2 \times \dots \times M_p$ and moreover the new manifold is equivalent (but not necessary homeomorphic) to the Cartesian product

$$G_{n_1, n_2, \dots, n_p, m}(F) \times (M_1 \times M_2 \times \dots \times M_p).$$

It follows from the fact that the set of matrices for the spreading of quadratic $n_i \times n_i$ matrices is the product (which is unique) of matrices of M_i and the Grassmann manifold $G_{n_i, m}$. Since M_i satisfies the property in Theorem 4.1, and the base manifold $G_{n_1, n_2, \dots, n_p, m}(F)$ can be decomposed into

$$\frac{m!}{n_1! n_2! \dots n_p! (m - n_1 - \dots - n_p)!}$$

cells of type F^i , we obtain that the inner sum $C_1 + \dots + C_p$ also satisfies the property of the Theorem 4.1.

This completes the proof of the theorem and moreover it gives a method for finding all of the cells. \square

In the paper [10] is given a decomposition of the full flag manifold $G_n(F)$ into $n!$ cells of type F^i . Indeed, the following theorem is proved in [10].

Theorem 4.2. *The manifold $G_n(F)$ is a disjoint union of $n!$ disjoint cells, such that for each sequence $(i_1, i_2, \dots, i_{n-1})$, for $0 \leq i_j \leq j$ and $1 \leq j \leq n-1$, there exists a cell $C_{i_1, \dots, i_{n-1}}$ which is homeomorphic to $F^{i_1} \times F^{i_2} \times \dots \times F^{i_{n-1}}$.*

Note that the Theorem 4.1 tells nothing about real manifolds because each real manifold can be decomposed into cells of type R^i . But there are complex manifolds which can not be decomposed into disjoint cells of type C^i . For example if the torus $T = S^1 \times S^1$ can be decomposed into some cells of type $C^1 = \mathbb{R}^2$ and $C^0 = \mathbb{R}^0$, then the Euler characteristic is a sum of such cells and it is positive number, which is a contradiction.

The cohomology modules for any manifolds constructed via the canonical forms in section 2 can be found easily. Indeed, we know the cohomology modules for the manifolds $G_{n,m}(F)$ and $G_{n_1, \dots, n_p, m}(F)$. Using the Leray-Hirsch theorem [1] we can find step by step all the cohomology modules for any such manifold. Indeed, according to the Theorem 2.2, we have the following theorem.

Theorem 4.3. *Let us denote by $P_t(M)$ the polynomial*

$$\dim H^0(M, \mathbb{R}) + t \dim H^1(M, \mathbb{R}) + t^2 \dim H^2(M, \mathbb{R}) + \dots + t^s \dim H^s(M, \mathbb{R}),$$

for a manifold M , where $s = \dim M$.

a) *If M is a real analytical manifold obtained via cfm for $F = \mathbb{R}$, then $P_t(M)$ is a product of polynomials of types*

$$P_t(G_{p,p+q}) = \frac{(1-t)(1-t^2) \dots (1-t^{p+q})}{(1-t)(1-t^2) \dots (1-t^p)(1-t)(1-t^2) \dots (1-t^q)};$$

b) *If M is a complex manifold obtained via cfm for $F = \mathbb{C}$, then $P_t(M)$ is a product of polynomials of types*

$$P_t(G_{p,p+q}) = \frac{(1-t^2)(1-t^4) \dots (1-t^{2(p+q)})}{(1-t^2)(1-t^4) \dots (1-t^{2p})(1-t^2)(1-t^4) \dots (1-t^{2q})};$$

c) *If M is a symplectic manifold obtained via cfm for $F = \mathbb{H}$, then $P_t(M)$ is a product of polynomials of types*

$$P_t(G_{p,p+q}) = \frac{(1-t^4)(1-t^8) \dots (1-t^{4(p+q)})}{(1-t^4)(1-t^8) \dots (1-t^{4p})(1-t^4)(1-t^8) \dots (1-t^{4q})}.$$

5. About a further generalization

Now will be presented a possible generalization for $F = \mathbb{C}$, which can be a subject of further research. We give the definition and the basic results about symmetric products of manifolds.

Let M be an arbitrary set and m be a positive integer. In the Cartesian product M^m we define a relation \approx such that $(x_1, \dots, x_m) \approx (y_1, \dots, y_m)$ iff y_1, y_2, \dots, y_m is an arbitrary permutation of x_1, x_2, \dots, x_m . This is equivalence relation and the equivalence class represented by (x_1, \dots, x_m) is denoted by $(x_1, \dots, x_m) / \approx$ and

the quotient space M^m / \approx is called *symmetric product* of M and is denoted by $M^{(m)}$.

If M is a topological space, then the quotient space $M^{(m)}$ is also a topological space. The space $M^{(m)}$ is introduced quite early [4], but mainly it was studied in [12]. The space $(\mathbb{R}^n)^{(m)}$ is a manifold only for $n = 2$ [12]. If $n = 2$, then $(\mathbb{R}^2)^{(m)} = \mathbb{C}^{(m)}$ is homeomorphic to \mathbb{C}^m . Indeed, using that \mathbb{C} is algebraically closed field, it is obvious that the mapping $\varphi : \mathbb{C}^{(m)} \rightarrow \mathbb{C}^m$ defined by

$$\varphi((z_1, \dots, z_m) / \approx) = (\sigma_1, \sigma_2, \dots, \sigma_m)$$

is a bijection, where σ_i ($1 \leq i \leq m$) is the i -th symmetric function of z_1, \dots, z_m , i.e.

$$\sigma_i(z_1, \dots, z_m) = \sum_{1 \leq j_1 < j_2 < \dots < j_i \leq m} z_{j_1} \cdot z_{j_2} \cdot \dots \cdot z_{j_i}.$$

The mapping φ is also a homeomorphism. Moreover, $M^{(m)}$ is a complex manifold if M is 1-dimensional complex manifold [7]. For example, if M is a sphere, i.e. the complex manifold CP^1 , then $M^{(m)}$ is the projective complex space CP^m . Using the permutation products it is easy to see how $M^{(m)} = CP^m$ decomposes into disjoint cells $\mathbb{C}^0, \mathbb{C}^1, \dots, \mathbb{C}^m$. Let $\xi \in M$. Then we define $((x_1, \dots, x_m) / \approx) \in M_i$ if exactly i of the elements x_1, \dots, x_m are equal to ξ . Thus

$$\begin{aligned} M^{(m)} &= M_0 \cup M_1 \cup \dots \cup M_m = (M \setminus \{\xi\})^{(m)} \cup (M \setminus \{\xi\})^{(m-1)} \cup \dots \cup (M \setminus \{\xi\})^{(0)} \\ &= \mathbb{C}^{(m)} \cup \mathbb{C}^{(m-1)} \cup \dots \cup \mathbb{C}^{(0)} = \mathbb{C}^m \cup \mathbb{C}^{m-1} \cup \dots \cup \mathbb{C}^0. \end{aligned}$$

Some recent results about symmetric products of manifolds are obtained in [11, 2]. This theory about symmetric products has an important role in the theory of the topological commutative vector valued groups [8, 9, 5].

We mentioned in the section 1 that the property 3⁰ can be omitted by assuming that the row vectors of the cfm are projective vectors, i.e. they are elements of CP^{m-1} . Indeed, for any such vector $v = (v_1, \dots, v_m) \in CP^{m-1}$ we joint a polynomial

$$P(z) = v_m z^{m-1} + v_{m-1} z^{m-2} + \dots + v_1$$

and hence its complex roots $(z_1, \dots, z_{m-1}) / \approx$ up to a permutation. Here $z_1, \dots, z_{m-1} \in C^* = \mathbb{C} \cup \{\infty\}$ such that if $v_m = v_{m-1} = \dots = v_{m-s+1} = 0$ and $v_{m-s} \neq 0$, then exactly s of the roots are equal to ∞ .

Now instead of the complex manifold $S^2 = \mathbb{C} \cup \{\infty\}$ we should consider an arbitrary 1-dimensional complex manifold M . Then any $1 \times m$ canonical form of matrices induces the complex manifold $M^{(m-1)}$ and it can be considered as a projective space over M . The idea for generalization is the following. For any cfm the vector rows should be considered as elements of $M^{(m-1)}$. The zero initial values of the vector rows correspond to the multiplicity of a chosen point ξ on the chosen 2-dimensional surface. If one manages to determine the corresponding orthogonality conditions, then a complex manifold which corresponds to the considered cfm and the basic 1-dimensional complex manifold M will be obtained.

At this moment we know only the projective space over given 2-dimensional real surface, which is the symmetric product of the surface, but do not know the Grassmann manifold over given 2-dimensional real surface.

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