International Electronic Journal of Geometry
Volume 1 No. 2 pp. 11-14 (2008) ©IEJG

THE NATURAL AFFINORS ON HIGHER ORDER PRINCIPAL PROLONGATIONS

JAN KUREK AND WŁODZIMIERZ M. MIKULSKI

(Communicated by Kazım İLARSLAN)

Abstract. Let $P \to M$ be a principal G-bundle with m-dimensional basis, where G is a Lie group. We describe all $\mathcal{PB}_m(G)$-natural affinors on the r-th order principal prolongation $W^r_m P$ of $P \to M$.

1. Introduction

We fix a Lie group G. Let $\mathcal{L}(G)$ be the Lie algebra of G and $e \in G$ be the unit element. The category of all principal G-bundles with m-dimensional bases and their (local) principal bundle isomorphisms with the identity group homomorphism will be denoted by $\mathcal{PB}_m(G)$.

Given $\mathcal{PB}_m(G)$-maps $\Phi, \Psi : P \to Q$ and a point $x \in M$ the following conditions are equivalent: (i) $j^r_{p_o}\Phi = j^r_{p_o}\Psi$ for some $p_o \in P_x$; (ii) $j^r_p\Phi = j^r_p\Psi$ for any $p \in P_x$. We write $j^r_x\Phi = j^r_x\Psi$ iff it is satisfied at least one of the equivalent conditions (i) or (ii), [2].

The r-th order principal prolongation $W^r_m P$ of a $\mathcal{PB}_m(G)$-object $P \to M$ is defined to be the space of all r-jets $j^r_0\varphi$ of local $\mathcal{PB}_m(G)$-maps $\varphi : \mathbf{R}^m \times G \to P$. By [2], $W^r_m P \to M$ is a principal bundle with the structure group $W^r_m G := J^r_0(\mathbf{R}^m \times G, \mathbf{R}^m \times G)_0$ and the fibred manifold $W^r_m P \to M$ coincides with the fibred product $P^r M \times_M J^r P$, where $P^r M = invJ^r_0(\mathbf{R}^m, M)$ is the r-th order frame bundle of M. Every $\mathcal{PB}_m(G)$-map $\Phi : P \to Q$ is extended (via composition of jets) into principal bundle (local) isomorphism $W^r_m\Phi : W^r_m P \to W^r_m Q$.

A $\mathcal{PB}_m(G)$-natural affinor on W^r_m is a family of $\mathcal{PB}_m(G)$-invariant tensor fields of type $(1, 1)$ (affinors)

$$A = A_P : TW^r_m P \to TW^r_m P$$

on $W^r_m P$ for any $\mathcal{PB}_m(G)$-object $P \to M$. The invariance means that for any $\mathcal{PB}_m(G)$-objects P and Q affinors A_P and A_Q are $W^r_m\Phi$ related (i.e. $TW^r_m\Phi \circ A_P = A_Q \circ TW^r_m\Phi$) for any $\mathcal{PB}_m(G)$-map $\Phi : P \to Q$.

A $\mathcal{PB}_m(G)$-natural affinor A is said to be of vertical type if $A_P : TW^r_m P \to VW^r_m P$ for any $\mathcal{PB}_m(G)$-object $P \to M$, where $VW^r_m P$ is the $(W^r_m P \to M)$-vertical subbundle in the tangent bundle $TW^r_m P$ of $W^r_m P$.

2000 *Mathematics Subject Classification.* 58A20, 58A32.
Key words and phrases. principal bundle, higher order principal prolongation, natural affinor.

In this paper we describe all $\mathcal{PB}_m(G)$-natural affinors on W^r_m.

Example 1.1. For any $\mathcal{PB}_m(G)$-object $P \to M$ we have the identity affinor $Id : TW^r_mP \to TW^r_mP$. Thus we have the identity $\mathcal{PB}_m(G)$-natural affinor Id on W^r_m.

To present a next example we need the following lemma.

Lemma 1.1 ([5]). *Any vector $v \in T_wW^r_mP$, $w \in (W^r_mP)_x$, $x \in M$ is of the form $\mathcal{W}^r_mX_w$ for some right invariant vector field $X \in \mathcal{X}_{G-inv}(P)$ on P, where $\mathcal{W}^r_mX$ is the flow lifting of $X \in \mathcal{X}_{G-inv}(P)$ to W^r_mP. Moreover j^r_xX is uniquely determined.*

Proof. (We cite the proof from [5].) We can assume that $P = \mathbf{R}^m \times G$ and w is over $(0, e)$. Since $W^r_m(\mathbf{R}^m \times G)$ is in usual way a sub-principal bundle of $P^r(\mathbf{R}^n \times G)$, then by well-known manifold version of the lemma, we find $X \in \mathcal{X}(\mathbf{R}^m \times G)$ such that $v = \mathcal{P}^rX_w$ and $j^r_{(0,e)}X$ is determined uniquely. Any right-invariant vector field Y gives $\mathcal{P}^rY_w$ which is tangent to W^r_mP. On the other hand the dimension of W^r_mP and the dimension of the space of r-jets j^r_0Y of right invariant Y are equal. Then the lemma follows from the dimension argument because the flow operator is linear. □

Example 1.2. Let $B : J^{r-1}_0(T_{G-inv}(\mathbf{R}^m \times G)) \to (J^r_0(T_{G-inv}(\mathbf{R}^m \times G)))_0$ be a linear map, where $J^{r-1}_0\ (T_{G-inv}(\mathbf{R}^m \times G)) = \{j^{r-1}_0X |\ X \in \mathcal{X}_{G-inv}\ (\mathbf{R}^m \times G)\}$ and $(J^r_0(T_{G-inv}(\mathbf{R}^m \times G)))_0 = \{j^r_0X \mid X \in \mathcal{X}_{G-inv}\ (\mathbf{R}^m \times G)\ , pr_{\mathbf{R}^m} \circ X_{(0,.)} = 0\}$, where $pr_{\mathbf{R}^m} : \mathbf{R}^m \times G \to \mathbf{R}^m$ is the projection. We define a vertical $\mathcal{PB}_m(G)$-natural affinor $A^B : TW^r_mP \to VW^r_mP$ on W^r_m by

$$A^B(v) = VW^r_m\Phi((\mathcal{W}^r_m\tilde{v})_\theta)\ ,\ v \in T_{j^r_0\Phi}W^r_mP\ ,\ j^r_0\Phi \in W^r_mP\ ,$$

where $\theta = j^r_0(id_{\mathbf{R}^m \times G}) \in W^r_m(\mathbf{R}^m \times G)$ is the element and $\tilde{v} \in \mathcal{X}_{G-inv}\ (\mathbf{R}^m \times G)$ is an arbitrary right invariant vector field on $\mathbf{R}^m \times G$ such that $j^r_0\tilde{v} = B(j^{r-1}_0((\Phi^{-1})_*\overline{v}))$ and $v = (\mathcal{W}^r_m\overline{v})_{j^r_0\Phi}$. One can standardly show that $A^B(v)$ is well-defined. (More precisely (by Lemma 1.1), $j^r_{\Phi(0)}\overline{v}$ is uniquely determined by v. Then $j^{r-1}_0((\Phi^{-1})_*\overline{v}) \in J^{r-1}_0(T_{G-inv}(\mathbf{R}^m \times G))$ is determined by v. Then $j^r_0(\tilde{v}) \in (J^r_0(T_{G-inv}(\mathbf{R}^m \times G)))_0$ is determined by v. Then $(\mathcal{W}^r_m\tilde{v})_\theta$ is determined by v and vertical. Then $A^B(v)$ is determined by v and vertical.) Using the linearity of the flow operator, we deduce that $A^B : TW^r_mP \to VW^r_mP$ is a vertical affinor on W^r_mP. Clearly the family A^B is a $\mathcal{PB}_m(G)$-natural affinor on W^r_m.

2. **The main result**

The main result of this paper is the following classification theorem.

Theorem 2.1. *Any $\mathcal{PB}_m(G)$-natural affinor on W^r_m is of the form*

$$A = \lambda Id + A^B$$

for a (uniquely determined by A) real number λ and a (uniquely determined by A) linear map $B : J^{r-1}_0(T_{G-inv}(\mathbf{R}^m \times G)) \to (J^r_0(T_{G-inv}(\mathbf{R}^m \times G)))_0$.

The proof of Theorem 2.1 will occupy the rest of this paper. We will use the following lemma.

Lemma 2.1. *Let $X, Y \in \mathcal{X}_{G-inv}(P)$ be right invariant vector fields on $p : P \to M$ and $x \in M$ be a point. Suppose that $j^r_x X = j^r_x Y$ and X is not-vertical over x. Then there exists a (localy defined) $\mathcal{PB}_m(G)$-map $\Phi : P \to P$ such that $j^{r+1}_x(\Phi) = j^{r+1}_x(id_P)$ and $\Phi_* X = Y$ near x.*

Proof. A direct modification of the proof of Lemma 42.4 in [2]. □

Lemma 2.2. *Let A be a $\mathcal{PB}_m(G)$-natural affinor on W^r_m. There is a unique real number λ such that $A - \lambda Id$ is of vertical type.*

Proof. Let X be a right-invariant vector field on $\mathbf{R}^m \times G$. Let $\mathcal{A}(X) := A \circ \mathcal{W}^r_m X$.

We can write $\mathcal{A}(X)_\theta = \mathcal{W}^r_m \tilde{X}_\theta$ for some right-invariant vector field $\tilde{X}$ (see Lemma 1.1), $\theta = j^r_0(id_{\mathbf{R}^m \times G})$. Suppose $pr_{\mathbf{R}^m} \circ X(0, e) \neq \mu pr_{\mathbf{R}^m} \circ \tilde{X}(0, e)$ for all $\mu \in \mathbf{R}$ and $pr_{\mathbf{R}^m} \circ \tilde{X}(0, e) \neq 0$ Then there is an $\mathcal{PB}_m(G)$-map $\Phi : \mathbf{R}^m \times G \to \mathbf{R}^m \times G$ preserving θ such that

$$J^r T\Phi(j^r_0 X) = j^r_0 X \text{ and } J^r T\Phi(j^r_0 \tilde{X}) \neq j^r_0 \tilde{X} \ .$$

Then

$$\mathcal{A}(X)_\theta = \mathcal{W}^r_m(\Phi_* \tilde{X})_\theta \neq \mathcal{W}^r_m(\tilde{X})_\theta = \mathcal{A}(X)_\theta \ .$$

This is a contradiction. Consequently, we have

$$T\pi^r \circ \mathcal{A}(X)_\theta = \lambda(j^r_0 X) pr_{\mathbf{R}^m} \circ X_{(0,e)} \tag{2.1}$$

for some (not necessarily unique and not necessarily smooth) map $\lambda : J^r_0(T_{G-inv}(\mathbf{R}^m \times G)) \to \mathbf{R}$ and all right-invariant vector fields on $\mathbf{R}^m \times G$, where $\pi^r : W^r_m(\mathbf{R}^m \times G) \to \mathbf{R}^m$ is the projection.

We are going to show that λ can be chosen smooth.

Of course (since the left hand side of (2.1) depends smoothly on $j^r_0 X$), the map $\Phi : J^r_0(T_{G-inv}(\mathbf{R}^m \times G)) \to \mathbf{R}$ given by

$$\Phi(j^r_0 X) = \lambda(j^r_0 X) X^1(0)$$

is smooth and $\Phi(j^r_0 X) = 0$ if $X^1(0) = 0$, where

$$X_{(0,e)} = \sum_i X^i(0) \frac{\partial}{\partial x^i}_0 \ + \ ...$$

and where ... is the vertical part of $X_{(0,e)}$. Then there is a smooth map $\Psi : J^r_0(T_{G-inv}(\mathbf{R}^m \times G)) \to \mathbf{R}$ such that $\Phi(j^r_0 X) = \Psi(j^r_0 X) X^1(0)$. Then we can define new $\lambda = \Psi$. This new λ is equal to the old one for $X^1(0) \neq 0$. Then for the new λ we have (2.1) if additionally $X^1(0) \neq 0$. Then we have (2.1) for all X in question because of the smooth and density arguments.

Since $\mathcal{A}(X)$ depends linearly on X, λ is constant.

Then $A((\mathcal{W}^r_m X)_\theta) - \lambda(\mathcal{W}^r_m X)_\theta$ is vertical. Then by Lemma 1.1, $A(v) - \lambda v$ is vertical for any $v \in T_\theta W^r_m(\mathbf{R}^m \times G)$. Then $A - \lambda Id$ is vertical because of the $\mathcal{PB}_m(G)$-invariance of $A - \lambda Id$. □

Proof of Theorem 2.1. Because of Lemma 2.2 we can assume that A is vertical. We define a $B : J^{r-1}_0(T_{G-inv}(\mathbf{R}^m \times G)) \to (J^r_0(T_{G-inv}(\mathbf{R}^m \times G)))_0$ by

$$B(j^{r-1}_0 X) = j^r_0 \tilde{X} \ ,$$

where $\tilde{X}$ is a right-invariant vector field on $\mathbf{R}^m \times G$ such that $(\mathcal{W}^r_m \tilde{X})_\theta = A((\mathcal{W}^r_m \overline{X})_\theta)$ and $\overline{X}$ is the unique right-invariant vector field on $\mathbf{R}^m \times G$ such that $j^{r-1}_0 X = j^{r-1}_0 \overline{X}$

and $\overline{X}$ has coefficients with respect to a basis of right invariant vector fields (consisting with the constant vector fields $\frac{\partial}{\partial x^i}$ on $\mathbf{R}^m$ and the right invariant vector fields on G corresponding to some basis $B_j \in T_eG$) being polynomials of degree $\leq r-1$.

Then $A((\mathcal{W}^r_m X)_\theta) = A^B((\mathcal{W}^r_m X)_\theta)$ for all right invariant vector fields on $\mathbf{R}^m \times G$ such that X has coefficients (with respect to the basis as above) being polynomials of degree $r-1$. Since the union of all orbits with respect to the $\mathcal{PB}_m(G)$-maps preserving θ of jets $j^r_0 X$ of right-invariant vector fields X on $\mathbf{R}^m \times G$ with coefficients (with respect to the basis as above) being polynomials of degree $\leq r-1$ is dense in $J^r_0(T_{G-inv}(\mathbf{R}^m \times G))$ (see Lemma 2.1), $A((\mathcal{W}^r_m X)_\theta) = A^B((\mathcal{W}^r_m X)_\theta)$ for all right-invariant vector fields X on $\mathbf{R}^m \times G$. Then $A(v) = A^B(v)$ for all $v \in T_\theta W^r_m(\mathbf{R}^m \times G)$ because of Lemma 1.1. Then $A = A^B$ because of the $\mathcal{PB}_m(G)$-invariance and the fact that W^r_m is a transitive bundle functor (i.e. $W^r_m P$ is the $\mathcal{PB}_m(G)$-orbit of θ). □

If $G = \{e\}$, then $\mathcal{PB}_m(\{e\})$ is the category $\mathcal{M}f_m$ of all m-dimensional manifolds and their embeddings, and $W^r_m = P^r$ is the r-th order frame bundle functor. Thus we reobtain the result of [4], where a classification of all $\mathcal{M}f_m$-natural affinors on the r-th order frame bundle $P^r M$ is given.

Natural affinors play a very important role in the differential geometry. They can be used to define generalized torsion of connections, [3]. In our situation any natural affinor $A : TW^r_m P \to TW^r_m P$ defines a torsion $\tau(\Gamma) := [\Gamma, A]$ of a principal connection $\Gamma : TW^r_m P \to VW^r_m P$ on $W^r_m P \to M$, where the bracket is the Frolicher-Nijenhuis one. That is why natural affinors on some natural bundles have been classified in many papers. Principal G-bundles play crucial role in the geometrization of physic, [1], [6], [7].

References

[1] Fatibene, L., Francaviglia, M., Natural and Gauge Natural Formalism for Classical Field Theories, Kluver, 2003.

[2] Kolář, I., Michor, P.W., Slovák, J., Natural Operations in Differential Geometry, Springer Verlag, 1993.

[3] Kolář, I., Modugno, M., Torsions of connectionons on some natural bundles, Differential Geom. Appl., 2(1992), 1-16.

[4] Kurek, J., Mikulski, W.M., The natural affinors on the r-th order frame bundle, Demonstratio Math. 41(3)(2008), to appear.

[5] Kurek, J., Mikulski, W.M., Canonical vector valued 1-forms on higher order principal prolongations, Lobachevskii Math. J., (2008), to appear.

[6] de Leon, M., Rodrigues, P.R., Generalized Classical Mechanics and Field Theory, North-Holland Math. Studies 112, 1985, Amsterdam.

[7] Palais, R., The Geometrization of Physics, Lecture Notes in Mathematics, Taiwan, 1981.

Institute of Mathematics, Maria Curie-Sklodowska University, Pl. M. Curie-Sklodowskiej 1, Lublin, POLAND

E-mail address: `kurek@hektor.umcs.lublin.pl`

Institute of Mathematics, Jagiellonian University, ul. Reymonta 4, Kraków, POLAND

E-mail address: `mikulski@im.uj.edu.pl`