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A NEW SUBCLASS OF UNIFORMLY SPIRALLIKE FUNCTIONS WITH FIXED COEFFICIENTS

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Abstract. In this paper a new subclass of uniformly spirallike functions is defined and several properties like coefficient estimate, closure theorems, distortion theorems, radii of starlikeness and convexity are studied.

1. Introduction and definitions

Let S denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic and univalent in the open unit disc $U = \{z \in \mathbb{C} : |z| \leq 1\}$. Also let S^* and $\mathcal{C}$ denote the subclasses of S that are respectively, starlike and convex. Motivated by certain geometric conditions, Goodman [2, 3] introduced an interesting subclass of starlike functions called uniformly starlike functions denoted by UST and an analogous subclass of convex functions called uniformly convex functions, denoted by UCV. From [6, 8] we have

$$f \in UCV \Leftrightarrow Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \left|\frac{zf''(z)}{f'(z)}\right|, z \in U.$$

In [8], Ronning introduced a new class S_p of starlike functions which has more manageable properties. The classes UCV and S_p were further extended by Kanas and Wisniowska in [4, 5] as $k - UCV(\alpha)$ and $k - ST(\alpha)$. The classes of uniformly spirallike and uniformly convex spirallike were introduced by Ravichandran et al [7]. This was further generalized in [11] as $UCSP(\alpha, \beta)$. In [12], Herb Silverman introduced the subclass T of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

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which are analytic and univalent in the unit disc U. Motivated by [13], new subclasses with negative coefficients $UCSPT(\alpha,\beta)$ and $SP_pT(\alpha,\beta)$ were introduced and studied in [10]. A function $f(z)$ defined by (1.1) is in $UCSPT(\alpha,\beta)$ if

$$Re\left\{e^{-i\alpha}\left(1+\frac{zf"(z)}{f'(z)}\right)\right\} \geq \left|\frac{zf"(z)}{f'(z)}\right| + \beta, \tag{1.2}$$

$|\alpha| < \frac{\pi}{2}, 0 \leq \beta < 1$. For the class $UCSPT(\alpha,\beta)$, [10] proved the following lemma.

Lemma 1.1. *A function $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ is in $UCSPT(\alpha,\beta)$ if and only if*

$$\sum_{n=2}^{\infty}(2n-\cos\alpha-\beta)n\,a_n \leq \cos\alpha-\beta. \tag{1.3}$$

Using (1.1), the functions $f(z) \in UCSPT(\alpha,\beta)$ will satisfy

$$a_2 \leq \frac{(cos\alpha-\beta)}{2(4-cos\alpha-\beta)}. \tag{1.4}$$

The subclass $UCSPT_c(\alpha,\beta)$ is the class of functions in $UCSPT(\alpha,\beta)$ of the form

$$f(z) = z - \frac{c(\cos\alpha-\beta)z^2}{2(4-\cos\alpha-\beta)} - \sum_{n=3}^{\infty} a_n z^n, \tag{1.5}$$

$(a_n \geq 0)$, where $0 \leq c \leq 1$ was studied in [1]. When $c = 1$ we get

$$UCSPT_1(\alpha,\beta) = UCSPT(\alpha,\beta).$$

As an extension of $UCSPT_c(\alpha,\beta)$ a new class of functions $k - UCSPT_c(\alpha,\beta)$ is defined and studied in this paper. Let $k - UCSPT_c(\alpha,\beta)$ be the class of functions of the form

$$f(z) = z - \frac{c(\cos\alpha-\beta)z^2}{2(2(k+1)-\cos\alpha-\beta)} - \sum_{n=3}^{\infty} a_n z^n, \tag{1.6}$$

$(a_n \geq 0)$, where $0 \leq c \leq 1$ and $0 < k \leq 1$.

2. Coefficient Estimate

Theorem 2.1. *The function f(z) defined by (1.5) belongs to $k - UCSPT_c(\alpha,\beta)$ if and only if*

$$\sum_{n=3}^{\infty}((k+1)n-\cos\alpha-\beta)na_n \leq (1-c)(\cos\alpha-\beta). \tag{2.1}$$

The result is sharp.

Proof. Taking

$$a_2 = \frac{c(\cos\alpha-\beta)}{2(2(k+1)-\cos\alpha-\beta)}, 0 \leq c \leq 1, \tag{2.2}$$

in (1.3) we get the required result. Also the result is sharp for the function

$$f(z) = z - \frac{c(\cos\alpha-\beta)z^2}{2(2(k+1)-\cos\alpha-\beta)} - \frac{(1-c)(\cos\alpha-\beta)z^n}{n((k+1)n-\cos\alpha-\beta)}, (n \geq 3). \tag{2.3}$$

□

Corollary 2.1.1. *If $f(z)$ defined by (1.5) is in the class $k - UCSPT_c(\alpha, \beta)$ then,*

$$a_n \leq \frac{(1-c)(\cos\alpha - \beta)}{n((k+1)n - \cos\alpha - \beta)}, (n \geq 3). \tag{2.4}$$

The result is sharp for the function $f(z)$ given in (2.3).

3. Closure Theorems

Theorem 3.1. *The class $k - UCSPT_c(\alpha, \beta)$ is closed under convex linear combination.*

Proof. Let $f(z)$ defined by (1.5) be in $k - UCSPT_c(\alpha, \beta)$. Now define $g(z)$ by

$$g(z) = z - \frac{c(\cos\alpha - \beta)z^2}{2(2(k+1) - \cos\alpha - \beta)} - \sum_{n=3}^{\infty} b_n z^n, (b_n \geq 0). \tag{3.1}$$

If $f(z)$ and $g(z)$ belong to $k - UCSPT_c(\alpha, \beta)$ then it is enough to prove that the function $H(z)$ defined by

$$H(z) = \lambda f(z) + (1-\lambda)g(z), (0 \leq \lambda \leq 1) \tag{3.2}$$

is also in $k - UCSPT_c(\alpha, \beta)$.

$$H(z) = z - \frac{c(\cos\alpha - \beta)z^2}{2(2(k+1) - \cos\alpha - \beta)} - \sum_{n=3}^{\infty} (\lambda a_n + (1-\lambda)b_n) z^n. \tag{3.3}$$

Using theorem (2.1) we get

$$\sum_{n=3}^{\infty} ((k+1)n - \cos\alpha - \beta) n (\lambda a_n + (1-\lambda)b_n) \leq (1-c)(\cos\alpha - \beta). \tag{3.4}$$

Hence $H(z)$ is in $k - UCSPT_c(\alpha, \beta)$. Thus $k - UCSPT_c(\alpha, \beta)$ is closed under convex linear combination. □

Theorem 3.2. *Let the functions*

$$f_j(z) = z - \frac{c(\cos\alpha - \beta)z^2}{2(2(k+1) - \cos\alpha - \beta)} - \sum_{n=3}^{\infty} a_{n,j} z^n, (a_{n,j} \geq 0), \tag{3.5}$$

be in the class $k = UCSPT_c(\alpha, \beta)$ for every j = 1,2,...m. Then the function $F(z)$ defined by

$$F(z) = \sum_{j=1}^{m} d_j f_j(z), (d_j \geq 0), \tag{3.6}$$

is also in the same class $k - UCSPT_c(\alpha, \beta)$ where

$$\sum_{j=1}^{m} d_j = 1. \tag{3.7}$$

Proof. Using (3.5) and (3.7) in (3.6) we have

$$F(z) = z - \frac{c(\cos\alpha - \beta)z^2}{2(2(k+1) - cos\alpha - \beta)} - \sum_{n=3}^{\infty}\left[\sum_{j=1}^{m} d_j a_n, j\right] z^n. \tag{3.8}$$

Each $f_j(z) \in k - UCSPT_c(\alpha, \beta)$ for j = 1,2,...m, theorem (2.1) gives

$$\sum_{n=3}^{\infty}((k+1)n - \cos\alpha - \beta)na_{n,j} \leq (1-c)(\cos\alpha - \beta), \tag{3.9}$$

for j =1,2,..m. Hence we get

$$\begin{aligned}\sum_{n=3}^{\infty} n((k+1)n - \cos\alpha - \beta)\left[\sum_{j=1}^{m} d_j a_{n,j}\right] &= \sum_{j=1}^{m} d_j\left[\sum_{n=3}^{\infty} n((k+1)n - \cos\alpha - \beta)a_{n,j}\right] \\ &\leq (1-c)(\cos\alpha - \beta).\end{aligned}$$

This implies $F(z) \in k - UCSPT_c(\alpha, \beta)$, by theorem(2.1). □

Theorem 3.3. *Let*

$$f_2(z) = z - \frac{c(\cos\alpha - \beta)z^2}{2(2(k+1) - \cos\alpha - \beta)} \tag{3.10}$$

and

$$f_n(z) = z - \frac{c(\cos\alpha - \beta)z^2}{2(2(k+1) - \cos\alpha - \beta)} - \frac{(1-c)(\cos\alpha - \beta)z^n}{n((k+1)n - \cos\alpha - \beta)}, \tag{3.11}$$

for n = 3,4,.... Then $f(z)$ is in $k - UCSPT_c(\alpha, \beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=2}^{\infty} \lambda_n f_n(z) \tag{3.12}$$

where $\lambda_n \geq 0$and $\sum_{n=2}^{\infty} \lambda_n = 1$.

Proof. First assume that $f(z)$ can be expressed in the form(3.12). Then we have

$$f(z) = z - \frac{c(\cos\alpha - \beta)z^2}{2(2(k+1) - \cos\alpha - \beta)} - \sum_{n=3}^{\infty}\frac{(1-c)(\cos\alpha - \beta)}{n((k+1)n - \cos\alpha - \beta)}\lambda_n z^n. \tag{3.13}$$

But

$$\begin{aligned}\sum_{n=3}^{\infty}\frac{(1-c)(\cos\alpha - \beta)}{n((k+1)n - \cos\alpha - \beta)}\lambda_n n((k+1)n - \cos\alpha - \beta) &= (1-c)(\cos\alpha - \beta)(1 - \lambda_2) \\ &\leq (1-c)(\cos\alpha - \beta).\end{aligned} \tag{3.14}$$

Hence from (2.1) it follows that $f(z) \in k - UCSPT_c(\alpha, \beta)$. Conversely, we assume that $f(z)$ defined by (1.6) is in the class $k - UCSPT_c(\alpha, \beta)$. Then by using (2.4), we get

$$a_n \leq \frac{(1-c)(\cos\alpha - \beta)}{n((k+1)n - \cos\alpha - \beta)}, (n = 3, 4, ...).$$

Taking $\lambda_n = \frac{n((k+1)n - \cos\alpha - \beta)a_n}{(1-c)(\cos\alpha - \beta)}, (n = 3, 4, ...)$ and $\lambda_2 = 1 - \sum_{n=3}^{\infty} \lambda_n$, we have (3.12). Hence the proof of theorem (3.3) is complete. □

Corollary 3.3.1. *The extreme points of the class $k - UCSPT_c(\alpha, \beta)$ are the functions*
$f_n(z), (n \geq 2)$ given by theorem (3.3) .

4. Distortion Theorems

In order to obtain the distortion bounds for the function $f(z) \in k{-}UCSPT_c(\alpha, \beta)$, we need the following lemmas.

Lemma 4.1. *Let the function $f_3(z)$ be defined by*

$$f_3(z) = z - \frac{c(\cos\alpha - \beta)z^2}{2(2(k+1) - \cos\alpha - \beta)} - \frac{(1-c)(\cos\alpha - \beta)z^3}{3(3(k+1) - \cos\alpha - \beta)}. \tag{4.1}$$

Then, for $0 \leq r < 1$ and $0 \leq c \leq 1$,

$$|f_3(re^{i\theta})| \geq r - \frac{c(\cos\alpha - \beta)r^2}{2(2(k+1) - \cos\alpha - \beta)} - \frac{(1-c)(\cos\alpha - \beta)r^3}{3(3(k+1) - \cos\alpha - \beta)}, \tag{4.2}$$

with equality for $\theta = 0$. For either $0 \leq c < c_0$ and $0 \leq r \leq r_0$ or $c_0 \leq c \leq 1$,

$$|f_3(re^{i\theta})| \leq r + \frac{c(\cos\alpha - \beta)r^2}{2(2(k+1) - \cos\alpha - \beta)} - \frac{(1-c)(\cos\alpha - \beta)r^3}{3(3(k+1) - \cos\alpha - \beta)}, \tag{4.3}$$

with equality for $\theta = \pi$. Further, for $0 \leq c < c_0$ and $r_0 \leq r < 1$,

$$\begin{aligned} |f_3(re^{i\theta})| \leq r \Bigg[& [1 + \frac{9c^2(\cos\alpha - \beta)(3(k+1) - \cos\alpha - \beta)}{16(1-c)(2(k+1) - \cos\alpha - \beta)^2}] \\ & + r^2(\cos\alpha - \beta)[\frac{2(1-c)}{3(3(k+1) - \cos\alpha - \beta)} \\ & - \frac{c^2(\cos\alpha - \beta)}{8(2(k+1) - \cos\alpha - \beta)^2}] \\ & + \frac{r^4(1-c)(\cos\alpha - \beta)^2}{(3(k+1) - \cos\alpha - \beta)}[\frac{(1-c)}{9(3(k+1) - \cos\alpha - \beta)} \\ & + \frac{c^2(cos\alpha - \beta)}{16(2(k+1) - \cos\alpha - \beta)^2}] \Bigg]^{1/2}, \end{aligned}$$

with equality for $\theta = \cos^{-1}\left[\frac{c(\cos\alpha-\beta)(1-c)r^2-3c(3(k+1)-\cos\alpha-\beta)}{8(1-c)(2(k+1)-\cos\alpha-\beta)r}\right]$, *where*

$$c_0 = \frac{1}{2(\cos\alpha-\beta)}\Bigg[(12\cos\alpha+10\beta-25(k+1)) + \sqrt{(12\cos\alpha+10\beta-25(k+1))^2+32(\cos\alpha-\beta)(2(k+1)-\cos\alpha-\beta)}\Bigg] \tag{4.4}$$

and

$$r_0 = \frac{1}{c(1-c)(\cos\alpha-\beta)}\Bigg[-4(1-c)(2(k+1)-\cos\alpha-\beta) + \sqrt{16(1-c)^2(2(k+1)-\cos\alpha-\beta)^2+3c^2(1-c)(3(k+1)-\cos\alpha-\beta)(\cos\alpha-\beta)}\Bigg]. \tag{4.5}$$

Proof. We employ the techniques used by Silverman and Silvia[13]. Since

$$\frac{\partial|f_3(re^{i\theta})|^2}{\partial\theta} = \frac{(\cos\alpha-\beta)r^3\sin\theta}{(2(k+1)-\cos\alpha-\beta)}\Bigg[c+\frac{8(1-c)(2(k+1)-\cos\alpha-\beta)r\cos\theta}{3(3(k+1)-\cos\alpha-\beta)} - \frac{c(1-c)r^2(\cos\alpha-\beta)}{3(3(k+1)-\cos\alpha-\beta)}\Bigg], \tag{4.6}$$

we see that $\frac{\partial|f_3(re^{i\theta})|^2}{\partial\theta}=0$, for θ_1=0, θ_2=π and

$$\theta_3 = \cos^{-1}\left[\frac{(\cos\alpha-\beta)c(1-c)r^2-3c(3(k+1)-\cos\alpha-\beta)}{8(1-c)(2(k+1)-\cos\alpha-\beta)r}\right], \tag{4.7}$$

since θ_3 is a valid root only when $-1 \le \cos\theta_3 \le 1$. Hence there is a third root if and only if $r_0 \le r < 1$ and $0 \le c \le c_0$. Thus the results of the theorem follow by comparing the extremal values $|f_3(re^{i\theta_k})|$, $(k=1,2,3)$ on the appropriate intervals. □

Lemma 4.2. *Let the function* $f_n(z)$ *be defined by (3.11) and* $n \ge 4$. *Then*

$$|f_n(re^{i\theta})| \le |f_n(-r)|. \tag{4.8}$$

Proof. Since $f_n(z) = z - \frac{c(\cos\alpha-\beta)z^2}{2(2(k+1)-\cos\alpha-\beta)} - \frac{(1-c)(\cos\alpha-\beta)z^n}{n((k+1)n-\cos\alpha-\beta)}$ and $\frac{r^n}{n}$ is a decreasing function of n, we have

$$\begin{aligned}|f_n(re^{i\theta})| &\le r + \frac{c(\cos\alpha-\beta)r^2}{2(2(k+1)-\cos\alpha-\beta)} + \frac{(1-c)(\cos\alpha-\beta)r^n}{n((k+1)n-\cos\alpha-\beta)}\\ &\le r + \frac{c(\cos\alpha-\beta)r^2}{2(2(k+1)-\cos\alpha-\beta)} + \frac{(1-c)(\cos\alpha-\beta)r^4}{4(4(k+1)-\cos\alpha-\beta)} = -f_4(-r),\end{aligned}$$

which gives (4.8) . □

Theorem 4.3. *Let the function $f(z)$ defined by (1.6) belong to the class $k-UCSPT_c(\alpha,\beta)$. Then for $0\le r<1$,*

$$|f(re^{i\theta})|\ge r-\frac{c(\cos\alpha-\beta)r^2}{2(2(k+1)-\cos\alpha-\beta)}-\frac{(1-c)(\cos\alpha-\beta)r^3}{3(3(k+1)-\cos\alpha-\beta)},$$

with equality for $f_3(z)$ at z=r and

$$|f(re^{i\theta})|\le max\{max_\theta|f_3(re^{i\theta})|,-f_4(-r)\},$$

where $max_\theta|f_3(re^{i\theta})|$ is given by lemma 4.1.

The proof is obtained by comparing the bounds of lemma 4.1 and lemma 4.2.

Corollary 4.3.1. *Let the function $f(z)$ be defined by (1.1) be in the class $k-UCSPT(\alpha,\beta)$. Then for $|z|=r<1$, we have*

$$r-\frac{(\cos\alpha-\beta)r^2}{2(2(k+1)-\cos\alpha-\beta)}\le|f(z)|\le r+\frac{(\cos\alpha-\beta)r^2}{2(2(k+1)-\cos\alpha-\beta)}.$$

The result is sharp.

Corollary 4.3.2. *Let the function $f(z)$ be defined by (1.5) be in the class $k-UCSPT_c(\alpha,\beta)$. Then the disk $|z|<1$ is mapped onto a domain that contains the disk*

$$|w|<\frac{6(3(k+1)-\cos\alpha-\beta)(2(k+1)-\cos\alpha-\beta)-(\cos\alpha-\beta)(4(k+1)+5c(k+1)-(c+2)(\cos\alpha-\beta))}{6(2(k+1)-\cos\alpha-\beta)(3(k+1)-\cos\alpha-\beta)}.$$

The result is sharp with the extremal function

$$f_3(z)=z-\frac{c(\cos\alpha-\beta)z^2}{2(2(k+1)-\cos\alpha-\beta)}-\frac{(1-c)(\cos\alpha-\beta)z^3}{3(3(k+1)-\cos\alpha-\beta)}.$$

Proof. The result follows by letting $r\to1$ in theorem 4.3. □

Lemma 4.4. *Let the function $f_3(z)$ be defined by (4.1) . Then for $0\le r<1$ and $0\le c\le1$,*

$$|f_3'(re^{i\theta})|\ge1-\frac{c(\cos\alpha-\beta)r}{(2(k+1)-\cos\alpha-\beta)}-\frac{(1-c)(\cos\alpha-\beta)r^2}{(3(k+1)-\cos\alpha-\beta)},$$

with equality for $\theta=0$. For either $0\le c<c_1$ and $o\le r\le r_1$ or $c_1\le c\le1$,

$$|f_3'(re^{i\theta})|\le1+\frac{c(\cos\alpha-\beta)r}{(2(k+1)-\cos\alpha-\beta)}-\frac{(1-c)(\cos\alpha-\beta)r^2}{(3(k+1)-\cos\alpha-\beta)},$$

with equality for $\theta=\pi$. Further,$0\le c<c_1$ and $r_1\le r<1$,

$$|f_3'(re^{i\theta})|\le\Bigg\{\left[1+\frac{c^2(\cos\alpha-\beta)(3(k+1)-\cos\alpha-\beta)}{4(1-c)(2(k+1)-\cos\alpha-\beta)^2}\right]$$

$$+(\cos\alpha-\beta)\left[\frac{2(1-c)}{(3(k+1)-\cos\alpha-\beta)}+\frac{c^2(\cos\alpha-\beta)}{2(2(k+1)-\cos\alpha-\beta)^2}\right]r^2$$

$$+\frac{(1-c)(\cos\alpha-\beta)^2}{3(k+1)-\cos\alpha-\beta}\Big[\frac{(1-c)}{(3(k+1)-\cos\alpha-\beta)}+\frac{c^2(\cos\alpha-\beta)}{4(2(k+1)-\cos\alpha-\beta)^2}\Big]r^4\Bigg\}^{1/2},$$

with equality for

$$\theta = \cos^{-1}\left[\frac{c(1-c)(\cos\alpha-\beta)r^2 - c(3(k+1)-\cos\alpha-\beta)}{4(1-c)r(2(k+1)-\cos\alpha-\beta)}\right],$$

where

$$c_1 = \frac{-(11(k+1)-6\cos\alpha-4\beta)}{2(\cos\alpha-\beta)} + \frac{\sqrt{(11(k+1)-6\cos\alpha-4\beta)^2+16(2(k+1)-\cos\alpha-\beta)(\cos\alpha-\beta)}}{2(\cos\alpha-\beta)}$$

and

$$r_1 = \frac{1}{c(1-c)(\cos\alpha-\beta)}\Big\{-2(1-c)(2(k+1)-\cos\alpha-\beta) + \sqrt{4(1-c)^2(2(k+1)-\cos\alpha-\beta)^2+c^2(1-c)(\cos\alpha-\beta)(3(k+1)-\cos\alpha-\beta)}\Big\}.$$

The proof of lemma(4.4) is given in the same way as lemma(4.1).

Theorem 4.5. *Let the function $f(z)$ defined by (1.6) be in the class $k-UCSPT_c(\alpha,\beta)$. Then for $0 \le r < 1$,*

$$|f'(re^{i\theta})| \ge 1 - \frac{c(\cos\alpha-\beta)r}{(2(k+1)-\cos\alpha-\beta)} - \frac{(1-c)(\cos\alpha-\beta)r^2}{(3(k+1)-\cos\alpha-\beta)},$$

with equality for $f_3'(z)$ at z=r and

$$\left|f'(re^{i\theta})\right| \le max\left\{max_\theta \left|f_3'(re^{i\theta})\right|, f_4'(-r)\right\},$$

where $max_\theta \left|f_3'(re^{i\theta})\right|$ is given by lemma (4.4).

Remark: For $c = 1$ in theorem 4.5 we obtain:

Corollary 4.5.1. *Let the function $f(z)$ defined by (1.1) be in the class $k-UCSPT(\alpha,\beta)$. Then for $|z| = r < 1$, we have*

$$1 - \frac{(\cos\alpha-\beta)r}{2(k+1)-\cos\alpha-\beta} \le |f'(z)| \le 1 + \frac{(\cos\alpha-\beta)r}{2(k+1)-\cos\alpha-\beta},$$

the result is sharp.

5. Radii of starlikeness and convexity

Theorem 5.1. *Let the function $f(z)$ defined by(1.6) be in the class $k-UCSPT_c(\alpha,\beta)$. Then $f(z)$ is starlike of order $\rho(0 \le \rho < 1)$ in the disc $|z| < r_1(\alpha,\beta,c,k,\rho)$ where $r_1(\alpha,\beta,c,k,\rho)$ is the largest value for which*

$$\frac{c(\cos\alpha-\beta)(2-\rho)r}{2(2(k+1)-\cos\alpha-\beta)} + \frac{(1-c)(\cos\alpha-\beta)(n-\rho)r^{n-1}}{n((k+1)n-\cos\alpha-\beta)} \le 1-\rho, \tag{5.1}$$

for $n \geq 3$. The result is sharp with the extremal function

$$f_n(z) = z - \frac{c(\cos\alpha - \beta)z^2}{2(2(k+1) - \cos\alpha - \beta)} - \frac{(1-c)(\cos\alpha - \beta)z^n}{n((k+1)n - \cos\alpha - \beta)}, \tag{5.2}$$

for some n.

Proof. It suffices to show that

$$\Big|\frac{zf'(z)}{f(z)} - 1\Big| \leq 1 - \rho, (o \leq \rho < 1),$$

for $|z| < r_1(\alpha, \beta, c, k, \rho)$. Note that

$$\begin{aligned}\Big|\frac{zf'(z)}{f(z)} - 1\Big| &\leq \frac{\frac{c(\cos\alpha-\beta)r}{2(2(k+1)-\cos\alpha-\beta)} + \sum_{n=3}^{\infty}(n-1)a_n r^{n-1}}{1 - \frac{c(\cos\alpha-\beta)r}{2(2(k+1)-\cos\alpha-\beta)} - \sum_{n=3}^{\infty} a_n r^{n-1}} \\ &\leq 1 - \rho,\end{aligned}$$

for $|z| \leq r$ if and only if

$$\frac{c(\cos\alpha - \beta)(2-\rho)r}{2(2(k+1) - \cos\alpha - \beta)} + \sum_{n=3}^{\infty}(n-\rho)a_n r^{n-1} \leq 1 - \rho.$$

Since $f(z)$ is in $k - UCSPT_c(\alpha, \beta)$ from (2.1) we may take

$$a_n = \frac{(1-c)(\cos\alpha - \beta)\lambda_n}{n((k+1)n - \cos\alpha - \beta)}, (n \geq 3),$$

where $\lambda_n \geq 0 (n \geq 3)$ and $\sum_{n=3}^{\infty} \lambda_n \leq 1$. For each fixed r, we choose the positive integer $n_0 = n_0(r)$ for which $\frac{(n-\rho)r^{n-1}}{n}$ is maximal. Then it follows that

$$\sum_{n=3}^{\infty}(n-\rho)a_n r^{n-1} \leq \frac{(1-c)(\cos\alpha - \beta)(n_0 - \rho)r^{n_0-1}}{n_0((k+1)n_0 - \cos\alpha - \beta)}.$$

Hence $f(z)$ is starlike of order ρ in $|z| < r_1(\alpha, \beta, c, k, \rho)$ provided that

$$\frac{c(\cos\alpha - \beta)(2-\rho)r}{2(2(k+1) - \cos\alpha - \beta)} + \frac{(1-c)(\cos\alpha - \beta)(n_0 - \rho)r^{n_0-1}}{n_0((k+1)n_0 - \cos\alpha - \beta)} \leq 1 - \rho.$$

We find the value $r_0 = r_0(\alpha, \beta, c, k, \rho)$ and the corresponding integer $n_0(r_0)$ so that

$$\frac{c(\cos\alpha - \beta)(2-\rho)r_0}{2(2(k+1) - \cos\alpha - \beta)} + \frac{(1-c)(\cos\alpha - \beta)(n_0 - \rho)r_0^{n_0-1}}{n_0((k+1)n_0 - \cos\alpha - \beta)} = 1 - \rho.$$

Then this value r_0 is the radius of starlikeness of order ρ for functions $f(z)$ belonging to the class $k - UCSPT_c(\alpha, \beta)$. □

We prove the following theorem concerning the radius of convexity of order ρ for functions in the class $k - UCSPT_c(\alpha, \beta)$.

Theorem 5.2. *Let the function $f(z)$ be defined by (1.6) be in the class $k-UCSPT_c(\alpha,\beta)$. Then $f(z)$ is convex of order $\rho(0 \leq \rho < 1)$ in the disc $|z| < r_2(\alpha,\beta,c,k,\rho)$, where $r_2(\alpha,\beta,c,k,\rho)$ is the largest value for which*

$$\frac{c(\cos\alpha-\beta)(2-\rho)r}{(2(k+1)-\cos\alpha-\beta)}+\frac{(1-c)(\cos\alpha-\beta)(n-\rho)r^{n-1}}{((k+1)n-\cos\alpha-\beta)} \leq 1-\rho,$$

for $n \geq 3$. The result is sharp for the function f(z) given by (5.2).

6. The class $k-UCSPT_{c_n,N}(\alpha,\beta)$

We now fix finitely many coefficients instead of fixing just the second coefficients. Let $UCSPT_{c_n,N}(\alpha,\beta)$ denote the class of functions in $UCSPT_c(\alpha,\beta)$ of the form

$$f(z)=z-\sum_{n=2}^{N}\frac{c_n(\cos\alpha-\beta)z^n}{n(2n-\cos\alpha-\beta)}-\sum_{n=N+1}^{\infty}a_nz^n,$$

where $0 \leq \sum_{n=2}^{N} c_n = c \leq 1$. Note that $k-UCSPT_{c_n,2}(\alpha,\beta)=k-UCSPT_c(\alpha,\beta)$.

Theorem 6.1. *The extreme points of the class $k-UCSPT_{c_n,N}(\alpha,\beta)$ are*

$$z-\sum_{n=2}^{N}\frac{c_n(\cos\alpha-\beta)z^n}{n((k+1)n-\cos\alpha-\beta)}$$

and

$$z-\sum_{n=2}^{N}\frac{c_n(\cos\alpha-\beta)z^n}{n((k+1)n-\cos\alpha-\beta)}-\frac{(1-c)(\cos\alpha-\beta)z^n}{n((k+1)n-\cos\alpha-\beta)},$$

for n=N+1,N+2,....

The characterization of the extreme points enables us to solve the standard extremal problems in the same manner as was done in $k-UCSPT_c(\alpha,\beta)$. The details are omitted.

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