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ON DECAY AND BLOW UP OF SOLUTIONS FOR A SYSTEM OF KIRCHHOFF TYPE EQUATIONS WITH DAMPING TERMS

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Abstract: In this paper, we investigate system of Kirchhoff type equations with bounded domain. We obtain decay of solutions by using multiplier method. Later, we will prove blow up results for negative inital energy.

Keywords: Decay, Blow up, Kirchhoff type equation.

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1. Introduction

In this paper, we consider the following initial boundary value system

$$\begin{cases} u_{tt} - M\left(\|\nabla u\|^2\right) \triangle u + \gamma_2 u_t + |u_t|^p u_t = F_u(u,v), & (x,t) \in \Omega \times (0,T), \\ v_{tt} - M\left(\|\nabla v\|^2\right) \triangle v + \gamma_2 v_t + |v_t|^q v_t = F_v(u,v), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = v(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), & x \in \Omega, \\ v(x,0) = v_0(x), \ v_t(x,0) = v_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in R^n $(n = 1,2,3)$, $p,q > 0$, $\gamma_2 > 0$. Let $\triangle = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ be the Laplace operator, and $M(s)$ be a nonnegative locally Lipschitz function, and $F : R^2 \longrightarrow R$ is a C^1 function given by

$$F(u,v) = a|u+v|^{r+2} + 2b|uv|^{\frac{r+2}{2}}, \tag{1.2}$$

where $r \geq 2$, $a > 1$ and $b > 0$, which implies

$$F_u(u,v) = (r+2)\left[a|u+v|^r(u+v) + b|u|^{\frac{r-2}{2}}|v|^{\frac{r+2}{2}}u\right],$$

$$F_v(u,v) = (r+2)\left[a|u+v|^r(u+v) + b|v|^{\frac{r-2}{2}}|u|^{\frac{r+2}{2}}v\right].$$

Also, we have

$$uF_u(u,v) + vF_v(u,v) = (r+2)F(u,v) \quad \forall (u,v) \in R^2. \tag{1.3}$$

Lemma 1.1 [10]. Let c_0 and c_1 be positive constants, such that

$$c_0\left(|u|^{r+2}+|v|^{r+2}\right)\leq F(u,v)\leq c_1\left(|u|^{r+2}+|v|^{r+2}\right). \tag{1.4}$$

Throughout this paper, we define $M(s)$ by

$$M(s)=\alpha+\beta s^{\gamma},\ s\geq 0,\ \alpha,\ \beta>0,\ \gamma\geq 0. \tag{1.5}$$

Obviously, $M(s)$ is a nonnegative locally Lipschitz function.

The problem (1.1) is a generalization of a model introduced by Kirchhoff [5]. More precisely, Kirchhoff proposed a model given by the equation for $f=g=0$,

$$\rho h\frac{\partial^2 u}{\partial t^2}+\delta\frac{\partial u}{\partial t}+g\left(\frac{\partial u}{\partial t}\right)=\left\{\rho_0+\frac{Eh}{2L}\int_0^L\left(\frac{\partial u}{\partial x}\right)^2dx\right\}\frac{\partial^2 u}{\partial x^2}+f(u), \tag{1.6}$$

for $0<x<L,\ t\geq 0$.

The qualitative analysis of solutions for a single Kirchhoff type equation

$$u_{tt}-M\left(\|\nabla u\|^2\right)\triangle u+g(u_t)=f(u),\ x\in\Omega,\ t>0, \tag{1.7}$$

has been discussed by many authors, see [3, 7, 12, 18, 20].

When $M(s)=1$, (1.1) become the following system

$$\begin{cases} u_{tt}-\triangle u+\gamma_2 u_t+|u_t|^p u_t=F_u(u,v), \\ v_{tt}-\triangle v+\gamma_2 v_t+|v_t|^q v_t=F_v(u,v). \end{cases} \tag{1.8}$$

Many authors studied the existence, blow up and decay of solutions of (1.8) (see [6, 11, 13, 14, 19, 21]). Also, many authors studied the existence and nonexistence of solutions (1.8) with $\gamma_2=0$ ([2, 16, 17]).

Motivated by previous works, we study the decay of solutions and the blow up of solutions with negative initial energy for the system of Kirchhoff type equations with damping terms.

The outline of this paper is as follows. In section 2, we give some lemmas and notations. In section 3, the decay of the solution is given. Section 4, we show the blow up properties of solution.

2. Preliminaries

In this paper, $\|.\|$ and $\|.\|_p$ denote the usual $L^2(\Omega)$ norm and $L^p(\Omega)$ norm, respectively. Firstly, we give the following lemmas:

Lemma 2.1 [8]. Suppose that, $0<p\ (n=1,2)$ or $0<p\leq 2\frac{n-1}{n-2}\ (n\geq 3)$ holds. Then there exists a positive constant $C>1$ depending only on Ω, such that

$$\|u\|_p^s\leq C\left(\|\nabla u\|^2+\|u\|_p^p\right),$$

for any $u\in H_0^1(\Omega),\ 2\leq s\leq p$.

Lemma 2.2 (Sobolev-Poincare inequality) [1]. Let p be a number with $2 \leq p < \infty$ $(n = 1, 2)$ or $2 \leq p \leq 2n/(n-2)$ $(n \geq 3)$, such that

$$\|u\|_q \leq C_* \|\nabla u\| \text{ for } u \in H_0^1(\Omega) \tag{2.1}$$

where $C_* = C_*(\Omega,\ q)$ is a constant.

We introduce the following functionals

$$\begin{aligned} J(t) &= J(u(t), v(t)) = \frac{\alpha}{2}\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) \\ &+ \frac{\beta}{2(\gamma+1)}\left(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)}\right) - \int_\Omega F(u, v)\, dx, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} I(t) &= I(u(t), v(t)) = \alpha\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) \\ &+ \beta\left(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)}\right) - (r+2)\int_\Omega F(u, v)\, dx. \end{aligned} \tag{2.3}$$

Next, we introduce the energy functional

$$\begin{aligned} E(t) &= \frac{1}{2}\left(\|u_t\|^2 + \|v_t\|^2\right) + \frac{\alpha}{2}\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) \\ &+ \frac{\beta}{2(\gamma+1)}\left(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)}\right) - \int_\Omega F(u, v)\, dx. \end{aligned} \tag{2.4}$$

We also define

$$W = \left\{(u, v) : (u, v) \in \left(H_0^1(\Omega) \cap H^2(\Omega)\right) \times \left(H_0^1(\Omega) \cap H^2(\Omega)\right),\ I(u, v) > 0\right\} \cup \{(0, 0)\}. \tag{2.5}$$

Lemma 2.3. Let $u(x, t)$ be the solution of (1.1). Then

$$E'(t) = -\gamma_2\left(\|u_t\|^2 + \|v_t\|^2\right) - \left(\|u_t\|_{p+2}^{p+2} + \|v_t\|_{q+2}^{q+2}\right) \leq 0. \tag{2.6}$$

Proof. By multiplying the first equation of (1.1) by u_t and the second equation by v_t, integrating over Ω, we obtain

$$E(t) - E(0) = -\int_0^t \left[\gamma_2\left(\|u_\tau\|^2 + \|v_\tau\|^2\right) + \left(\|u_\tau\|_{p+2}^{p+2} + \|v_\tau\|_{q+2}^{q+2}\right)\right] d\tau \text{ for } t \geq 0. \tag{2.7}$$

We state a local existence result without a proof here (see [4, 15, 18])

Theorem 2.1 . Suppose that $\min\{p, q\} > r$ such that

$$\begin{cases} 0 < p, q, \quad 0 < r, \qquad n = 1, 2 \\ 0 < p, q \leq \frac{2}{n-2}, \quad 0 < r, \qquad n \geq 3, \end{cases}$$

and let $(u_0, v_0) \in \left(H_0^1(\Omega) \cap H^2(\Omega)\right) \times \left(H_0^1(\Omega) \cap H^2(\Omega)\right)$, $(u_1, v_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$ be given. Then the problem (1.1) has a unique local solution

$$u, v \in C\left([0, T); H_0^1(\Omega) \cap H^2(\Omega)\right) \text{ and } u_t, v_t \in C\left([0, T); H_0^1(\Omega)\right),$$

for any fixed time $T>0$.

3. The decay result

In this section, we consider the energy decay of the solution to (1.1). For this purpose, we use the functional

$$\Phi(t)=E(t)+\epsilon\int_{\Omega}(uu_t+vv_t)\,dx+\frac{\epsilon\gamma_2}{2}\left(\|u\|^2+\|v\|^2\right), \tag{3.1}$$

where ϵ is a positive constant. Inspiring by idea in [9], we will show in the next lemma that $\Phi(t)$ and $E(t)$ are equivalent.

Lemma 3.1. For $\epsilon>0$ small enough, the relation

$$\alpha_1 E(t)\leq\Phi(t)\leq\alpha_2(E(t))^{\frac{1}{\gamma+1}} \tag{3.2}$$

holds for two positive constants α_1 and α_2.

Proof. Applying Young inequality and Sobolev embedding theorem, we obtain

$$\begin{aligned}\Phi(t) &\leq E(t)+\frac{\epsilon}{2}\left(\int_{\Omega}u^2dx+\int_{\Omega}u_t^2dx+\int_{\Omega}v^2dx+\int_{\Omega}v_t^2dx\right)+\frac{\epsilon\gamma_2}{2}\left(\|u\|^2+\|v\|^2\right)\\ &\leq\left(1+\frac{\epsilon}{2}\right)E(t)+C_*\frac{\epsilon\gamma_2}{2}C^{\frac{1}{\gamma+1}}\left(\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)}\right)^{\frac{1}{(\gamma+1)}},\end{aligned}$$

where $(a+b)^{\lambda}\leq C\left(a^{\lambda}+b^{\lambda}\right),\ a,b>0$ was used. After that we obtain

$$\begin{aligned}\Phi(t) &\leq\left(1+\frac{\epsilon}{2}\right)E(t)+C_*\frac{\epsilon\gamma_2}{2}C^{\frac{1}{\gamma+1}}(E(t))^{\frac{1}{(\gamma+1)}}\\ &\leq\left(\left(1+\frac{\epsilon}{2}\right)(E(t))^{\frac{\gamma}{\gamma+1}}+C_*\frac{\epsilon\gamma_2}{2}C^{\frac{1}{\gamma+1}}\right)(E(t))^{\frac{1}{(\gamma+1)}}\\ &\leq\left(\left(1+\frac{\epsilon}{2}\right)(E(0))^{\frac{\gamma}{\gamma+1}}+C_*\frac{\epsilon\gamma_2}{2}C^{\frac{1}{\gamma+1}}\right)(E(t))^{\frac{1}{(\gamma+1)}}\\ &=\alpha_2(E(t))^{\frac{1}{(\gamma+1)}},\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\Phi(t) &\geq E(t)-\epsilon\left[\frac{1}{4\tau}\left(\|u_t\|^2+\|v_t\|^2\right)+\tau\left(\|u\|^2+\|v\|^2\right)\right]+\frac{\epsilon\gamma_2}{2}\left(\|u\|^2+\|v\|^2\right)\\ &\geq E(t)-\frac{\epsilon}{4\tau}\left(\|u_t\|^2+\|v_t\|^2\right)+\epsilon\left(\frac{\gamma_2}{2}-\tau\right)\left(\|u\|^2+\|v\|^2\right)\\ &\geq E(t)-\frac{\epsilon}{4\tau}\left(\|u_t\|^2+\|v_t\|^2\right)\\ &=J(t)+\left(\frac{1}{2}-\frac{\epsilon}{4\tau}\right)\left(\|u_t\|^2+\|v_t\|^2\right)\\ &\geq J(t)+\frac{\alpha_1}{2}\left(\|u_t\|^2+\|v_t\|^2\right)\\ &\geq\alpha_1 E(t).\end{aligned} \tag{3.4}$$

for small enough τ. This completes the proof.

Theorem 3.1. Suppose that $\min\{p,q,r\}>2\gamma$ and

$$\frac{\beta(2\gamma+1)}{2(\gamma+1)}>2c_*^{r+2}c_1(r+1)\left(\frac{2(r+2)(\gamma+1)}{\beta(r-2\gamma)}E(0)\right)^{\frac{r-2\gamma}{2(\gamma+1)}}$$

such that (2.5) is satisfied and let $(u_0, v_0) \in W$ be given. Then the solution satisfies

$$E(t) \leq \begin{cases} Ke^{-kt}, & \gamma = 0, \\ (kt+K)^{-\frac{1}{\gamma}}, & \gamma > 0, \end{cases} \tag{3.5}$$

where K and k are positive constants which will be defined later.

Proof. Now differentiate (3.1) and use Eq. (1.1) and Young inequality, we have

$$\begin{aligned} \Phi'(t) &= -\gamma_2\left(\|u_t\|^2+\|v_t\|^2\right)-\left(\|u_t\|_{p+2}^{p+2}+\|v_t\|_{q+2}^{q+2}\right)+\epsilon\left(\|u_t\|^2+\|v_t\|^2\right) \\ &\quad -\epsilon\alpha\left(\|\nabla u\|^2+\|\nabla v\|^2\right)-\epsilon\beta\left(\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)}\right) \\ &\quad -\epsilon\left(\int_\Omega uu_t\,|u_t|^p\,dx+\int_\Omega vv_t\,|v_t|^q\,dx\right)+\epsilon(r+2)\int_\Omega F(u,v)\,dx \\ &\leq -(\gamma_2-\epsilon)\left(\|u_t\|^2+\|v_t\|^2\right)-\left(\|u_t\|_{p+2}^{p+2}+\|v_t\|_{q+2}^{q+2}\right) \\ &\quad -\epsilon\alpha\left(\|\nabla u\|^2+\|\nabla v\|^2\right)-\epsilon\beta\left(\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)}\right) \\ &\quad +\epsilon\left(\delta\|u\|_{p+2}^{p+2}+c(\delta)\|u_t\|_{p+2}^{p+2}+\delta\|v\|_{q+2}^{q+2}+c(\delta)\|v_t\|_{q+2}^{q+2}\right) \\ &\quad +\epsilon(r+2)\int_\Omega F(u,v)\,dx. \end{aligned} \tag{3.6}$$

By using the definition of the $E(t)$, we get

$$\begin{aligned} \Phi'(t) &\leq -\epsilon E(t)-\left(\gamma_2-\frac{3\epsilon}{2}\right)\left(\|u_t\|^2+\|v_t\|^2\right)-\frac{\epsilon\alpha}{2}\left(\|\nabla u\|^2+\|\nabla v\|^2\right) \\ &\quad -(1-c(\delta))\left(\|u_t\|_{p+2}^{p+2}+\|v_t\|_{q+2}^{q+2}\right)+\epsilon\left(\delta\|u\|_{p+2}^{p+2}+\delta\|v\|_{q+2}^{q+2}\right) \\ &\quad +\epsilon(r+1)\int_\Omega F(u,v)\,dx-\frac{\epsilon\beta(2\gamma+1)}{2(\gamma+1)}\left(\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)}\right). \end{aligned}$$

By use of $F(u,v) \leq c_1\left(|u|^{r+2}+|v|^{r+2}\right)$, we have

$$\begin{aligned} \Phi'(t) &\leq -\epsilon E(t)+\epsilon\delta\left(\|u\|_{p+2}^{p+2}+\|v\|_{q+2}^{q+2}\right)+\epsilon c_1(r+1)\left(\|u\|_{r+2}^{r+2}+\|v\|_{r+2}^{r+2}\right) \\ &\quad -\frac{\epsilon\beta(2\gamma+1)}{2(\gamma+1)}\left(\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)}\right). \end{aligned} \tag{3.7}$$

Since $I(t) > 0$,

$$\begin{aligned} J(t) &= \frac{r}{2(r+2)}\left[\alpha\left(\|\nabla u\|^2+\|\nabla v\|^2\right)+\frac{\beta(r-2\gamma)}{(\gamma+1)r}\left(\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)}\right)\right]+\frac{1}{r+2}I(t) \\ &\geq \frac{r}{2(r+2)}\left[\alpha\left(\|\nabla u\|^2+\|\nabla v\|^2\right)+\frac{\beta(r-2\gamma)}{(\gamma+1)r}\left(\|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)}\right)\right]. \end{aligned} \tag{3.8}$$

Thus,

$$\begin{aligned} \|\nabla u\|^{2(\gamma+1)}+\|\nabla v\|^{2(\gamma+1)} &\leq \frac{2(r+2)(\gamma+1)}{\beta(r-2\gamma)}J(t) \\ &\leq \frac{2(r+2)(\gamma+1)}{\beta(r-2\gamma)}E(t) \\ &\leq \frac{2(r+2)(\gamma+1)}{\beta(r-2\gamma)}E(0). \end{aligned} \tag{3.9}$$

From Poincare inequality, we have

$$\begin{aligned} \|u\|_{p+2}^{p+2} &\leq c_*^{p+2}\|\nabla u\|^{p+2} \\ &= c_*^{p+2}\|\nabla u\|^{p-2\gamma}\|\nabla u\|^{2(\gamma+1)} \\ &= c_*^{p+2}\left(\|\nabla u\|^{2(\gamma+1)}\right)^{\frac{p-2\gamma}{2(\gamma+1)}}\|\nabla u\|^{2(\gamma+1)} \\ &\leq c_*^{p+2}\left(\frac{2(r+2)(\gamma+1)}{\beta(r-2\gamma)}E(0)\right)^{\frac{p-2\gamma}{2(\gamma+1)}}\|\nabla u\|^{2(\gamma+1)}, \end{aligned} \tag{3.10}$$

and similarly

$$\|v\|_{q+2}^{q+2} \leq c_*^{q+2}\left(\frac{2(r+2)(\gamma+1)}{\beta(r-2\gamma)}E(0)\right)^{\frac{q-2\gamma}{2(\gamma+1)}}\|\nabla v\|^{2(\gamma+1)}. \tag{3.11}$$

Furthermore,

$$\|u\|_{r+2}^{r+2} \leq c_*^{r+2}\left(\frac{2(r+2)(\gamma+1)}{\beta(r-2\gamma)}E(0)\right)^{\frac{r-2\gamma}{2(\gamma+1)}}\|\nabla u\|^{2(\gamma+1)},$$

and similarly

$$\|v\|_{r+2}^{r+2} \leq c_*^{r+2}\left(\frac{2(r+2)(\gamma+1)}{\beta(r-2\gamma)}E(0)\right)^{\frac{r-2\gamma}{2(\gamma+1)}}\|\nabla v\|^{2(\gamma+1)}. \tag{3.13}$$

Substituting (3.9)-(3.13) into (3.7), we have

$$\begin{aligned} \Phi'(t) &\leq -\epsilon E(t) - \epsilon\left[\frac{\beta(2\gamma+1)}{2(\gamma+1)} - 2c_*^{r+2}c_1(r+1)\left(\frac{2(r+2)(\gamma+1)}{\beta(r-2\gamma)}E(0)\right)^{\frac{r-2\gamma}{2(\gamma+1)}}\right. \\ &\quad \left.-\delta m\right]\left(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)}\right) \end{aligned} \tag{3.14}$$

where $m = \max\left\{c_*^{p+2}\left(\frac{2(r+2)(\gamma+1)}{\beta(r-2\gamma)}E(0)\right)^{\frac{p-2\gamma}{2(\gamma+1)}}, c_*^{q+2}\left(\frac{2(r+2)(\gamma+1)}{\beta(r-2\gamma)}E(0)\right)^{\frac{q-2\gamma}{2(\gamma+1)}}\right\}$. Thus from the assumptions of the theorem and by choosing a sufficiently small $\delta > 0$, we obtain that

$$\Phi'(t) \leq -\epsilon E(t) \leq -\frac{\epsilon}{(\alpha_2)^{\gamma+1}}(\Phi(t))^{\gamma+1}. \tag{3.15}$$

We separate (3.15) into two cases.

Case 1: $\gamma = 0$, then a simply integration of (3.15) over $(0,t)$ yields

$$E(t) \leq \Phi(t) \leq \Phi(0)e^{-kt},$$

where $k = \frac{\epsilon}{(\alpha_2)^{\gamma+1}}$.

Case 2: $\gamma > 0$, a simply integration of (3.15) over $(0,t)$ yields

$$E(t) \leq \Phi(t) \leq \left(kt + \Phi^{-\gamma}(0)\right)^{-\frac{1}{\gamma}},$$

where $k = \frac{\epsilon\gamma}{(\alpha_2)^{\gamma+1}}$. This completes the proof.

4. Blow up

In this section, we state and prove blow up result.

Theorem 4.1. Suppose that $r > \max\{2\gamma, p, q\}$, $E(0) < 0$, and there exists a constant τ such that $\tau \leq \frac{2\alpha\gamma}{\gamma_2 C_*}$, where C_* is the constant of the Sobolev embedding theorem. Then the solution of this system blows up in finite time T^*, and

$$T^* \leq \frac{1-\sigma}{\xi\sigma\Psi^{\frac{\sigma}{1-\sigma}}(0)},$$

where $\Psi(t)$ and σ are given in (4.1) and (4.2) respectively.

Proof. Define $H(t) = -E(t)$, then $E(0) < 0$ and (2.6) gives $H(t) \geq H(0) > 0$. Define

$$\Psi(t) = H^{1-\sigma}(t) + \varepsilon\left(\int_\Omega u u_t dx + \int_\Omega v v_t dx\right), \tag{4.1}$$

where ε is a small constant to be chosen later and

$$0 < \sigma \leq \min\left\{\frac{r-p}{(r+2)(p+1)}, \frac{r-q}{(r+2)(q+1)}\right\}. \tag{4.2}$$

A direct differentiation of $\Psi(t)$ gives

$$\begin{aligned}\Psi'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\left(\|u_t\|^2 + \|v_t\|^2\right) - \varepsilon\alpha\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) \\ &\quad -\varepsilon\beta\left(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)}\right) + \varepsilon(r+2)\int_\Omega F(u,v)\,dx \\ &\quad -\varepsilon\gamma_2\left(\int_\Omega u u_t dx + \int_\Omega v v_t dx\right) - \varepsilon\left(\int_\Omega u u_t |u_t|^p dx + \int_\Omega v v_t |v_t|^q dx\right). \end{aligned} \tag{4.3}$$

From definition of $H(t)$, it follows that

$$\begin{aligned} -\beta\left(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)}\right) &= 2(\gamma+1)H(t) + (\gamma+1)\left(\|u_t\|^2 + \|v_t\|^2\right) \\ &\quad +\alpha(\gamma+1)\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) \\ &\quad -2(\gamma+1)\int_\Omega F(u,v)\,dx. \end{aligned} \tag{4.4}$$

Substitute (4.4) into (4.3) to obtain

$$\begin{aligned}\Psi'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\left(\|u_t\|^2 + \|v_t\|^2\right) \\ &\quad -\varepsilon\alpha\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) + 2\varepsilon(\gamma+1)H(t) \\ &\quad +\varepsilon(\gamma+1)\left(\|u_t\|^2 + \|v_t\|^2\right) + \varepsilon\alpha(\gamma+1)\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) \\ &\quad -2\varepsilon(\gamma+1)\int_\Omega F(u,v)\,dx + \varepsilon(r+2)\int_\Omega F(u,v)\,dx \\ &\quad -\varepsilon\gamma_2\left(\int_\Omega u u_t dx + \int_\Omega v v_t dx\right) - \varepsilon\left(\int_\Omega u u_t |u_t|^p dx + \int_\Omega v v_t |v_t|^q dx\right). \end{aligned} \tag{4.5}$$

Now, we make use of the following Young's inequality

$$XY \leq \frac{\delta^k X^k}{k} + \frac{\delta^l Y^l}{l},$$

$X, Y \geq 0,\ \delta > 0,\ k, l \in R^+$ such that $\frac{1}{k} + \frac{1}{l} = 1$. We have

$$\int_\Omega u u_t dx \leq \frac{\tau}{2}\|u\|^2 + \frac{1}{2\tau}\|u_t\|^2, \qquad \int_\Omega v v_t dx \leq \frac{\tau}{2}\|v\|^2 + \frac{1}{2\tau}\|v_t\|^2,$$

$$\begin{aligned}\int_\Omega uu_t\,|u_t|^p\,dx &\le \frac{\delta_1^{p+2}}{p+2}\,\|u\|_{p+2}^{p+2}+\frac{(p+1)\,\delta_1^{-\frac{p+2}{p+1}}}{p+2}\,\|u_t\|_{p+2}^{p+2}\\ &\le \frac{\delta_1^{p+2}}{p+2}\,\|u\|_{p+2}^{p+2}+\frac{(p+1)\,\delta_1^{-\frac{p+2}{p+1}}}{p+2}H'(t)\end{aligned}$$

and similarly

$$\int_\Omega vv_t\,|v_t|^q\,dx \le \frac{\delta_2^{q+2}}{q+2}\,\|v\|_{q+2}^{q+2}+\frac{(q+1)\,\delta_2^{-\frac{q+2}{q+1}}}{q+2}H'(t),$$

where $\delta_1,\ \delta_2$ are constants depending on the time t that will be specified later. Thus, (4.5) becomes

$$\begin{aligned}\Psi'(t) \ \ge\ & (1-\sigma)\,H^{-\sigma}(t)\,H'(t)+\varepsilon\left(\|u_t\|^2+\|v_t\|^2\right)\\ &+\varepsilon\alpha\gamma\left(\|\nabla u\|^2+\|\nabla v\|^2\right)+2\varepsilon(\gamma+1)\,H(t)+\varepsilon(\gamma+1)\left(\|u_t\|^2+\|v_t\|^2\right)\\ &+\varepsilon(r-2\gamma)\int_\Omega F(u,v)\,dx-\frac{\varepsilon\gamma_2\tau}{2}\left(\|u\|^2+\|v\|^2\right)-\frac{\varepsilon\gamma_2}{2\tau}\left(\|u_t\|^2+\|v_t\|^2\right)\\ &-\varepsilon\left(\frac{(p+1)\,\delta_1^{-\frac{p+2}{p+1}}}{p+2}+\frac{(q+1)\,\delta_2^{-\frac{q+2}{q+1}}}{q+2}\right)H'(t)-\varepsilon\left(\frac{\delta_1^{p+2}}{p+2}\,\|u\|_{p+2}^{p+2}+\frac{\delta_2^{q+2}}{q+2}\,\|v\|_{q+2}^{q+2}\right)\end{aligned}\tag{4.6}$$

By using Sobolev-Poincare's inequality, we have

$$\begin{aligned}\Psi'(t) \ \ge\ & (1-\sigma)\,H^{-\sigma}(t)\,H'(t)+\varepsilon\left(\alpha\gamma-\frac{\gamma_2\tau C_*}{2}\right)\left(\|\nabla u\|^2+\|\nabla v\|^2\right)+2\varepsilon(\gamma+1)\,H(t)\\ &+\varepsilon\left(\gamma+2-\frac{\gamma_2}{2\tau}\right)\left(\|u_t\|^2+\|v_t\|^2\right)+\varepsilon(r-2\gamma)\int_\Omega F(u,v)\,dx\\ &-\varepsilon\left(\frac{(p+1)\,\delta_1^{-\frac{p+2}{p+1}}}{p+2}+\frac{(q+1)\,\delta_2^{-\frac{q+2}{q+1}}}{q+2}\right)H'(t)-\varepsilon\left(\frac{\delta_1^{p+2}}{p+2}\,\|u\|_{p+2}^{p+2}+\frac{\delta_2^{q+2}}{q+2}\,\|v\|_{q+2}^{q+2}\right)\end{aligned}\tag{4.7}$$

Therefore by taking δ_1 and δ_2 so that $\delta_1^{-\frac{p+2}{p+1}}=k_1H^{-\sigma}(t)$, $\delta_2^{-\frac{q+2}{q+1}}=k_2H^{-\sigma}(t)$, where $k_1,\ k_2>0$ are specified later, we get

$$\delta_1^{p+2}=k_1^{-(p+1)}H^{\sigma(p+1)}(t)\le k_1^{-(p+1)}c_1^{\sigma(p+1)}\left(\|u\|_{r+2}^{r+2}+\|v\|_{r+2}^{r+2}\right)^{\sigma(p+1)},\tag{4.8}$$

and

$$\delta_2^{q+2}=k_2^{-(q+1)}H^{\sigma(q+1)}(t)\le k_2^{-(q+1)}c_1^{\sigma(q+1)}\left(\|u\|_{r+2}^{r+2}+\|v\|_{r+2}^{r+2}\right)^{\sigma(q+1)},\tag{4.9}$$

since $H(t)=-E(t)\le\int_\Omega F(u,v)\,dx\le c_1\left(|u|^{r+2}+|u|^{r+2}\right)$.

Substituting (4.8) and (4.9) into (4.7), we have

$$\begin{aligned}\Psi'(t) \geq\ & \left(1-\sigma-\frac{\varepsilon(p+1)k_1}{p+2}-\frac{\varepsilon(q+1)k_2}{q+2}\right)H^{-\sigma}(t)H'(t)+2\varepsilon(\gamma+1)H(t)\\ &+\varepsilon\left(\alpha\gamma-\frac{\gamma_2\tau C_*}{2}\right)\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\\ &+\varepsilon\left(2+\gamma-\frac{\gamma_2}{2\tau}\right)\left(\|u_t\|^2+\|v_t\|^2\right)+\varepsilon(r-2\gamma)\int_\Omega F(u,v)\,dx\\ &-\frac{\varepsilon k_1^{-(p+1)}c_1^{\sigma(p+1)}}{p+2}\left(\|u\|_{r+2}^{r+2}+\|v\|_{r+2}^{r+2}\right)^{\sigma(p+1)}\|u\|_{p+2}^{p+2}\\ &-\frac{\varepsilon k_2^{-(q+1)}c_1^{\sigma(q+1)}}{q+2}\left(\|u\|_{r+2}^{r+2}+\|v\|_{r+2}^{r+2}\right)^{\sigma(q+1)}\|v\|_{q+2}^{q+2}. \end{aligned} \tag{4.10}$$

Since $r>\max\{p,q\}$, we obtain

$$\|u\|_{p+2}^{p+2}\leq C\|u\|_{r+2}^{p+2}\leq C\left(\|u\|_{r+2}+\|v\|_{r+2}\right)^{p+2},$$
$$\|v\|_{q+2}^{q+2}\leq C\|v\|_{r+2}^{q+2}\leq C\left(\|u\|_{r+2}+\|v\|_{r+2}\right)^{q+2}.$$

Thus

$$\begin{aligned}\Psi'(t) \geq\ & \left(1-\sigma-\frac{\varepsilon(p+1)k_1}{p+2}-\frac{\varepsilon(q+1)k_2}{q+2}\right)H^{-\sigma}(t)H'(t)+2\varepsilon(\gamma+1)H(t)\\ &+\varepsilon\left(\alpha\gamma-\frac{\gamma_2\tau C_*}{2}\right)\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\\ &+\varepsilon\left(2+\gamma-\frac{\gamma_2}{2\tau}\right)\left(\|u_t\|^2+\|v_t\|^2\right)+\varepsilon(r-2\gamma)\int_\Omega F(u,v)\,dx\\ &-\frac{\varepsilon k_1^{-(p+1)}c_1^{\sigma(p+1)}C}{p+2}\left(\|u\|_{r+2}+\|v\|_{r+2}\right)^{\sigma(r+2)(p+1)+p+2}\\ &-\frac{\varepsilon k_2^{-(q+1)}c_1^{\sigma(q+1)}C}{q+2}\left(\|u\|_{r+2}+\|v\|_{r+2}\right)^{\sigma(r+2)(q+1)+q+2}, \end{aligned} \tag{4.11}$$

where $(a+b)^\lambda\leq C\left(a^\lambda+b^\lambda\right),\ a,b>0$ is used. From (4.2), we have $2\leq\sigma(p+1)(r+2)+p+2\leq r+2,\ 2\leq\sigma(q+1)(r+2)+q+2\leq r+2$. By using Lemma 2.1, we have

$$\|u\|_{r+2}^{\sigma(p+1)(r+2)+p+2}\leq C\left(\|\nabla u\|^2+\|u\|_{r+2}^{r+2}\right),$$
$$\|v\|_{r+2}^{\sigma(q+1)(r+2)+q+2}\leq C\left(\|\nabla v\|^2+\|v\|_{r+2}^{r+2}\right).$$

Thus

$$\begin{aligned}\Psi'(t) \geq\ & \left(1-\sigma-\frac{\varepsilon(p+1)k_1}{p+2}-\frac{\varepsilon(q+1)k_2}{q+2}\right)H^{-\sigma}(t)H'(t)+2\varepsilon(\gamma+1)H(t)\\ &+\varepsilon\left(\alpha\gamma-\frac{\gamma_2\tau C_*}{2}\right)\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\\ &+\varepsilon\left(2+\gamma-\frac{\gamma_2}{2\tau}\right)\left(\|u_t\|^2+\|v_t\|^2\right)+\varepsilon(r-2\gamma)\int_\Omega F(u,v)\,dx\\ &+\varepsilon\left(-\frac{k_1^{-(p+1)}c_1^{\sigma(p+1)}C}{p+2}-\frac{k_2^{-(q+1)}c_1^{\sigma(q+1)}C}{q+2}\right)\left(\|u\|_{r+2}^{r+2}+\|v\|_{r+2}^{r+2}\right)\\ &+\varepsilon\left(-\frac{k_1^{-(p+1)}c_1^{\sigma(p+1)}C}{p+2}-\frac{k_2^{-(q+1)}c_1^{\sigma(q+1)}C}{q+2}\right)\left(\|\nabla u\|^2+\|\nabla v\|^2\right). \end{aligned} \tag{4.12}$$

By using the $c_0\left(|u|^{r+2}+|v|^{r+2}\right)\leq F(u,v)$ in (4.12) we obtain

$$\begin{aligned}\Psi'(t) &\geq \left(1-\sigma-\frac{\varepsilon(p+1)k_1}{p+2}-\frac{\varepsilon(q+1)k_2}{q+2}\right)H^{-\sigma}(t)H'(t)\\ &+2\varepsilon(\gamma+1)H(t)+\varepsilon\left(2+\gamma-\frac{\gamma_2}{2\tau}\right)\left(\|u_t\|^2+\|v_t\|^2\right)\\ &+\varepsilon\left(c_0(r-2\gamma)-\frac{k_1^{-(p+1)}c_1^{\sigma(p+1)}C}{p+2}-\frac{k_2^{-(q+1)}c_1^{\sigma(q+1)}C}{q+2}\right)\left(\|u\|_{r+2}^{r+2}+\|v\|_{r+2}^{r+2}\right)\\ &+\varepsilon\left(\alpha\gamma-\frac{\gamma_2\tau C_*}{2}-\frac{k_1^{-(p+1)}c_1^{\sigma(p+1)}C}{p+2}-\frac{k_2^{-(q+1)}c_1^{\sigma(q+1)}C}{q+2}\right)\left(\|\nabla u\|^2+\|\nabla v\|^2\right),\end{aligned} \tag{4.13}$$

where $r>2\gamma$ is used. We choose $k_1,\ k_2$ large enough so that

$$c_0(r-2\gamma)-\frac{k_1^{-(p+1)}c_1^{\sigma(p+1)}C}{p+2}-\frac{k_2^{-(q+1)}c_1^{\sigma(q+1)}C}{q+2}>\frac{c_0(r-2\gamma)}{2}$$

and

$$\alpha\gamma-\frac{\gamma_2\tau C_*}{2}-\frac{k_1^{-(p+1)}c_1^{\sigma(p+1)}C}{p+2}-\frac{k_2^{-(q+1)}c_1^{\sigma(q+1)}C}{q+2}>\frac{\alpha\gamma}{2}-\frac{\gamma_2\tau C_*}{4}.$$

Then, we choose ε small enough so that $1-\sigma-\frac{\varepsilon(p+1)k_1}{p+2}-\frac{\varepsilon(q+1)k_2}{q+2}\geq 0$. Thus, we have

$$\begin{aligned}\Psi'(t) &\geq \varepsilon\left(2+\gamma-\frac{\gamma_2}{2\tau}\right)\left(\|u_t\|^2+\|v_t\|^2\right)+2\varepsilon(\gamma+1)H(t)\\ &+\varepsilon\left(\frac{\alpha\gamma}{2}-\frac{\gamma_2\tau C_*}{4}\right)\left(\|\nabla u\|^2+\|\nabla v\|^2\right)+\varepsilon\frac{c_0(r-2\gamma)}{2}\left(\|u\|_{r+2}^{r+2}+\|v\|_{r+2}^{r+2}\right)\\ &\geq \eta\left(\|u_t\|^2+\|v_t\|^2+H(t)+\|\nabla u\|^2+\|\nabla v\|^2+\|u\|_{r+2}^{r+2}+\|v\|_{r+2}^{r+2}\right),\end{aligned} \tag{4.14}$$

where $\eta=\min\left\{\varepsilon\left(2+\gamma-\frac{\gamma_2}{2\tau}\right),\ 2\varepsilon(\gamma+1),\ \varepsilon\left(\frac{\alpha\gamma}{2}-\frac{\gamma_2\tau C_*}{4}\right),\ \varepsilon\frac{c_0(r-2\gamma)}{2}\right\}$. Consequently we have

$$\Psi(t)\geq\Psi(0)=H^{1-\sigma}(0)+\varepsilon\left(\int_\Omega u_0u_1dx+\int_\Omega v_0v_1dx\right)>0,\ \forall t\geq 0. \tag{4.15}$$

On the other hand, by the Hölder's inequality, we get

$$\begin{aligned}\left|\int_\Omega uu_tdx+\int_\Omega vv_tdx\right|^{\frac{1}{1-\sigma}} &\leq \|u\|^{\frac{1}{1-\sigma}}\|u_t\|^{\frac{1}{1-\sigma}}+\|v\|^{\frac{1}{1-\sigma}}\|v_t\|^{\frac{1}{1-\sigma}}\\ &\leq C\left(\|u\|_{r+2}^{\frac{1}{1-\sigma}}\|u_t\|^{\frac{1}{1-\sigma}}+\|v\|_{r+2}^{\frac{1}{1-\sigma}}\|v_t\|^{\frac{1}{1-\sigma}}\right).\end{aligned} \tag{4.16}$$

Young inequality gives

$$\left|\int_\Omega uu_tdx+\int_\Omega vv_tdx\right|^{\frac{1}{1-\sigma}}\leq C\left(\|u\|_{r+2}^{\frac{\mu}{1-\sigma}}+\|u_t\|^{\frac{\theta}{1-\sigma}}+\|v\|_{r+2}^{\frac{\mu}{1-\sigma}}+\|v_t\|^{\frac{\theta}{1-\sigma}}\right), \tag{4.17}$$

for $\frac{1}{\mu}+\frac{1}{\theta}=1$. We take $\theta=2(1-\sigma)$, to get $\mu=\frac{2(1-\sigma)}{1-2\sigma}\leq r+2$ by (4.2). Therefore (4.17) becomes

$$\left|\int_\Omega uu_tdx+\int_\Omega vv_tdx\right|^{\frac{1}{1-\sigma}}\leq C\left(\|u_t\|^2+\|v_t\|^2+\|u\|_{r+2}^{\frac{2}{1-2\sigma}}+\|v\|_{r+2}^{\frac{2}{1-2\sigma}}\right). \tag{4.18}$$

By using Lemma 2.1, we obtain

$$\left|\int_{\Omega} uu_t dx + \int_{\Omega} vv_t dx\right|^{\frac{1}{1-\sigma}} \leq C\left(\|u_t\|^2 + \|v_t\|^2 + \|u\|_{r+2}^{r+2} + \|v\|_{r+2}^{r+2} + \|\nabla u\|^2 + \|\nabla v\|^2\right). \quad (4.19)$$

Thus

$$\begin{aligned} \Psi^{\frac{1}{1-\sigma}}(t) &= \left[H^{1-\sigma}(t) + \varepsilon\left(\int_{\Omega} uu_t dx + \int_{\Omega} vv_t dx\right)\right]^{\frac{1}{1-\sigma}} \\ &\leq 2^{\frac{\sigma}{1-\sigma}}\left(H(t) + \varepsilon^{\frac{1}{1-\sigma}}\left|\int_{\Omega} uu_t dx + \int_{\Omega} vv_t dx\right|^{\frac{1}{1-\sigma}}\right) \\ &\leq C\left(\|u_t\|^2 + \|v_t\|^2 + H(t) + \|u\|_{r+2}^{r+2} + \|v\|_{r+2}^{r+2} + \|\nabla u\|^2 + \|\nabla v\|^2\right). \quad (4.20) \end{aligned}$$

By combining of (4.14) and (4.20) we arrive

$$\Psi'(t) \geq \xi \Psi^{\frac{1}{1-\sigma}}(t), \quad (4.21)$$

where ξ is a positive constant. A simple integration yields

$$\Psi^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{\Psi^{-\frac{\sigma}{1-\sigma}}(0) - \frac{\xi\sigma t}{1-\sigma}},$$

which implies that the solution blows up in a finite time T^*, with

$$T^* \leq \frac{1-\sigma}{\xi\sigma\Psi^{\frac{\sigma}{1-\sigma}}(0)}.$$

References

[1] Adams, R.A., Fournier, J.J.F., *Sobolev Spaces,* Academic Press, New York, 2003.

[2] Agre, K., Rammaha, M.A., "Systems of nonlinear wave equations with damping and source terms", *Diff. Integral Eqns.,* 19, 1235–1270, 2006.

[3] Benaissa, A., Messaoudi, S.A., "Blow-up of solutions for the Kirchhoff equation of q-Laplacian type with nonlinear dissipation", *Colloquium Mathematicum,* 94, 103-109, 2002.

[4] Georgiev, V., Todorova, G., "Existence of a solution of the wave equation with nonlinear damping and source term", J. Differ. Equations, 109 (1994) 295–308.

[5] Kirchhoff, G., *Mechanik,* Teubner, Leipzig, 1883.

[6] Korpusov, M.O., "Blow up the solution of a nonlinear system of equations with positive energy", *Theoretical and Mathematical Physics,* 171, 725-738, 2012.

[7] Matsuyama, T., Ikehata, R., "On global solutions and energy decay for the wave equations of the Kirchhoff type with nonlinear damping terms", *J. Math. Anal. Appl.,* 204, 729-753, 1996.

[8] Messaoudi, S.A., "Blow up in a nonlinearly damped wave equation", *Math. Nachr.,* 231, 105-111, 2001.

[9] Messaoudi, S.A., "On the decay of solutions for a class of quasilinear hyperbolic equations with non-linear damping and source terms", *Mat. Meth. Appl. Sci.,* 28, 1819-1828, 2005.

[10] Messaoudi, S.A., Said-Houari, B., "Global nonexistence of positive initial-energy solutions of a system of nonlinear viscoelastic wave equations with damping and source terms", *J. Math. Anal. Appl.,* 365, 277–287, 2010.

[11] Miranda, M.M., Medeiros, L.A., "On the existence of global solutions of a coupled nonlinear Klein-Gordon equations", *Funkcial Ekvac.,* 30, 147-161, 1987.

[12] Ono, K., "Global existence, decay, and blow-up of solutions for some mildly degenerate nonlinear Kirchhoff strings", *J. Differential Equations,* 137, 273-301, 1997.

[13] Pişkin, E., "Uniform decay and blow-up of solutions for coupled nonlinear Klein-Gordon equations with nonlinear damping terms", *Math. Meth. Appl. Sci.,* 37, 3036-3047, 2014.

[14] Pişkin, E., "Blow-up of solutions for coupled nonlinear Klein-Gordon equations with weak damping terms", *Math. Sci. Letters,* 3, 189-191, 2014.

[15] Pişkin, E., "Existence, decay and blow up of solutions for the extensible beam equation with nonlinear damping and source terms", *Open Math.,* 13, 408-420, 2015.

[16] Said-Houari, B., "Global nonexistence of positive initial-energy solutions of a system of nonlinear wave equations with damping and source terms", *Diff. Integral Eqns.,* 23, 79–92, 2010.

[17] Said-Houari, B., "Global existence and decay of solutions of a nonlinear system of wave equations", *Appl. Anal.,* 91, 475-489, 2012.

[18] Taniguchi, T., "Existence and asymptotic behaviour of solutions to weakly damped wave equations of Kirchhoff type with nonlinear damping and source terms", *J. Math. Anal. Appl.,* 361, 566–578, 2010.

[19] Wu, S.T., "Blow-up results for system of nonlinear Klein-Gordon equations with arbitrary positive initial energ", *Electron. J. Diff. Equations,* 2012, 1-13, 2012.

[20] Wu, S.T., Tsai, L.Y., "Blow-up solutions for some nonlinear wave equations of Kirchhoff type with some dissipation", *Nonlinear Anal.,* 65, 243-264, 2006.

[21] Ye, Y., "Global existence and asymptotic stability for coupled nonlinear Klein-Gordon equations with nonlinear damping terms", *Dynamical Syst.,* 28, 287-298, 2013.