Communications in Advanced Mathematical Sciences
Vol. II, No. 2, 154-160, 2019
Research Article
e-ISSN:2651-4001
DOI: 10.33434/cams.553548

On $\mathscr{I}_2$-Cauchy Double Sequences in Fuzzy Normed Spaces

Erdinç Dündar [1]*, Muhammed Recai Türkmen[2]

Abstract
In this paper, we investigate relationship between $\mathscr{I}_2$-convergence and $\mathscr{I}_2$-Cauchy double sequences in fuzzy normed spaces. After, we introduce the concepts of $\mathscr{I}_2^*$-Cauchy double sequences and study relationships between $\mathscr{I}_2$-Cauchy and $\mathscr{I}_2^*$-Cauchy double sequences in fuzzy normed spaces.

Keywords: Double sequences, Fuzzy normed space, $\mathscr{I}_2$-Cauchy, $\mathscr{I}_2$-convergence.
2010 AMS: Primary 34C41, 40A35, Secondary 40G15

[1] *Department of Mathematics, Faculty of Science and Literature, Afyon Kocatepe University, 03200, Afyonkarahisar, Turkey, ORCID: 0000-*
[2] *Department of Mathematics, Faculty of Education, Afyon Kocatepe University, 03200, Afyonkarahisar, Turkey, ORCID: 0000-*
***Corresponding author**: edundar@aku.edu.tr
Received: 14 April 2019, **Accepted:** 20 May 2019, **Available online:** 27 June 2019

1. Introduction and background

Throughout the paper $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers, respectively. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [1] and Schoenberg [2]. A lot of developments have been made in this area after the various studies of researchers [3, 4]. The idea of $\mathscr{I}$-convergence was introduced by Kostyrko et al. [5] as a generalization of statistical convergence which is based on the structure of the ideal $\mathscr{I}$ of subset of the set of natural numbers $\mathbb{N}$. Das et al. [6] introduced the concept of $\mathscr{I}$-convergence of double sequences in a metric space and studied some properties of this convergence. A lot of developments have been made in this area after the works of [7, 8, 9, 10].

The concept of ordinary convergence of a sequence of fuzzy numbers was firstly introduced by Matloka [11] and proved some basic theorems for sequences of fuzzy numbers. Nanda [12] studied the sequences of fuzzy numbers and showed that the set of all convergent sequences of fuzzy numbers are a complete metric space. Şençimen and Pehlivan [13] introduced the notions of statistically convergent sequence and statistically Cauchy sequence in a fuzzy normed linear space. Hazarika [14] studied the concepts of $\mathscr{I}$-convergence, $\mathscr{I}^*$-convergence and $\mathscr{I}$-Cauchy sequence in a fuzzy normed linear space. Dündar and Talo [15, 16] introduced the concepts of $\mathscr{I}_2$-convergence and $\mathscr{I}_2$-Cauchy sequence for double sequences of fuzzy numbers and studied some properties and relations of them. Hazarika and Kumar [17] introduced the notion of $\mathscr{I}_2$-convergence and $\mathscr{I}_2$-Cauchy double sequences in a fuzzy normed linear space. Dündar and Türkmen [18] studied some properties of $\mathscr{I}_2$-convergence and $\mathscr{I}_2^*$-convergence of double sequences in fuzzy normed spaces. A lot of developments have been made in this area after the various studies of researchers [19, 20, 21, 22].

Now, we recall the concept of ideal, convergence, statistical convergence, ideal convergence of sequence, double sequence and fuzzy normed and some basic definitions (see [1, 3, 4, 13, 15, 20, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34]).

Fuzzy sets are considered with respect to a nonempty base set X of elements of interest. The essential idea is that each element $x \in X$ is assigned a membership grade $u(x)$ taking values in $[0,1]$, with $u(x) = 0$ corresponding to nonmembership,

$0 < u(x) < 1$ to partial membership, and $u(x) = 1$ to full membership. According to Zadeh [35], a fuzzy subset of X is a nonempty subset $\{(x, u(x)) : x \in X\}$ of $X \times [0,1]$ for some function $u : X \to [0,1]$. The function u itself is often used for the fuzzy set.

A fuzzy set u on $\mathbb{R}$ is called a fuzzy number if it has the following properties:
1. u is normal, that is, there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;
2. u is fuzzy convex, that is, for $x, y \in \mathbb{R}$ and $0 \le \lambda \le 1$, $u(\lambda x + (1-\lambda)y) \ge \min[u(x), u(y)]$;
3. u is upper semicontinuous;
4. $supp\, u = cl\{x \in \mathbb{R} : u(x) > 0\}$, or denoted by $[u]_0$, is compact.

Let $L(\mathbb{R})$ be set of all fuzzy numbers. If $u \in L(\mathbb{R})$ and $u(t) = 0$ for $t < 0$, then u is called a non-negative fuzzy number. We write $L^*(\mathbb{R})$ by the set of all non-negative fuzzy numbers. We can say that $u \in L^*(\mathbb{R})$ iff $u_\alpha^- \ge 0$ for each $\alpha \in [0,1]$. Clearly we have $\widetilde{0} \in L(\mathbb{R})$. For $u \in L(\mathbb{R})$, the α level set of u is defined by

$$[u]_\alpha = \begin{cases} \{x \in \mathbb{R} : u(x) \ge \alpha\}, & \text{if } \alpha \in (0,1] \\ supp\, u, & \text{if } \alpha = 0. \end{cases}$$

A partial order $\preceq$ on $L(\mathbb{R})$ is defined by $u \preceq v$ if $u_\alpha^- \le v_\alpha^-$ and $u_\alpha^+ \le v_\alpha^+$ for all $\alpha \in [0,1]$.

Arithmetic operation for $t \in \mathbb{R}$, $\oplus, \ominus, \odot$ and $\oslash$ on $L(\mathbb{R}) \times L(\mathbb{R})$ are defined by
$(u \oplus v)(t) = \sup_{s\in\mathbb{R}} \{u(s) \wedge v(t-s)\}$, $\quad (u \ominus v)(t) = \sup_{s\in\mathbb{R}} \{u(s) \wedge v(s-t)\}$,
$(u \odot v)(t) = \sup_{s\in\mathbb{R}, s\neq 0} \{u(s) \wedge v(t/s)\}$ and $(u \oslash v)(t) = \sup_{s\in\mathbb{R}} \{u(st) \wedge v(s)\}$.
For $k \in \mathbb{R}^+$, ku is defined as $ku(t) = u(t/k)$ and $0u(t) = \tilde{0}$, $t \in \mathbb{R}$.
Some arithmetic operations for $\alpha-$level sets are defined as follows:
$u, v \in L(\mathbb{R})$ and $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$ and $[v]_\alpha = [v_\alpha^-, v_\alpha^+]$, $\alpha \in (0,1]$. Then,
$[u \oplus v]_\alpha = [u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+]$, $[u \ominus v]_\alpha = [u_\alpha^- - v_\alpha^+, u_\alpha^+ - v_\alpha^-]$,
$[u \odot v]_\alpha = [u_\alpha^-.v_\alpha^-, u_\alpha^+.v_\alpha^+]$ and $\left[\tilde{1} \oslash u\right]_\alpha = \left[\frac{1}{u_\alpha^+}, \frac{1}{u_\alpha^-}\right]$, $u_\alpha^- > 0$.
For $u, v \in L(\mathbb{R})$, the supremum metric on $L(\mathbb{R})$ defined as

$$D(u,v) = \sup_{0\le\alpha\le1} \max\left\{\left|u_\alpha^- - v_\alpha^-\right|, \left|u_\alpha^+ - v_\alpha^+\right|\right\}.$$

It is known that D is a metric on $L(\mathbb{R})$ and $(L(\mathbb{R}), D)$ is a complete metric space.

A sequence $x = (x_k)$ of fuzzy numbers is said to be convergent to the fuzzy number x_0, if for every $\varepsilon > 0$ there exists a positive integer k_0 such that $D(x_k, x_0) < \varepsilon$ for $k > k_0$ and a sequence $x = (x_k)$ of fuzzy numbers convergent to levelwise to x_0 if and only if $\lim_{k\to\infty} [x_k]_\alpha = [x_0]_\alpha^-$ and $\lim_{k\to\infty} [x_k]_\alpha = [x_0]_\alpha^+$, where $[x_k]_\alpha = \left[(x_k)_\alpha^-, (x_k)_\alpha^+\right]$ and $[x_0]_\alpha = \left[(x_0)_\alpha^-, (x_0)_\alpha^+\right]$, for every $\alpha \in (0,1)$.

Let X be a vector space over $\mathbb{R}$, $\|.\| : X \to L^*(\mathbb{R})$ and the mappings $L; R$ (respectively, left norm and right norm) : $[0,1] \times [0,1] \to [0,1]$ be symetric, nondecreasing in both arguments and satisfy $L(0,0) = 0$ and $R(1,1) = 1$.

The quadruple $(X, \|.\|, L, R)$ is called fuzzy normed linear space (briefly FNS) and $\|.\|$ a fuzzy norm if the following axioms are satisfied

1. $\|x\| = \widetilde{0}$ iff $x = 0$,
2. $\|rx\| = |r| \odot \|x\|$ for $x \in X$, $r \in \mathbb{R}$,
3. For all $x, y \in X$
   (a) $\|x+y\|(s+t) \ge L(\|x\|(s), \|y\|(t))$, whenever $s \le \|x\|_1^-$, $t \le \|y\|_1^-$ and $s+t \le \|x+y\|_1^-$,
   (b) $\|x+y\|(s+t) \le R(\|x\|(s), \|y\|(t))$, whenever $s \ge \|x\|_1^-$, $t \ge \|y\|_1^-$ and $s+t \ge \|x+y\|_1^-$.

Let $(X, \|.\|_C)$ be an ordinary normed linear space. Then, a fuzzy norm $\|.\|$ on X can be obtained by

$$\|x\|(t) = \begin{cases} 0, & \text{if } 0 \le t \le a\|x\|_C \text{ or } t \ge b\|x\|_C \\ \frac{t}{(1-a)\|x\|_C} - \frac{a}{1-a}, & \text{if } a\|x\|_C \le t \le \|x\|_C \\ \frac{-t}{(b-1)\|x\|_C} + \frac{b}{b-1}, & \text{if } \|x\|_C \le t \le b\|x\|_C \end{cases}$$

where $\|x\|_C$ is the ordinary norm of x $(\neq 0)$, $0 < a < 1$ and $1 < b < \infty$. For $x = 0$, define $\|x\| = \widetilde{0}$. Hence, $(X, \|.\|)$ is a fuzzy normed linear space.

Let us consider the topological structure of an FNS $(X, \|.\|)$. For any $\varepsilon > 0, \alpha \in [0,1]$ and $x \in X$, the $(\varepsilon, \alpha)-$ neighborhood of x is the set $\mathscr{N}_x(\varepsilon, \alpha) = \{y \in X : \|x-y\|_\alpha^+ < \varepsilon\}$.

Let $(X,\|.\|)$ be an FNS. A sequence $(x_n)_{n=1}^{\infty}$ in X is convergent to $x\in X$ with respect to the fuzzy norm on X and we denote by $x_n \overset{FN}{\to} x$, provided that $(D)-\lim_{n\to\infty}\|x_n-x\|=\widetilde{0}$; i.e., for every $\varepsilon>0$ there is an $N(\varepsilon)\in\mathbb{N}$ such that $D\left(\|x_n-x\|,\widetilde{0}\right)<\varepsilon$ for all $n\geq N(\varepsilon)$. This means that for every $\varepsilon>0$ there is an $N(\varepsilon)\in\mathbb{N}$ such that for all $n\geq N(\varepsilon)$, $\sup\limits_{\alpha\in[0,1]}\|x_n-x\|_{\alpha}^{+}=\|x_n-x\|_{0}^{+}<\varepsilon$.

Let $(X,\|.\|)$ be an FNS. Then a double sequence (x_{jk}) is said to be convergent to $x\in X$ with respect to the fuzzy norm on X if for every $\varepsilon>0$ there exist a number $N=N(\varepsilon)$ such that $D\left(\|x_{jk}-x\|,\widetilde{0}\right)<\varepsilon$, for all $j,k\geq N$.

In this case, we write $x_{jk}\overset{FN}{\longrightarrow}x$. This means that, for every $\varepsilon>0$ there exist a number $N=N(\varepsilon)$ such that $\sup\limits_{\alpha\in[0,1]}\|x_{jk}-x\|_{\alpha}^{+}=\|x_{jk}-x\|_{0}^{+}<\varepsilon$, for all $j,k\geq N$. In terms of neighnorhoods, we have $x_{jk}\overset{FN}{\longrightarrow}x$ provided that for any $\varepsilon>0$, there exists a number $N=N(\varepsilon)$ such that $x_{jk}\in\mathscr{N}_x(\varepsilon,0)$, whenever $j,k\geq N$.

Let $X\neq\emptyset$. A class $\mathscr{I}$ of subsets of X is said to be an ideal in X provided:

(i) $\emptyset\in\mathscr{I}$, (ii) $A,B\in\mathscr{I}$ implies $A\cup B\in\mathscr{I}$, (iii) $A\in\mathscr{I}$, $B\subset A$ implies $B\in\mathscr{I}$.

$\mathscr{I}$ is called a nontrivial ideal if $X\notin\mathscr{I}$. A nontrivial ideal $\mathscr{I}$ in X is called admissible if $\{x\}\in\mathscr{I}$ for each $x\in X$.

A nontrivial ideal $\mathscr{I}_2$ of $\mathbb{N}\times\mathbb{N}$ is called strongly admissible if $\{i\}\times\mathbb{N}$ and $\mathbb{N}\times\{i\}$ belong to $\mathscr{I}_2$ for each $i\in\mathbb{N}$. It is evident that a strongly admissible ideal is also admissible. Throughout the paper we take $\mathscr{I}_2$ as a strongly admissible ideal in $\mathbb{N}\times\mathbb{N}$.

Let $\mathscr{I}_2^0=\{A\subset\mathbb{N}\times\mathbb{N}:(\exists m(A),(i,j)\geq m(A)\Rightarrow(i,j)\notin A)\}$. Then $\mathscr{I}_2^0$ is a nontrivial strongly admissible ideal and clearly an ideal $\mathscr{I}_2$ is strongly admissible if and only if $\mathscr{I}_2^0\subset\mathscr{I}_2$.

Let $X\neq\emptyset$. A non empty class $\mathscr{F}$ of subsets of X is said to be a filter in X provided:

(i) $\emptyset\notin\mathscr{F}$, (ii) $A,B\in\mathscr{F}$ implies $A\cap B\in\mathscr{F}$, (iii) $A\in\mathscr{F}$, $A\subset B$ implies $B\in\mathscr{F}$.

Let $\mathscr{I}$ is a nontrivial ideal in X, $X\neq\emptyset$, then the class $\mathscr{F}(\mathscr{I})=\{M\subset X:(\exists A\in\mathscr{I})(M=X\backslash A)\}$ is a filter on X, called the filter associated with $\mathscr{I}$.

Let (X,ρ) be a linear metric space and $\mathscr{I}_2\subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence $x=(x_{mn})$ in X is said to be $\mathscr{I}_2$-convergent to $L\in X$, if for any $\varepsilon>0$ we have $A(\varepsilon)=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\rho(x_{mn},L)\geq\varepsilon\}\in\mathscr{I}_2$ and we write $\mathscr{I}_2-\lim\limits_{m,n\to\infty}x_{mn}=L$.

Let $(X,\|.\|)$ be fuzzy normed space. A sequence $x=(x_m)_{m\in\mathbb{N}}$ in X is said to be $\mathscr{I}-$ convergent to $L\in X$ with respect to fuzzy norm on X if for each $\varepsilon>0$, the set $A(\varepsilon)=\left\{m\in\mathbb{N}:\|x_m-L\|_0^{+}\geq\varepsilon\right\}$ belongs to $\mathscr{I}$. In this case, we write $x_m\overset{F\mathscr{I}}{\longrightarrow}L$. The element L is called the $\mathscr{I}-$limit of (x_m) in X.

Let $(X,\|.\|)$ be a fuzzy normed space. A double sequence $x=(x_{mn})_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ in X is said to be $\mathscr{I}_2-$ convergent to $L_1\in X$ with respect to fuzzy norm on X if for each $\varepsilon>0$, the set $A(\varepsilon)=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|x_{mn}-L_1\|_0^{+}\geq\varepsilon\right\}\in\mathscr{I}_2$. In this case, we write $x_{mn}\overset{F\mathscr{I}_2}{\longrightarrow}L_1$ or $x_{mn}\to L_1\,(F\mathscr{I}_2)$ or $F\mathscr{I}_2-\lim\limits_{m,n\to\infty}x_{mn}=L_1$. The element L_1 is called the $F\mathscr{I}_2-$limit of (x_{mn}) in X. In terms of neighborhoods, we have $x_{mn}\overset{F\mathscr{I}_2}{\longrightarrow}L_1$ provided that for each $\varepsilon>0$, $\{(m,n)\in\mathbb{N}\times\mathbb{N}:x_{mn}\notin\mathscr{N}_{L_1}(\varepsilon,0)\}\in\mathscr{I}_2$. A useful interpretation of the above definition is the following;

$$x_{mn}\overset{F\mathscr{I}_2}{\longrightarrow}L_1\Leftrightarrow F\mathscr{I}_2-\lim_{m,n\to\infty}\|x_{mn}-L_1\|_0^{+}=0.$$

Note that $F\mathscr{I}_2-\lim\limits_{m,n\to\infty}\|x_{mn}-L_1\|_0^{+}=0$ implies that

$$F\mathscr{I}_2-\lim\|x_{mn}-L_1\|_{\alpha}^{-}\quad=\quad F\mathscr{I}_2-\lim\|x_{mn}-L_1\|_{\alpha}^{+}=0,$$

for each $\alpha\in[0,1]$, since $0\leq\|x_{mn}-L_1\|_{\alpha}^{-}\leq\|x_{mn}-L_1\|_{\alpha}^{+}\leq\|x_{mn}-L_1\|_{0}^{+}$ holds for every $m,n\in\mathbb{N}$ and for each $\alpha\in[0,1]$.

Let $(X,\|.\|)$ be a fuzzy normed space. A double sequence $x=(x_{mn})$ in X is said to be $\mathscr{I}_2$-Cauchy (or $F\mathscr{I}_2$-Cauchy) double sequence with respect to the fuzzy norm on X if, for each $\varepsilon>0$, there exists integers $p=p(\varepsilon)$ and $q=q(\varepsilon)$ such that the set $\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|x_{mn}-x_{pq}\|_0^{+}\geq\varepsilon\right\}$ belongs to $\mathscr{I}_2$.

We say that an admissible ideal $\mathscr{I}_2\subset 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the property (AP2), if for every countable family of mutually disjoint sets $\{A_1,A_2,...\}$ belonging to $\mathscr{I}_2$, there exists a countable family of sets $\{B_1,B_2,...\}$ such that $A_j\cap B_j\in\mathscr{I}_2^0$, i.e., $A_j\cap B_j$ is included in the finite union of rows and columns in $\mathbb{N}\times\mathbb{N}$ for each $j\in\mathbb{N}$ and $B=\bigcup_{j=1}^{\infty}B_j\in\mathscr{I}_2$ (hence $B_j\in\mathscr{I}_2$ for each $j\in\mathbb{N}$).

Lemma 1.1. *([27], Theorem 3.3) Let $\{P_i\}_{i=1}^{\infty}$ be a countable collection of subsets of $\mathbb{N}\times\mathbb{N}$ such that $P_i\in\mathscr{F}(\mathscr{I}_2)$ for each i, where $\mathscr{F}(\mathscr{I}_2)$ is a filter associated with a strongly admissible ideal $\mathscr{I}_2$ with the property (AP2). Then there exists a set $P\subset\mathbb{N}\times\mathbb{N}$ such that $P\in\mathscr{F}(\mathscr{I}_2)$ and the set $P\backslash P_i$ is finite for all i.*

Lemma 1.2. *([17], Theorem 3.5) Let $(X,\|.\|)$ be fuzzy normed space and $\mathscr{I}_2$ be a admissible ideal. Then, every $\mathscr{I}_2$-convergent sequence is $\mathscr{I}_2$-Cauchy sequence.*

2. Main results

In this section, we investigate relationship between $\mathscr{I}_2$-convergence and $\mathscr{I}_2$-Cauchy double sequences in fuzzy normed spaces. After, we introduce the concepts of $\mathscr{I}_2^*$-Cauchy double sequences and study relationships between $\mathscr{I}_2$-Cauchy and $\mathscr{I}_2^*$-Cauchy double sequences in fuzzy normed spaces.

Theorem 2.1. *Let $(X,\|.\|)$ be a fuzzy normed space. Then, a double sequence (x_{mn}) is $F\mathscr{I}_2$-convergent if and only if it is $F\mathscr{I}_2$-Cauchy double sequence.*

Proof. Hazarika and Kumar proved that every $F\mathscr{I}_2$-convergent sequence is $F\mathscr{I}_2$-Cauchy sequence in Lemma 1.2.

Assume that (x_{mn}) is $F\mathscr{I}_2$-Cauchy double sequence. We prove that (x_{mn}) is $F\mathscr{I}_2$-convergent. To this effect, let (ε_{pq}) be a strictly decreasing sequence of numbers converging to zero. Since (x_{mn}) is $F\mathscr{I}_2-$Cauchy double sequence, there exist two strictly increasing sequences (k_p) and (l_q) of positive integers such that the set

$$A(\varepsilon_{pq}) = \left\{(m,n)\in\mathbb{N}\times\mathbb{N} : \left\|x_{mn}-x_{k_pl_q}\right\|_0^+ \geq \varepsilon_{pq}\right\}$$

belongs to $\mathscr{I}_2$, $(p,q\in\mathbb{N})$. This implies that

$$\emptyset \neq \left\{(m,n)\in\mathbb{N}\times\mathbb{N} : \left\|x_{mn}-x_{k_pl_q}\right\|_0^+ < \varepsilon_{pq}\right\} \tag{2.1}$$

belongs to $\mathscr{F}(\mathscr{I}_2)$, $(p,q\in\mathbb{N})$. Let p,q,s,t be four positive integers such that $p\neq q$ and $s\neq t$. By (2.1), both the sets

$$D(\varepsilon_{pq}) = \{(m,n)\in\mathbb{N}\times\mathbb{N} : \left\|x_{mn}-x_{k_pl_q}\right\|_0^+ < \varepsilon_{pq}\}$$

and

$$C(\varepsilon_{st}) = \{(m,n)\in\mathbb{N}\times\mathbb{N} : \|x_{mn}-x_{k_sl_t}\|_0^+ < \varepsilon_{st}\}$$

are non empty sets in $\mathscr{F}(\mathscr{I}_2)$. Since $\mathscr{F}(\mathscr{I}_2)$ is a filter on $\mathbb{N}\times\mathbb{N}$, therefore $\emptyset\neq D(\varepsilon_{pq})\cap C(\varepsilon_{st})\in\mathscr{F}(\mathscr{I}_2)$. Thus, for each pair (p,q) and (s,t) of positive integers with $p\neq q$ and $s\neq t$, we can select a pair $\left(m_{(p,q),(s,t)},n_{(p,q),(s,t)}\right)\in\mathbb{N}\times\mathbb{N}$ such that

$$\left\|x_{m_{pqst}n_{pqst}}-x_{k_pl_q}\right\|_0^+ < \varepsilon_{pq} \text{ and } \left\|x_{m_{pqst}n_{pqst}}-x_{k_sl_t}\right\|_0^+ < \varepsilon_{st}.$$

It follows that

$$\begin{aligned}\left\|x_{k_pl_q}-x_{k_sl_t}\right\|_0^+ &\leq \left\|x_{m_{pqst}n_{pqst}}-x_{k_pl_q}\right\|_0^+ + \left\|x_{m_{pqst}n_{pqst}}-x_{k_sl_t}\right\|_0^+ \\ &< \varepsilon_{pq}+\varepsilon_{st}\to 0, \text{ as } p,q,s,t\to\infty.\end{aligned}$$

This implies that $\left(x_{k_pl_q}\right)$ $(p,q\in\mathbb{N})$ is a Cauchy double sequence in fuzzy normed space, therefore it satisfies the Cauchy convergence criterion. Thus, the sequence $\left(x_{k_pl_q}\right)$ converges to a finite limit L_1 that is,

$$\lim_{p,q\to\infty} x_{k_pl_q} = L_1.$$

Also, we have $\varepsilon_{pq}\to 0$ as $p,q\to\infty$, so for each $\varepsilon>0$ we can choose the positive integers p_0,q_0 such that for $p\geq p_0$ and $q\geq q_0$,

$$\varepsilon_{p_0q_0} < \frac{\varepsilon}{2} \text{ and } \left\|x_{k_pl_q}-L_1\right\|_0^+ < \frac{\varepsilon}{2}. \tag{2.2}$$

Now, we define the set

$$A(\varepsilon) = \left\{(m,n)\in\mathbb{N}\times\mathbb{N} : \|x_{mn}-L_1\|_0^+ \geq \varepsilon\right\}.$$

We prove that $A(\varepsilon)\subset A\left(\varepsilon_{p_0q_0}\right)$. Let $(m,n)\in A(\varepsilon)$, then by second half of (2.2) we have

$$\begin{aligned}\varepsilon \leq \|x_{mn}-L_1\|_0^+ &\leq \left\|x_{mn}-x_{k_{p_0}l_{q_0}}\right\|_0^+ + \left\|x_{k_{p_0}l_{q_0}}-L_1\right\|_0^+ \\ &\leq \left\|x_{mn}-x_{k_{p_0}l_{q_0}}\right\|_0^+ + \frac{\varepsilon}{2}.\end{aligned}$$

This implies that

$$\frac{\varepsilon}{2} \leq \left\| x_{mn} - x_{k_{p_0} l_{q_0}} \right\|_0^+$$

and therefore by first half of (2.2) we have

$$\varepsilon_{p_0 q_0} \leq \left\| x_{mn} - x_{k_{p_0} l_{q_0}} \right\|_0^+.$$

This implies that $(m,n) \in A(\varepsilon_{p_0 q_0})$ and therefore $A(\varepsilon)$ is contained in $A(\varepsilon_{p_0 q_0})$. Since $A(\varepsilon_{p_0 q_0})$ belongs to $\mathscr{I}_2$ therefore, $A(\varepsilon)$ belongs to $\mathscr{I}_2$. This proves that (x_{mn}) is $F\mathscr{I}_2$-convergent to L_1. □

Definition 2.2. *Let $(X, \|.\|)$ be a fuzzy normed space. A double sequence $x = (x_{mn})$ in X is said to be $\mathscr{I}_2^*$-Cauchy (or $F\mathscr{I}_2^*$-Cauchy) double sequence with respect to fuzzy norm on X if, there exists a set $M \in \mathscr{F}(\mathscr{I}_2)$ (i.e., $H = \mathbb{N} \times \mathbb{N} \backslash M \in \mathscr{I}_2$) and $k_0 = k_0(\varepsilon)$ such that for every $\varepsilon > 0$ and for $(m,n),(s,t) \in M$, $\|x_{mn} - x_{st}\|_0^+ < \varepsilon$, whenever $m,n,s,t > k_0$. In this case, we write $\lim_{m,n,s,t\to\infty} \|x_{mn} - x_{st}\|_0^+ = 0$.*

Theorem 2.3. *Let $\mathscr{I}_2$ be an admissible ideal of $\mathbb{N} \times \mathbb{N}$. If a double sequence (x_{mn}) in X is an $F\mathscr{I}_2^*$-Cauchy sequence, then it is $F\mathscr{I}_2$-Cauchy sequence.*

Proof. Suppose that (x_{mn}) is an $F\mathscr{I}_2^*$-Cauchy sequence. Then, there exists a set $M \in \mathscr{F}(\mathscr{I}_2)$ (i.e., $H = \mathbb{N} \times \mathbb{N} \backslash M \in \mathscr{I}_2$) and $k_0 = k_0(\varepsilon)$ such that for every $\varepsilon > 0$ and for $(m,n),(s,t) \in M$ $\|x_{mn} - x_{st}\|_0^+ < \varepsilon$, whenever $m,n,s,t \geq k_0$. Then,

$$\begin{aligned} A(\varepsilon) &= \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - x_{st}\|_0^+ \geq \varepsilon\} \\ &\subset H \cup [M \cap ((\{1,...,k_0\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,...,k_0\}))]. \end{aligned}$$

Since $\mathscr{I}_2$ be an admissible ideal, then

$$H \cup [M \cap ((\{1,...,k_0\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,...,k_0\}))] \in \mathscr{I}_2.$$

Therefore, we have $A(\varepsilon) \in \mathscr{I}_2$. This shows that (x_{mn}) is $F\mathscr{I}_2$-Cauchy sequence in X. □

Theorem 2.4. *Let $\mathscr{I}_2$ be an admissible ideal of $\mathbb{N} \times \mathbb{N}$ with the property $(AP2)$ and (x_{mn}) be a double sequence in X. Then, the concepts $\mathscr{I}_2-$Cauchy double sequence with respect to fuzzy norm on X and $\mathscr{I}_2^*$-Cauchy double sequence with respect to fuzzy norm on X coincide.*

Proof. If a double sequence is $F\mathscr{I}_2^*$-Cauchy, then it is $F\mathscr{I}_2$-Cauchy by Theorem 2.3, where $\mathscr{I}_2$ need not have the property $(AP2)$. Now, it is sufficient to prove that a double sequence (x_{mn}) in X is a $F\mathscr{I}_2^*$-Cauchy double sequence under assumption that it is an $F\mathscr{I}_2$-Cauchy double sequence. Let (x_{mn}) be an $F\mathscr{I}_2$-Cauchy double sequence in X. Then, there exists $s = s(\varepsilon), t = t(\varepsilon) \in \mathbb{N}$ such that for every $\varepsilon > 0$,

$$A(\varepsilon) = \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - x_{st}\|_0^+ \geq \varepsilon\right\} \in \mathscr{I}_2.$$

Let

$$P_i = \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - x_{s_i t_i}\|_0^+ < \frac{1}{i}\right\},$$

where $s_i = s(1 \backslash i)$, $(i \in \mathbb{N})$, $t_i = t(1 \backslash i)$. It is clear that $P_i \in \mathscr{F}(\mathscr{I}_2)$ for all $i \in \mathbb{N}$. Since $\mathscr{I}_2$ has the property $(AP2)$, then by Lemma 1.1 there exists a set $P \subset \mathbb{N} \times \mathbb{N}$ such that $P \in \mathscr{F}(\mathscr{I}_2)$ and $P \backslash P_i$ is finite for all $i \in \mathbb{N}$. Now we show that

$$\lim_{m,n,s,t\to\infty} \|x_{mn} - x_{st}\|_0^+ = 0,$$

for $(m,n),(s,t) \in P$. To prove this, let $\varepsilon > 0$ and $j \in \mathbb{N}$ such that $j > 2/\varepsilon$. If $(m,n),(s,t) \in P$ then $P \backslash P_i$ is a finite set, so there exists $N = N(j)$ such that $(m,n),(s,t) \in P_j$ for all $m,n,s,t > N(j)$. Therefore,

$$\|x_{mn} - x_{s_i t_i}\|_0^+ < \frac{1}{j} \text{ and } \|x_{st} - x_{s_i t_i}\|_0^+ < \frac{1}{j},$$

for all $m,n,s,t > N(j)$. Hence it follows that

$$\begin{aligned} \|x_{mn} - x_{st}\|_0^+ &\leq \|x_{mn} - x_{s_it_i}\|_0^+ + \|x_{st} - x_{s_it_i}\|_0^+ \\ &\leq \frac{1}{j} + \frac{1}{j} = \frac{2}{j} < \varepsilon, \end{aligned}$$

for all $m,n,s,t > N(j)$. Thus, for any $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that for $m,n,s,t > N(j)$ and $(m,n),(s,t) \in P$ we have

$$\|x_{mn} - x_{st}\|_0^+ < \varepsilon.$$

This shows that the double sequence (x_{mn}) in X is an $F\mathscr{I}_2^*$-Cauchy double sequence in fuzzy normed spaces. □

References

[1] H. Fast, *Sur la convergence statistique*, Colloq. Math., **2** (1951), 241–244.

[2] I.J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly **66** (1959), 361–375.

[3] M. Mursaleen, O.H.H. Edely, *Statistical convergence of double sequences*, J. Math. Anal. Appl. **288** (2003), 223–231.

[4] D. Rath, B.C. Tripaty, *On statistically convergence and statistically Cauchy sequences*, Indian J. Pure Appl. Math. **25**(4) (1994), 381–386.

[5] P. Kostyrko, T. Šalát, W. Wilczyński, *$\mathscr{I}$-convergence*, Real Anal. Exchange, **26**(2) (2000), 669–686.

[6] P. Das, P. Kostyrko, W. Wilczyński, P. Malik, *$\mathscr{I}$ and $\mathscr{I}^*$-convergence of double sequences*, Math. Slovaca, **58**(5) (2008), 605–620.

[7] P. Das, P. Malik, *On extremal $\mathscr{I}$-limit points of double sequences*, Tatra Mt. Math. Publ., **40** (2008), 91–102.

[8] V. Kumar, *On $\mathscr{I}$ and $\mathscr{I}^*$-convergence of double sequences*, Math. Commun. **12** (2007), 171–181.

[9] T. Šalát, B.C. Tripaty, M. Ziman, *On $\mathscr{I}$-convergence field*, Ital. J. Pure Appl. Math. **17** (2005), 45–54.

[10] B. Tripathy, B.C. Tripathy, *On $\mathscr{I}$-convergent double sequences*, Soochow J. Math. **31** (2005), 549–560.

[11] M. Matloka, *Sequences of fuzzy numbers*, Busefal, **28** (1986), 28–37.

[12] S. Nanda, *On sequences of fuzzy numbers*, Fuzzy Sets Syst. **33** (1989), 123–126.

[13] C. Ṣençimen, S. Pehlivan, *Statistical convergence in fuzzy normed linear spaces*, Fuzzy Sets and Systems, **159** (2008), 361–370.

[14] B. Hazarika, *On ideal convergent sequences in fuzzy normed linear spaces*, Afrika Mat., **25**(4) (2013), 987–999.

[15] E. Dündar, Ö. Talo, *$\mathscr{I}_2$-convergence of double sequences of fuzzy numbers*, Iran. J. Fuzzy Syst., **10**(3) (2013), 37–50

[16] E. Dündar, Ö. Talo, Ö., *$\mathscr{I}_2$-Cauchy Double Sequences of Fuzzy Numbers*, Gen. Math. Notes, **16**(2) (2013), 103–114.

[17] B. Hazarika, V. Kumar, *Fuzzy real valued $\mathscr{I}$-convergent double sequences in fuzzy normed spaces*, J. Intell. Fuzzy Syst., **26** (2014), 2323–2332.

[18] E. Dündar, M. R. Türkmen, *On $\mathscr{I}_2$-Convergence and $\mathscr{I}_2^*$-Convergence of Double Sequences in Fuzzy Normed Spaces*, Konuralp J. Math., (in press).

[19] V. Kumar, K. Kumar, *On the ideal convergence of sequences of fuzzy numbers*, Inform. Sci. **178** (2008), 4670–4678.

[20] S.A. Mohiuddine, H. Ṣevli, M. Cancan, *Statistical convergence of double sequences in fuzzy normed spaces*, Filomat, **26**(4) (2012), 673–681.

[21] M. R. Türkmen, E. Dündar, *On Lacunary Statistical Convergence of Double Sequences and Some Properties in Fuzzy Normed Spaces*, J. Intell. Fuzzy Syst., **36**(1) (2019), 467-472.

[22] R. Saadati, *On the $\mathscr{I}$-fuzzy topological spaces*, Chaos, Solitons Fractals, **37** (2008), 1419–1426.

[23] T.Bag, S.K. Samanta, *Fixed point theorems in Felbin's type fuzzy normed linear spaces*, J. Fuzzy Math., **16**(1) (2008), 243–260.

[24] B. Bede, S.G. Gal, *Almost periodic fuzzy-number-valued functions*, Fuzzy Sets Syst., **147**(2004), 385–403.

[25] P. Diamond, P. Kloeden, *Metric Spaces of Fuzzy Sets-Theory and Applications*, World Scientific Publishing, Singapore, **(1994).**

[26] E. Dündar, B. Altay, *$\mathscr{I}_2$-convergence and $\mathscr{I}_2$-Cauchy of double sequences*, Acta Math. Sci., **34**(2) (2014), 343–353.

[27] E. Dündar, B. Altay, *On some properties of $\mathscr{I}_2$-convergence and $\mathscr{I}_2$-Cauchy of double sequences,* Gen. Math. Notes, **7**(1) (2011), 1–12.

[28] J.-X. Fang, H. Huang, *On the level convergence of a sequence of fuzzy numbers*, Fuzzy Sets and Systems, **147** (2004), 417–415.

[29] C. Felbin, *Finite-dimensional fuzzy normed linear space*, Fuzzy Sets and Systems, **48**(2) (1992), 239–248.

[30] Ö. Kişi, *$\mathscr{I}_2-\lambda^-$-statistically convergence of double sequences in fuzzy normed spaces,* J. Intell. Fuzzy Syst., **36**(2) (2019), 1–12.

[31] M. Mizumoto, K. Tanaka, *Some properties of fuzzy numbers*, Advances in Fuzzy Set Theory and Applications, North-Holland (Amsterdam), **1979**, 153–164.

[32] A. Nabiev, S. Pehlivan, M. Gürdal, *On $\mathscr{I}$-Cauchy sequences*, Taiwanese J. Math. **11**(2) (2007) 569–5764.

[33] A. Pringsheim, *Zur theorie der zweifach unendlichen Zahlenfolgen*, Math. Ann. **53** (1900), 289–321.

[34] E. Savaş, M. Mursaleen, *On statistically convergent double sequences of fuzzy numbers*, Inform. Sci. **162** (2004), 183–192

[35] L.A. Zadeh, *Fuzzy sets*, Inform. Control **8**(1965), 338–353.