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Ordered weak φ-contractions in cone metric spaces over Banach algebras and fixed point theorems

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Abstract

In this work, we introduce the class of ordered weak φ-contractions in cone metric spaces over Banach algebras and prove some fixed point results for the mappings belonging to this new class. Our results generalize and extend some known fixed point results in cone metric spaces to the spaces equipped with a partial order. Some examples are given which illustrate the results proved herein.

Keywords: Cone metric space; ordered weak φ-contraction; fixed point.
2010 MSC: 47H10; 54H25.

1. Introduction

In 1922, Banach [1] proved the famous Banach contraction principle and showed how it can be used to solve existence problems of integral equations. The Banach contraction principle and its generalizations are at the center of various research activities, and are the main source of the inspiration of metric fixed point theory.

Theorem 1.1 (Banach)**.** *Let (X, d) be a complete metric space and $T: X \to X$ be a contraction mapping, i.e., satisfies the following condition: there exists $\lambda \in [0, 1)$ such that*

$$d(Tx, Ty) \leq \lambda d(x, y) \tag{1}$$

for all $x, y \in X$. Then T has a unique fixed point in X, i.e., there exists a unique $x^* \in X$ such that $Tx^* = x^*$.

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Here, the constant λ is called the control constant and the above theorem is also known as the Banach contraction principle.

The Banach contraction principle is generalized by several authors in various directions. Rus [23, 24] and Berinde [2] generalized the Banach contraction principle by introducing the notion of φ-contractions. In such generalizations, a function $\varphi : [0,\infty) \to [0,\infty)$ (control function) is used instead the control constant $\lambda \in (0, 1)$. Importance of using such function in place of constants can be found in the papers [2, 23, 24]. In this connection, see also the paper by Reich and Zaslavski [21]

Huang and Zhang [8] introduced the notion of cone metric spaces and generalized the concept of metric spaces. In cone metric space, the usual real-valued metric function was replaced by a vector-valued function which takes the values in a Banach space. They proved some fixed point results in such spaces for various types of contractive mappings. Because of some similarities with the fixed point results in usual metric spaces, the fixed point results in cone metric spaces were the consequences of their usual metric versions (see, e.g., [3, 6, 7, 14]). Recently, Liu and Xu [16] introduced the concept of a cone metric space over Banach algebra, and proved some fixed point results in such spaces. They showed that the fixed point results in this new setting cannot be derived from their usual metric versions. Several authors followed this idea of generalization of metric spaces and proved various version of fixed point theorems in such spaces (see [9, 10, 11, 12, 4, 17, 18, 25]).

On the other hand, Ran and Reurings [20], and Nieto and Lopez [19] investigated the existence and uniqueness of fixed points of contractions in the spaces endowed with a partial order. In such results the contractive conditions on mappings were weakened by using a partial order defined on the space.

Recently, Li and Huang [15] proved some fixed point results for weak φ-contractions in cone metric spaces over Banach algebra and provided some applications of such mappings. In this paper, we prove an ordered version of result of Li and Huang [15] which generalizes and unifies the results of Li and Huang [15], Ran and Reurings [20], and Nieto and Lopez [19] in cone metric spaces over Banach algebras. To justify the new results, some examples are provided.

2. Preliminaries

The following notions and facts will be needed in this paper.

Definition 2.1 ([23])**.** *A function $\varphi : [0,\infty) \to [0,\infty)$ is called a comparison if it satisfies the following two conditions:*

(1) φ is monotone nondecreasing, i.e., $0 \le t_1 \le t_2 \Rightarrow \varphi(t_1) \le \varphi(t_2)$;

(2) $\{\varphi^n(t)\}$ $(t > 0)$ converges to 0 as $n \to \infty$.

It is obvious that $\varphi(t) < t$ for each $t > 0$, $\varphi(0) = 0$ and $\lim_{t\to 0} \varphi(t) = 0$.

Definition 2.2 ([23])**.** *Let (X, d) be a metric space. A mapping $T : X \to X$ is called a φ-contraction if there exists a comparison $\varphi : [0,\infty) \to [0,\infty)$ such that*

$$d(Tx, Ty) \le \varphi(d(x, y)) \quad \textit{for all} \quad x, y \in X.$$

Theorem 2.3 ([23])**.** *Let (X, d) be a metric space and $T : X \to X$ be a φ-contraction. Then T has a unique fixed point in X. Moreover, for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.*

Definition 2.4 ([16])**.** *Let $\mathcal{A}$ be a Banach algebra with a unit e and a zero element θ. A nonempty closed subset P of $\mathcal{A}$ is called a cone if the following conditions hold:*

(1) $\{\theta, e\} \subset P$;

(2) $\forall \alpha, \beta \in [0,\infty) \Rightarrow \alpha P + \beta P \subseteq P$;

(3) $P^2 = PP \subset P$;

(4) $P \cap (-P) = \{\theta\}$.

A cone P is called a solid cone if $P^\circ \ne \emptyset$, where P° stands for the interior of P.

We can always define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \ll y$ to indicate that $y - x \in P^\circ$. We shall also write $\|\cdot\|$ as the norm on $\mathcal{A}$. A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in \mathcal{A}$, $\theta \preceq x \preceq y$ implies $\|x\| \leq K\|y\|$.

We always suppose that $\mathcal{A}$ is a Banach algebra with a unit e, P is a solid cone in $\mathcal{A}$, and $\preceq$ is partial ordering with respect to P.

Definition 2.5 ([16]). *Let X be a nonempty set and $\mathcal{A}$ be a Banach algebra. A mapping $d : X \times X \to \mathcal{A}$ is called a cone metric if it satisfies:*
(i) $\theta \preceq d(x, y)$, for all $x, y \in X$, $d(x, y) = \theta \Leftrightarrow x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y \in X$.
In this case, the pair (X, d) is called a cone metric space over Banach algebra $\mathcal{A}$.

Definition 2.6 ([5]). *A sequence $\{u_n\}$ in a Banach algebra $\mathcal{A}$ is said to be a c-sequence if for each $c \gg \theta$, there exists $N \in \mathbb{N}$ such that $u_n \ll c$ for all $n > N$.*

Definition 2.7 ([11]). *Let (X, d) be a cone metric space over Banach algebra $\mathcal{A}$ and $\{x_n\}$ be a sequence in X. We say that*
(i) $\{x_n\}$ converges to $x \in X$ if $\{d(x_n, x)\}$ is a c-sequence and in this case we write $x_n \to x$ as $n \to \infty$;
(ii) $\{x_n\}$ is a Cauchy sequence if $\{d(x_n, x_m)\}$ is a c-sequence for n, m;
(iii) (X, d) is complete if every Cauchy sequence in X is convergent.

It is obvious that the limit of a convergent sequence in a cone metric space (X, d) over Banach algebra $\mathcal{A}$ is unique. We say that the mapping $T \colon X \to X$ is continuous if $Tx_n \to Tx$ as $n \to \infty$, whenever, $x_n \to x$ as $n \to \infty$ for some sequence $\{x_n\}$ in X.

Lemma 2.8 ([13]). *Let $\mathcal{A}$ be a Banach algebra and $u, v, w \in \mathcal{A}$. Then*
(1) $u \ll w$ if $u \preceq v \ll w$ or $u \ll v \preceq w$;
(2) $u = \theta$ if $\theta \preceq u \ll c$ for each $c \gg \theta$.

Lemma 2.9 ([22]). *Let $\mathcal{A}$ be a Banach algebra with its unit e. Then the spectral radius of $u \in \mathcal{A}$ equals to $\rho(u) = \lim_{n\to\infty} \|u^n\|^{\frac{1}{n}}$.*

Lemma 2.10 ([10]). *Let P be a cone in a Banach algebra $\mathcal{A}$, $\{u_n\}$ and $\{v_n\}$ be two c-sequences in $\mathcal{A}$, and $\alpha, \beta \in P$ be vectors, then $\{\alpha u_n + \beta v_n\}$ is a c-sequence in $\mathcal{A}$.*

Lemma 2.11 ([9]). *Let P be a cone and $k \in P$ with $\rho(k) < 1$. Then $\{k^n\}$ is a c-sequence.*

Definition 2.12 ([15]). *Let $\mathcal{A}$ be a Banach algebra and P be a cone in $\mathcal{A}$. A mapping $\varphi : P \to P$ is called a weak comparison if the following conditions hold:*
(i) φ is nondecreasing with respect to $\preceq$, namely, $t_1, t_2 \in P$, $t_1 \preceq t_2 \Rightarrow \varphi(t_1) \preceq \varphi(t_2)$;
(ii) $\{\varphi^n(t)\}$ $(t \in P)$ is a c-sequence in P;
(iii) if $\{u_n\}$ is a c-sequence in P, then $\{\varphi(u_n)\}$ is also a c-sequence in P.

Remark 2.13 ([15]). *By Definition 2.12, we have $\varphi(\theta) = \theta$. Indeed, by (i) of Definition 2.12, we have $\theta \preceq \varphi(\theta) \preceq \varphi^n(\theta)$. Since $\{\varphi^n(\theta)\}$ is a c-sequence, then by Lemma 2.8, it may be verified that $\varphi(\theta) = \theta$.*

If $\mathcal{A} = \mathbb{R}$ and $P = [0, \infty)$, then the above definition is reduced into the Definition 2.1

Example 2.14 ([15]). *Let $\mathcal{A}$ be a Banach algebra, P be a cone in $\mathcal{A}$, and $k \in P$. Take $\varphi(t) = kt$ $(t \in P)$, where $\rho(k) < 1$. Then by Lemma 2.10 and Lemma 2.11, φ is a weak comparison.*

Example 2.15 ([15]). *Let M be a compact set of $\mathbb{R}^n$ and $\mathcal{A} = C(M)$, where $C(M)$ denotes the set of all continuous functions on M. Let $P = \{u \in \mathcal{A} : u(t) \geq 0, t \in M\}$ and define a mapping $\varphi : P \to P$ by $\varphi(u) = \frac{u}{u+1}$. Then φ is a weak comparison.*

Definition 2.16 ([15]). *Let (X,d) be a cone metric space over Banach algebra $\mathcal{A}$. Let P be a cone and $\varphi : P \to P$ be a weak comparison. Then a mapping $T : X \to X$ is called a weak φ-contraction if*

$$d(Tx,Ty) \preceq \varphi(d(x,y)) \quad \text{for all} \quad x,y \in X. \tag{2}$$

Clearly, the above definition generalizes the Definition 2.2.

3. Main Results

First we introduce some definitions which will be needed in the sequel.

Suppose X is a nonempty set, $T\colon X \to X$ a mapping and $x_0 \in X$. Then, the sequence $\{x_n\}$ is called a Picard sequence $\{x_n\}$ generated by T if $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}$.

Definition 3.1. *Let (X,d) be a cone metric space over Banach algebra $\mathcal{A}$, $T\colon X \to X$ be a mapping and $\sqsubseteq$ be a partial order on X. Then, X is said to be T-orbitally ordered complete if a nondecreasing Picard sequence $\{x_n\}$, $x_0 \in X$ generated by T is a Cauchy sequence then there exist $x^* \in X$ and a subsequence $\{x_{n_k}\}$ such that $\{d(x_{n_k}, x^*)\}$ is a c-sequence and $x_{n_k} \sqsubseteq x^*$ for all $k \in X$.*

The following two examples shows that the T-orbital ordered completeness and the completeness of the space are mutual independent concepts.

Example 3.2. *Let $\mathcal{A} = C^1_{\mathbb{R}}[0,1]$ and define a norm on $\|\cdot\|$ by $\|x\| = \|x\|_\infty + \|x'\|_\infty$ for $x \in \mathcal{A}$. Define multiplication in $\mathcal{A}$ as just pointwise multiplication. Then $\mathcal{A}$ is a real unital Banach algebra with unit $e = 1$. The set $P = \{x \in \mathcal{A}\colon x \geq 0\}$ is a cone in $\mathcal{A}$. Let $X = \mathbb{Q} \cap [0,1]$. Define $d\colon X \times X \to \mathcal{A}$ by $d(x,y) = |x-y|e^t$ for all $x,y \in X$, then (X,d) is a cone metric space over Banach algebra $\mathcal{A}$. Define a mapping $T\colon X \to X$ by $Tx = \dfrac{x}{2}$ for all $x \in X$ and partial order $\sqsubseteq$ by $\sqsubseteq = \{(x,y) \in X \times X\colon y \leq x\}$. Then it is obvious that every nondecreasing Picard sequence in X is a monotonic nonincreasing sequence with respect to the ordinary order in X and must converge to some point $x^* \in X$ with $x_{n_k} \sqsubseteq x^*$. Therefore, X is T-orbitally ordered complete.*

On the other hand, it is easy to see that the space X is not a complete metric space.

Example 3.3. *Let $\mathcal{A}$ and P are same as in the previous example. Suppose, $X = [0,1]$ and define $d\colon X \times X \to \mathcal{A}$ by $d(x,y) = |x-y|e^t$ for all $x,y \in X$, then (X,d) is a cone metric space with Banach algebra $\mathcal{A}$. Define a mapping $T\colon X \to X$ by $Tx = \dfrac{x}{2}$ for all $x \in X$ and partial order $\sqsubseteq$ by $\sqsubseteq = \{(x,y) \in X \times X\colon y \leq x\} \setminus \{(x,0)\colon x \in (0,1]\}$. Then, although, every nondecreasing Picard sequence $\{x_n\}$ in X generated by T with $x_0 \in (0,1]$, is a monotonic nonincreasin sequence with respect to the ordinary order in X and must converge to some point $x^* \in X$, but there exists no subsequence $\{x_{n_k}\}$ such that $x_{n_k} \sqsubseteq x^*$. Therefore, X is not T-orbitally ordered complete.*

On the other hand, it is easy to see that the space X is a complete cone metric space.

Definition 3.4. *Let (X,d) be a cone metric space over Banach algebra $\mathcal{A}$ and $\sqsubseteq$ be a partial order on X. Let P be a cone and $\varphi : P \to P$ be a weak comparison. Then a mapping $T : X \to X$ is called an ordered weak φ-contraction if*

$$d(Tx,Ty) \preceq \varphi(d(x,y)) \text{ for all } x,y \in X \text{ with } x \sqsubseteq y. \tag{3}$$

It is obvious that the above contractive condition is weaker than the condition (2). The following example verifies this fact.

Example 3.5. *Let $A = \mathbb{R}^2$ with the norm $\|(x_1,x_2)\| = |x_1| + |x_2|$. Define the multiplication on A by $xy = (x_1y_1, x_1y_2 + x_2y_1)$ for all $x = (x_1,x_2), y = (y_1,y_2) \in A$. Then, A is a Banach algebra with unit $e = (1,0)$. Let $P = \{(x_1,x_2) \in \mathbb{R}^2\colon x_1,x_2 \geq 0\}$. Then P is a positive cone. Let $X = [0,1] \times [0,1]$ and define the cone metric $d\colon X \times X \to P$ by*

$$d((x_1,x_2),(y_1,y_2)) = (|x_1 - y_1|, |x_2 - y_2|) \in P.$$

Then, (X,d) is a cone metric space. Let $\mathbb{Q}\cap[0,1)=\mathbb{Q}_1$ and define the mappings $T\colon X\to X$ and a partial order $\sqsubseteq$ on X by:

$$T(x_1,x_2)=\begin{cases}(a,a), & \text{if } x_1,x_2\in\mathbb{Q}_1;\\ (x_1,x_2), & \text{otherwise.}\end{cases}$$

and $\sqsubseteq=\{(x,y)\in X\times X\colon x=(x_1,x_2)\preceq y=(y_1,y_2), x_1,x_2,y_1,y_2\in\mathbb{Q}_1\}$, where $a\in[0,1]$ is fixed and Δ is the diagonal of $X\times X$. Then, T is an ordered weak φ-contraction with arbitrary φ on P. On the other hand, T is not a weak φ-contraction.

Next, we state a fixed point theorem for ordered weak φ-contraction on cone metric spaces.

Theorem 3.6. *Let (X,d) be a cone metric space over Banach algebra $\mathcal{A}$ and $\sqsubseteq$ be a partial order on X. Let $T:X\to X$ be an ordered weak φ-contraction and the following conditions are satisfied:*

(i) *X is T-orbitally ordered complete;*
(ii) *T is nondecreasing with respect to $\sqsubseteq$;*
(iii) *there exists $x_0\in X$ such that $x_0\sqsubseteq Tx_0$.*

Then T has a fixed point in X. Moreover, the iterative sequence $\{T^nx_0\}$ converges to a fixed point of T.

Proof. Suppose $x_0\in X$ is such that $x_0\sqsubseteq Tx_0$. Let $x_n=T^nx_0,\ n\geq 1$. We shall show that the Picard sequence $\{T^nx_0\}=\{x_n\}$ is nondecreasing with respect to $\sqsubseteq$. Then, since $x_0\sqsubseteq Tx_0$ we have $x_0\sqsubseteq x_1$ and T is nondecreasing it implies that $Tx_0\sqsubseteq Tx_1$, i.e., $x_1\sqsubseteq x_2$. Again, since T is nondecreasing we have $Tx_1\sqsubseteq Tx_2$, i.e., $x_2\sqsubseteq x_3$. Continuing this process we obtain

$$x_n\sqsubseteq x_{n+1}\quad\text{for all}\quad n\geq 1.$$

Thus, $\{x_n\}$ is nondecreasing with respect to $\sqsubseteq$.

Next, we show that the sequence $\{x_n\}$ is a Cauchy sequence. Then, for any $c\gg\theta$, by definition of φ, there exists $n_0\in\mathbb{N}$ such that $\varphi^{n_0}(c)\ll c$. Since $x_n\sqsubseteq x_{n+1}$ for all $n\geq 1$ by transitivity of $\sqsubseteq$ we have $x_n\sqsubseteq x_{n+r}$ for all $n\geq 1, r\geq 0$, and as T is an ordered weak φ-contraction, it follows that

$$\begin{aligned} d(x_n,x_{n+n_0}) &= d(Tx_{n-1},Tx_{n+n_0-1})\\ &\preceq \varphi(d(x_{n-1},x_{n+n_0-1})). \end{aligned}\tag{4}$$

Again, since $x_n\sqsubseteq x_{n+r}$ for all $n\geq 1, r\geq 0$ we can replace n by $n-1$ in the above inequality and then we obtain

$$d(x_{n-1},x_{n+n_0-1}) \preceq \varphi(d(x_{n-2},x_{n+n_0-2})).$$

Since φ is nondecreasing we obtain

$$\varphi(d(x_{n-1},x_{n+n_0-1})) \preceq \varphi^2(d(x_{n-2},x_{n+n_0-2})).$$

The above inequality with (4) yields

$$d(x_n,x_{n+n_0})\preceq\varphi^2(d(x_{n-2},x_{n+n_0-2})).$$

Proceeding in similar manner we obtain the following inequality:

$$d(x_n,x_{n+n_0})\preceq\varphi^n(d(x_0,x_{n_0})).\tag{5}$$

Similarly, we can obtain

$$d(x_{n+n_0},T^{n_0}x)=d(x_{n_0},x_{n+n_0})\preceq\varphi^{n_0}(d(x_n,x)).\tag{6}$$

As, $d(x_0, x_{n_0}) \in P$ the sequence $\{\varphi^n(d(x_0, x_{n_0}))\}$ is a c-sequence, and by Lemma 2.8 and inequality (5) it follows that $\{d(x_n, x_{n+n_0})\}$ is a c-sequence, and so, by choice of n_0 there exists $n_1 \in \mathbb{N}$ such that

$$d(x_n, x_{n+n_0}) \ll c - \varphi^{n_0}(c) \quad \text{for all} \quad n \geq n_1.$$

Define $B_{\sqsubseteq}(x_n, c) = \{x \in X : d(x_n, x) \ll c, x_n \sqsubseteq x\}, n \geq n_1$, then obviously $B_{\sqsubseteq}(x_n, c) \neq \emptyset$. Choose $x \in B_{\sqsubseteq}(x_n, c)$ with $n \geq n_1$, tnen by (5) and the definition of $B_{\sqsubseteq}(x_n, c)$ we get

$$\begin{aligned} d(x_n, T^{n_0}x) &\preceq d(x_n, x_{n+n_0}) + d(x_{n+n_0}, T^{n_0}x) \\ &\ll c - \varphi^{n_0}(c) + \varphi^{n_0}(d(x_n, x)) \\ &\preceq c - \varphi^{n_0}(c) + \varphi^{n_0}(c) \\ &= c. \end{aligned}$$

This means that $B_{\sqsubseteq}(x_n, c)$ for $n \geq n_1$ is T^{n_0}-invariant. Now, for any $k \in \mathbb{N}$, we have $d(x_n, x_{n+kn_0}) \ll c$.

Since $x_n \sqsubseteq x_{n+r}$ for all $n \geq 1, r \geq 0$ by using (3) and properties of φ we obtain

$$d(x_i, x_{i+1}) \preceq \varphi^i(x_0, x_1), \quad d(x_{i+1}, x_{i+2}) \preceq \varphi^i(x_1, x_2), \ldots$$

for all $i \in \mathbb{N}$. Therefore,

$$\begin{gathered} d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+n_0-1}, x_{n+n_0}) \\ \preceq \varphi^n(d(x_0, x_1)) + \varphi^n(d(x_1, x_2)) + \cdots + \varphi^n(d(x_{n_0-1}, x_{n_0})). \end{gathered}$$

Since each sequence $\{\varphi^n(d(x_i, x_{i+1}))\}$, $i = 0, 1, \ldots, n_0 - 1$ is a c-sequences, therefore by Lemma 2.8, $\{\varphi^n(d(x_0, x_1)) + \varphi^n(d(x_1, x_2)) + \cdots + \varphi^n(d(x_{n_0-1}, x_{n_0}))\}$ is a c-sequence, and so, $\{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+n_0-1}, x_{n+n_0})\}$ is also a c-sequence. It follows that for the above $c \gg \theta$, there exists $n_2 \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+n_0-1}, x_{n+n_0}) \ll c$$

for all $n > n_2$.

Let $n_3 = \max\{n_1, n_2\}$, choose $m, n > n_3$, and assume that

$$k_m = \left[\frac{m - n_3}{n_0}\right], \quad k_n = \left[\frac{n - n_3}{n_0}\right],$$

where $[\cdot]$ denotes the integer part. Then, as

$$n_3 \leq m - k_m n_0 < n_3 + n_0, \quad n_3 \leq n - k_n n_0 < n_3 + n_0,$$

we must have

$$d(x_m, x_n) \preceq d(x_m, x_{m-k_m n_0}) + d(x_{m-k_m n_0}, x_{n-k_n n_0}) + d(x_{n-k_n n_0}, x_n) \ll 3c.$$

Therefore, $\{x_n\}$ is a Cauchy sequence. Since X is T-orbitally ordered complete, there exist $x^* \in X$ and a subsequence $\{x_{n_k}\}$ such that $\{d(x_{n_k}, x^*)\}$ is a c-sequence and $x_{n_k} \sqsubseteq x^*$ for all $k \in X$.

We shall show that u is a fixed point of T. Since $\{d(x_{n_k}, x^*)\}$ is a c-sequence and $x_{n_k} \sqsubseteq x^*$ for all $k \in X$ we obtain from (3) that

$$\begin{aligned} d(Tx^*, x^*) &\preceq d(Tx^*, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k}) + d(x_{n_k}, x^*) \\ &= d(Tx^*, Tx_{n_k}) + d(x_{n_k+1}, x_{n_k}) + d(x_{n_k}, x^*) \\ &\preceq \varphi(d(x^*, x_{n_k})) + \varphi^{n_k}(d(x_1, x_0)) + d(x^*, x_n). \end{aligned}$$

Since $\{d(x^*, x_{n-1})\}$ and $\{\varphi^{n_k}(d(x_1, x_0))\}$ are c-sequences, it follows from the definition of φ and Lemma 2.8 that $\{d(Tx^*, x^*)\}$ is a c-sequence. Therefore, $d(Tx^*, x^*) = \theta$, i.e., $Tx^* = x^*$. Thus, x^* is a fixed point of T. $\square$

The following example shows that the fixed point in the above theorem may not be unique and it shows the existence of mappings on which the result of [15] is be applicable while the new result is.

Example 3.7. *Let M be a compact set of $\mathbb{R}^n$ and $\mathcal{A} = C(M)$, where $C(M)$ denotes the set of all continuous functions on M. Let $P = \{u(t) \in \mathcal{A} : u(t) \geq 0, t \in M\}$ and define a mapping $\varphi : P \to P$ by $\varphi(u) = \frac{u}{u+1}$. Then φ is a weak comparison. Let $X = \mathcal{A}$ and define a mapping $d : X \times X \to \mathcal{A}$ by*

$$d(u(t), v(t)) = |u(t) - v(t)|,\ t \in M$$

. Then (X, d) is a cone metric space over Banach algebra. Let $\mathcal{Q} = \{u(t) \in \mathcal{A}\colon 0 \leq u(t) \in \mathbb{Q} \text{ for all } t \in M\}$ and

$$\sqsubseteq = \{(u(t), v(t)) \in X \times X \colon u(t), v(t) \in \mathcal{Q}, v(t) \leq u(t) \text{ for all } t \in M\}.$$

Define a mapping $T : X \to X$ by

$$Tu = \begin{cases} \frac{u}{u+1}, & \text{if } u \in \mathcal{Q}; \\ u \sin u, & \text{otherwise.} \end{cases}$$

Clearly, T is not a weak φ-contraction. For instance, if $u = \dfrac{\pi}{2}, v = \pi$, then we have $d(Tu, Tv) = d(u, v)$, and so, there is no weak comparison φ such that $d(Tu, Tv) \preceq \varphi(d(u, v))$. On the other hand, T is an ordered weak φ-contraction. Because, if $u \sqsubseteq v$ we have $u(t), v(t) \in \mathcal{Q}$ and $0 \leq v(t) \leq u(t)$ for all $t \in M$, and we have

$$\begin{aligned} d(Tu, Tv) &= \left| \frac{u}{u+1} - \frac{v}{v+1} \right| \preceq \frac{v-u}{v-u+1} = \frac{d(u, v)}{d(u, v)+1} \\ &= \varphi\left(d(x, y)\right). \end{aligned}$$

It is easy to see that X is T-orbitally ordered complete.

If $u \sqsubseteq v$ we have $u(t), v(t) \in \mathcal{Q}$ and $0 \leq u(t) \leq v(t)$ for all $t \in M$, and so, $\frac{u(t)}{u(t)+1}, \frac{v(t)}{v(t)+1} \in \mathcal{Q}$ and $0 \leq \frac{u(t)}{u(t)+1} \leq \frac{v(t)}{v(t)+1}$ for all $t \in M$, i.e., $Tu(t), Tv(t) \in \mathcal{Q}$ and $0 \leq Tu(t) \leq Tv(t)$ for all $t \in M$. It shows that $Tu \sqsubseteq Tv$, and so, T is nondecreasing.

Finally, for any $u_0(t) \in \mathcal{Q}$ we have $Tu_0(t) \in \mathcal{Q}$ and $\frac{u_0(t)}{u_0(t)+1} \leq u_0(t)$ for all $t \in M$, i.e., $u_0(t) \sqsubseteq Tu_0(t)$.

Thus, all the conditions of Theorem 3.6 are satisfied, and T has a fixed point in X. Indeed, T has infinitely many fixed points in X and the set of all fixed points of T is given by $\mathfrak{F}(T) = \left\{0, (1 + 4n)\dfrac{\pi}{2} \colon n \in \mathbb{Z}\right\}$.

We now ensure the uniqueness of fixed point of mapping T by giving a necessary condition.

Theorem 3.8. *Suppose that all the conditions of Theorem 3.6 are satisfied. In addition, suppose, for each pair $x, y \in X$ there exists $z \in X$ such that $x \sqsubseteq z$ and $y \sqsubseteq z$, then T has a unique fixed point in X.*

Proof. The existence of the fixed point $x^* \in X$ follows from Theorem 3.6. Suppose, there are two fixed point $x^*, y^* \in X$ of T and $x^* \neq y^*$. Then by assumption, there exists $z \in X$ such that $x^* \sqsubseteq z$ and $y^* \sqsubseteq z$. Since T is monotonic we obtain $x^* = T^n x^* \sqsubseteq T^n z$ and $y^* = T^n y^* \sqsubseteq T^n z$ for all $n \in \mathbb{N}$. Since T is an ordered weak φ-contraction we obtain: for every $n \in \mathbb{N}$

$$\begin{aligned} d(x^*, T^n z) &= d(T^n x^*, T^n z) = d(TT^{n-1}x^*, TT^{n-1}z) \\ &\preceq \varphi\left(d(T^{n-1}x^*, T^{n-1}z)\right) \\ &= \varphi\left(d(x^*, T^{n-1}z)\right). \end{aligned}$$

Using the properties of φ and the above inequality successively we obtain

$$d(x^*, T^n z) \preceq \varphi^n(x^*, z).$$

Since $\{\varphi^n(x^*, z)\}$ is a c-sequence, it follows from the above inequality and Lemma 2.8 that for every given $c \in \mathcal{A}$ with $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that $d(x^*, T^n z) \ll c$ for all $n > n_0$. Therefore, $T^n z \to x^*$ as $n \to \infty$. Similarly, we can show that $T^n z \to y^*$ as $n \to \infty$. By uniqueness of limit, we must have $x^* = y^*$. This contradiction shows that the fixed point must be unique. □

The following corollary is an ordered version of the main result of Liu and Xu [16].

Corollary 3.9. *Let (X,d) be a cone metric space over Banach algebra $\mathcal{A}$, $\sqsubseteq$ be a partial order on X and $T : X \to X$ be a mapping. Suppose that the following conditions are satisfied:*

(i) *there exists $k \in P$ such that*

$$d(Tx,Ty) \preceq kd(x,y)$$

for all $x, y \in X$ with $x \sqsubseteq y$, where $\rho(k) < 1$;

(ii) *X is T-orbitally ordered complete;*

(iii) *T is nondecreasing with respect to $\sqsubseteq$;*

(iv) *there exists $x_0 \in X$ such that $x_0 \sqsubseteq Tx_0$.*

Then T has a fixed point in X. Moreover, the iterative sequence $\{T^n x_0\}$ converges to a fixed point of T. In addition, suppose, for each pair $x, y \in X$ there exists $z \in X$ such that $x \sqsubseteq z$ and $y \sqsubseteq z$, then T has a unique fixed point in X.

Proof. Let $\varphi(t) = kt$ for all $t \in P$, then φ is a weak comparison. Now the result follows from Theorem 3.8. □

Remark 3.10. *The above corollary is an improved ordered vector-valued version of Theorem 2.1 of [16], as we have removed the assumption of normality of cones of Theorem 2.1 of [16], as well as, we use the T-orbital ordered completeness, instead, the completeness of space X.*

In the next theorem, we use the continuity of mapping T and the completeness of the space, instead, T-orbital ordered completeness of space.

Theorem 3.11. *Let (X,d) be a complete cone metric space over Banach algebra $\mathcal{A}$ and $\sqsubseteq$ be a partial order on X. Let $T : X \to X$ be an ordered weak φ-contraction and the following conditions are satisfied:*

(i) *T is continuous;*

(ii) *T is nondecreasing with respect to $\sqsubseteq$;*

(iii) *there exists $x_0 \in X$ such that $x_0 \sqsubseteq Tx_0$.*

Then T has a fixed point in X. Moreover, the iterative sequence $\{T^n x_0\}$ converges to a fixed point of T.

Proof. Following the lines of proof of Theorem 3.6 we obtain the sequence $\{x_n\} = \{T^n x_0\}$ is a nondecreasing Cauchy sequence. By completeness of X, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Now, by continuity of T we have

$$\begin{aligned} & \lim_{n\to\infty} x_{n+1} = x^* \\ \Longrightarrow \quad & \lim_{n\to\infty} Tx_n = x^* \\ \Longrightarrow \quad & Tx^* = x^*. \end{aligned}$$

Thus, x^* is a fixed point of T. □

Following the arguments as used in the proof of Theorem 3.8, one can prove the following result.

Theorem 3.12. *Suppose that all the conditions of Theorem 3.11 are satisfied. In addition, suppose, for each pair $x, y \in X$ there exists $z \in X$ such that $x \sqsubseteq z$ and $y \sqsubseteq z$, then T has a unique fixed point in X.*

References

[1] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fund. Math., **3** (1922), 133–181.

[2] V. Berinde, Generalized contractions and applications (Romannian), Editura Cub Press 22, Baia Mare (1997).

[3] H. Çakallı, A. Sönmez, Ç. Genç, On an equivalence of topological vector space valued cone metric spaces and metric spaces, Appl. Math. Lett., 25, (2012) 429-433.

[4] M. Cvetković, V. Rakočević, Extensions of Perov theorem, Carpathian J. Math., **31** (2015), 181–188.

[5] M. Dordević, D. Dorić, Z. Kadelburg, S. Radenović, D. Spasić, Fixed point results under c-distance in tvs-cone metric spaces, Fixed Point Theory Appl., **2011** (2011), 9 pages.

[6] W.S. Du, A note on cone metric fixed point theory and its equivalence, Nonlinear Anal., 72(5), (2010) 2259-2261.

[7] Y. Feng, W. Mao, The equivalence of cone metric spaces and metric spaces, Fixed Point Theory, 11(2), (2010) 259-264.

[8] L.-G. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl., **332** (2) (2007), 1468–1476.

[9] H. Huang, S. Radenović, Common fixed point theorems of generalized Lipschitz mappings in cone b-metric spaces over Banach algebras and applications, J. Nonlinear Sci. Appl., **8** (5) (2015), 787–799.

[10] H. Huang, S. Radenović, Some fixed point results of generalized Lipschitz mappings on cone b-metric spaces over Banach algebras, J. Comput. Anal. Appl., **20** (3) (2016), 566–583.

[11] H. Huang, S. Radenović, G. Deng, A sharp generalization on cone b-metric space over Banach algebra, J. Nonlinear Sci. Appl., **10** (2) (2017), 429–435.

[12] H. Huang, S. Xu, H. Liu, S. Radenović, Fixed point theorems and T-stability of Picard iteration for generalized Lipschitz mappings in cone metric spaces over Banach algebras, J. Comput. Anal. Appl., **20** (5) (2016), 869–888.

[13] S. Janković, Z. Kadelburg, S. Radenović, On cone metric spaces: A survey, Nonlinear Anal., **74** (7) (2011), 2591–2601.

[14] Z. Kadelburg, S. Radenović, V. Rakočević, A note on the equivalence of some metric and cone metric fixed point results, Appl. Math. Lett., 24, (2011) 370-374.

[15] B. Li, H. Huang, Fixed point results for weak φ-contractions in cone metric spaces over Banach algebras and applications, Journal of Function Spaces, Volume 2017 (2017), Article ID 5054603, 6 pages.

[16] H. Liu, S. Xu, Cone metric spaces over Banach algebras and fixed point theorems of generalized Lipschitz mappings, Fixed Point Theory Appl., **2013** (2013), 10 pages.

[17] S.K. Malhotra, J.B. Sharma, S. Shukla, Fixed points of α-admissible mappings in cone metric spaces with Banach algebra, Inter. J. Anal. Appl., **9** (1) (2015), 9–18.

[18] S.K. Malhotra, J.B. Sharma, S. Shukla, Some fixed point theorems for G-contractions in cone b-metric spaces over Banach algebra, J. Nonlinear Funct. Anal. 2017 (2017), Article ID 9, pages 17.

[19] J.J. Nieto and R.R. Lopez, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equation, Order (2005), 223-239.

[20] A.C.M. Ran and M.C.B. Reurings, A fixed point theorem in partially ordered sets and some application to matrix equations, Proc. Amer. Math. Sco., **132** (2004), 1435-1443.

[21] S. Reich and A.J. Zaslavski, The set of noncontractive mappings is σ-porous in the space of all nonexpansive mappings, C. R. Acad. Sci. Paris 333 (2001), 539-541.

[22] W. Rudin, Functional Analysis, 2nd edn. McGraw-Hill, New York (1991).

[23] I. A. Rus, Generalized contractions, Seminar on Fixed Point Theory, **3** (1983), 1–130.

[24] I. A. Rus, Generalized contractions and applications, Cluj University Press, Cluj-Napoca, (2011), 198 pp, ISBN 973-8095-71-9.

[25] S. Shukla, S. Balasubramanian, M. Pavlović, A generalized Banach fixed point theorem, Bull. Malays. Math. Sci. Soc., **39** (4) (2016), 1529–1539.