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Exponential Growth of Solutions for Nonlinear Coupled Viscoelastic Wave Equations

Erhan Pişkin[1*] and Şeyhmus Altındağ[1]

[1] *Department of Mathematics, Dicle University, 21280 Diyarbakır, Turkey*
[*] *Corresponding author*

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Abstract

In this work, we consider an initial-boundary value problem related to the nonlinear coupled viscoelastic equations

$$\begin{cases} |u_t|^j u_{tt} - \Delta u_{tt} - div\left(|\nabla u|^{\alpha-2}\nabla u\right) - \Delta u + \int\limits_0^t g(t-s)\Delta u ds + |u_t|^{m-1}u_t = f_1(u,v), \\ |v_t|^j v_{tt} - \Delta v_{tt} - div\left(|\nabla v|^{\beta-2}\nabla v\right) - \Delta v + \int\limits_0^t h(t-s)\Delta v ds + |v_t|^{r-1}v_t = f_2(u,v). \end{cases}$$

We will show the exponential growth of solutions with positive initial energy.

1. Introduction

In this work we consider the following coupled system of viscoelastic wave equations:

$$\begin{cases} |u_t|^j u_{tt} - \Delta u_{tt} - div\left(|\nabla u|^{\alpha-2}\nabla u\right) - \Delta u + \int\limits_0^t g(t-s)\Delta u ds + |u_t|^{m-1}u_t = f_1(u,v), & (x,t)\in\Omega\times(0,T), \\ |v_t|^j v_{tt} - \Delta v_{tt} - div\left(|\nabla v|^{\beta-2}\nabla v\right) - \Delta v + \int\limits_0^t h(t-s)\Delta v ds + |v_t|^{r-1}v_t = f_2(u,v), & (x,t)\in\Omega\times(0,T), \\ u(x,t) = v(x,t) = 0, & (x,t)\in\partial\Omega\times(0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x\in\Omega, \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^n $(n=1,2,3)$ with smooth boundary $\partial\Omega$, the constants $j>0$, $\alpha\geq 2$, $\beta\geq 2$, $m\geq 1$, $r\geq 1$. Here, $f_1(u,v)$ and $f_2(u,v)$ are nonlinear functions defined as

$$\begin{cases} f_1(u,v) = a|u+v|^{2(p+1)}(u+v) + b|u|^p u|v|^{p+2}, \\ f_2(u,v) = a|u+v|^{2(p+1)}(u+v) + b|v|^p v|u|^{p+2} \end{cases} \tag{1.2}$$

in which the constants $a>0$, $b>0$, and p satisfies

$$\begin{cases} p>-1, & n=1,2, \\ -1<p\leq 1, & n=3. \end{cases} \tag{1.3}$$

Let

$$f_1(u,v) = \frac{\partial F(u,v)}{\partial u} \text{ and } f_2(u,v) = \frac{\partial F(u,v)}{\partial v},$$

Email addresses and ORCID numbers: **episkin@dicle.edu.tr, https://orcid.org/0000-0001-6587-4479 (E. Pişkin), altindag.s@hotmail.com, https://orcid.org/0000-0002-1863-2909 (Ş. Altındağ)**

where

$$F(u,v)=\frac{1}{2(p+2)}\left[a\left|u+v\right|^{2(p+2)}(u+v)+2b\left|uv\right|^{p+2}\right].$$

There are two positive constants $c_0,\ c_1$ such that

$$c_0\left(|u|^{2(r+2)}+|v|^{2(r+2)}\right)\le 2(r+2)F(u,v)\le c_1\left(|u|^{2(r+2)}+|v|^{2(r+2)}\right).$$

As a special case, for $\alpha=\beta=2$, the system (1.1) becomes the following system

$$\begin{cases} |u_t|^j u_{tt}-\Delta u_{tt}-\Delta u+\int\limits_0^t g(t-s)\Delta u ds+|u_t|^{m-1}u_t=f_1(u,v),\\ |v_t|^j v_{tt}-\Delta v_{tt}-\Delta v+\int\limits_0^t h(t-s)\Delta v ds+|v_t|^{r-1}v_t=f_2(u,v).\end{cases} \tag{1.4}$$

Liu [1] proved decay of the solutions for system (1.4) under some appropriate functions f_1 and f_2. Later, Said-Houari [2] studied exponential growth of the solutions for system (1.4). When $j=0$ and without the $\Delta u_{tt},\ \Delta v_{tt}$ terms, the system (1.4) has been investigated by some authors and results concerning local and global existence, blow up, decay of the solutions were obtained [3, 4, 5, 6, 7, 8]. Hao et al. [9] considered global nonexistence of the solution of (1.1), with negative initial energy.
Motivated by the above papers, in this work we prove the exponential growth of solutions for the problem (1.1), with positive initial energy. This work is organized as follows: In section 2, we present some lemmas and notations needed later of this paper. In section 3, exponential growth of the solution is proved.

2. Preliminaries

In this part, we give some assumptions and lemmas which will be used throughout this paper. Let $\|.\|$ and $\|.\|_p$ denote the usual $L^2(\Omega)$ norm and $L^p(\Omega)$ norm, respectively.
Now, we make the following assumptions on the C^1-nonnegative and nonincreasing relaxation functions g and h :

$$1-\int\limits_0^\infty g(s)ds=l>0,\quad 1-\int\limits_0^\infty h(s)ds=k>0 \tag{2.1}$$

and $\forall s\ge 0$

$$g'(s)\le 0,\quad h'(s)\le 0. \tag{2.2}$$

Let us define

$$\begin{aligned} I(t)=I(u,v)&=\left(1-\int\limits_0^t g(s)ds\right)\|\nabla u\|^2+\left(1-\int\limits_0^t h(s)ds\right)\|\nabla v\|^2\\ &\quad -2(p+2)\int\limits_\Omega F(u,v)dx+(g\circ\nabla u+h\circ\nabla v)+\frac{1}{\alpha}\|\nabla u\|_\alpha^\alpha+\frac{1}{\beta}\|\nabla v\|_\beta^\beta, \end{aligned} \tag{2.3}$$

$$\begin{aligned} J(t)=J(u,v)&=\frac{1}{2}\left(1-\int\limits_0^t g(s)ds\right)\|\nabla u\|^2+\frac{1}{2}\left(1-\int\limits_0^t h(s)ds\right)\|\nabla v\|^2\\ &\quad -\int\limits_\Omega F(u,v)dx+\frac{1}{2}(g\circ\nabla u+h\circ\nabla v)+\frac{1}{\alpha}\|\nabla u\|_\alpha^\alpha+\frac{1}{\beta}\|\nabla v\|_\beta^\beta \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} E(t)&=\frac{1}{j+2}\left(\|u_t\|_{j+2}^{j+2}+\|v_t\|_{j+2}^{j+2}\right)+\frac{1}{2}\left(\|\nabla u_t\|^2+\|\nabla v_t\|^2\right)\\ &\quad+\frac{1}{2}\left(1-\int\limits_0^t g(s)ds\right)\|\nabla u\|^2+\frac{1}{2}\left(1-\int\limits_0^t h(s)ds\right)\|\nabla v\|^2\\ &\quad-\int\limits_\Omega F(u,v)dx+\frac{1}{2}(g\circ\nabla u+h\circ\nabla v)+\frac{1}{\alpha}\|\nabla u\|_\alpha^\alpha+\frac{1}{\beta}\|\nabla v\|_\beta^\beta \end{aligned} \tag{2.5}$$

where

$$(\phi\circ\psi)(t)=\int\limits_0^t\phi(t-\tau)\int\limits_\Omega|\psi(t)-\psi(\tau)|^2 dxd\tau.$$

Lemma 2.1. *$E(t)$ is a nonincreasing function for $t \geq 0$ and*

$$\begin{aligned} E'(t) &= -\left(\|u_t\|_{m+1}^{m+1} + \|v_t\|_{m+1}^{m+1}\right) + \frac{1}{2}\left(g' \circ \nabla u + h' \circ \nabla v\right) \\ &\quad - \frac{1}{2}\left(g(t)\|\nabla u\|^2 + h(t)\|\nabla v\|^2\right) \\ &\leq 0. \end{aligned} \tag{2.6}$$

Proof. Multiplying the first and second equation of (1.1) by u_t and v_t, respectively, integrating over $\Omega \times [0,t]$, then adding them together and integrating by parts, we obtain (2.6). □

3. Exponential growth of solutions

In this part, we are going to consider the exponential growth of the solution for the problem (1.1).
Firstly, we give following two lemmas.

Lemma 3.1. *[10, 11]. Suppose that (1.3) holds. Let (u,v) for $\eta > 0$*

$$\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2} \leq \eta \left[\frac{1}{\alpha}\|\nabla u\|_\alpha^\alpha + \frac{1}{\beta}\|\nabla v\|_\beta^\beta + I_1\|\nabla u\|^2 + I_2\|\nabla v\|^2\right]^{p+2},$$

where

$$I_1 = \int_{\Omega_1} |u_t|\left(|u|^{2p+3} + |v|^{2p+3} + |u|^{p+1}|v|^{p+2}\right)dx,$$

$$I_2 = \int_{\Omega_2} |v_t|\left(|u|^{2p+3} + |v|^{2p+3} + |u|^{p+2}|v|^{p+1}\right)dx$$

and

$$\begin{aligned} \Omega_1 &= \{(x,t) : |u(x,t)| \leq 1,\ |v(x,t)| \leq 1\}, \\ \Omega_2 &= \{(x,t) : |u(x,t)| \leq 1,\ |v(x,t)| \geq 1\}. \end{aligned}$$

Lemma 3.2. *[10, 11]. Suppose that (1.3) holds. Let (u,v) be the solution of problem (1.1). Assume further that $E(0) < E_1$ and*

$$\left[\frac{1}{\alpha}\|\nabla u_0\|_\alpha^\alpha + \frac{1}{\beta}\|\nabla v_0\|_\beta^\beta + I(0)\right]^{\frac{1}{2}} > \alpha_1.$$

Then, there exists a constant $\alpha_2 > \alpha_1$ such that

$$\left[\frac{1}{\alpha}\|\nabla u\|_\alpha^\alpha + \frac{1}{\beta}\|\nabla v\|_\beta^\beta + I(t)\right]^{\frac{1}{2}} > \alpha_2,$$

$$\left(\|u+v\|_{2(p+2)}^{2(p+2)} + \|uv\|_{p+2}^{p+2}\right)^{\frac{1}{2(p+2)}} > B\alpha_2,$$

for all $t \in (0,T)$, where

$$B = \eta^{\frac{1}{2(p+2)}},\ \alpha_1 = B^{-\frac{p+2}{p+1}},\ E_1 = \left(\frac{1}{2} - \frac{1}{2(p+2)}\right)\alpha_1^2.$$

Theorem 3.3. *Suppose that (1.3) holds. Assume further that*

$$\max\{j+2, m+1, r+1\} < 2(p+2),$$

$$E(0) < E_1$$

and (2.1), (2.2) hold. There exist constant γ such that

$$\max\{\alpha,\beta\} < \gamma < 2(p+2)$$

and

$$\min\{l,k\} > \frac{1/(2\gamma)}{(\gamma/2) - 1 + 1/(2\gamma)}.$$

Then, any solution of (1.1) grows exponentially.

Proof. We define the functional

$$H(t)=E_1-E(t). \tag{3.1}$$

From (2.1), (2.5) and Lemma 3.2, we have

$$\begin{aligned}
0<H(0)&\le H(t)\\
&\le E_1-E(t)\\
&=E_1-\frac{1}{j+2}\left(\|u_t\|_{j+2}^{j+2}+\|v_t\|_{j+2}^{j+2}\right)-\frac{1}{2}\left(\|\nabla u_t\|^2+\|\nabla v_t\|^2\right)\\
&-\frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|^2-\frac{1}{2}\left(1-\int_0^t h(s)\,ds\right)\|\nabla v\|^2\\
&+\int_\Omega F(u,v)\,dx-\frac{1}{2}(g\circ\nabla u+h\circ\nabla v)-\frac{1}{\alpha}\|\nabla u\|_\alpha^\alpha-\frac{1}{\beta}\|\nabla v\|_\beta^\beta\\
&<E_1-\frac{1}{2}\alpha_2^2+\frac{1}{2(p+2)}\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\right)\\
&<\frac{C_1}{2(p+2)}\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\right).
\end{aligned} \tag{3.2}$$

Let us define the functional

$$L(t)=H(t)+\frac{\varepsilon}{j+1}\int_\Omega\left(|u_t|^j u_t u+|v_t|^j v_t v\right)dx-\varepsilon\int_\Omega(\Delta u u_t+\Delta v v_t)\,dx, \tag{3.3}$$

where ε is a small positive constants to be determined later.
By differentiating with respect to t and using (3.3) and (1.1), we have

$$\begin{aligned}
L'(t)&=H'(t)+\varepsilon\int_\Omega\left[\left(|u_t|^j u_{tt}u+|v_t|^j v_{tt}v\right)+\frac{1}{j+1}\left(|u_t|^{j+2}+|v_t|^{j+2}\right)\right]dx\\
&+\varepsilon\left(\|\nabla u_t\|^2+\|\nabla v_t\|^2\right)-\varepsilon\int_\Omega(u\Delta u_{tt}+v\Delta v_{tt})\,dx\\
&=H'(t)+\frac{\varepsilon}{j+1}\left(\|u_t\|_{j+2}^{j+2}+\|v_t\|_{j+2}^{j+2}\right)-\varepsilon\int_\Omega\left(u|u_t|^{m-1}u_t+v|v_t|^{r-1}v_t\right)dx\\
&+\varepsilon\left(\|\nabla u_t\|^2+\|\nabla v_t\|^2\right)-\varepsilon\left(\|\nabla u\|^2+\|\nabla v\|^2\right)-\varepsilon\left(\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\beta^\beta\right)\\
&+2\varepsilon(p+2)\int_\Omega F(u,v)dx+\varepsilon\left(\int_0^t g(s)\,ds\right)\|\nabla u\|^2+\varepsilon\left(\int_0^t h(s)\,ds\right)\|\nabla v\|^2\\
&+\varepsilon\int_0^t g(t-s)\int_\Omega\nabla u[\nabla u(s)-\nabla u(t)]\,dxds\\
&+\varepsilon\int_0^t h(t-s)\int_\Omega\nabla v[\nabla v(s)-\nabla v(t)]\,dxds.
\end{aligned} \tag{3.4}$$

Using Cauchy-Schwarz and Young's inequalities, we get

$$\begin{aligned}
\int_0^t g(t-s)\int_\Omega\nabla u[\nabla u(s)-\nabla u(t)]\,dxds&\le\int_0^t g(t-s)\left(\int_\Omega|\nabla u(t)|^2dx\right)^{\frac{1}{2}}\left(\int_\Omega|\nabla u(s)-\nabla u(t)|^2dx\right)^{\frac{1}{2}}ds\\
&\le\int_0^t g(t-s)\|\nabla u(t)\|\,\|\nabla u(s)-\nabla u(t)\|\,ds\\
&\le\int_0^t g(t-s)\left(\lambda\|\nabla u(s)-\nabla u(t)\|^2+\frac{1}{4\lambda}\|\nabla u(t)\|^2\right)ds\\
&\le\lambda\int_0^t g(t-s)\|\nabla u(s)-\nabla u(t)\|^2ds+\frac{1}{4\lambda}\int_0^t g(t-s)\|\nabla u(t)\|^2ds\\
&\le\lambda(g\circ\nabla u)+\frac{1}{4\lambda}\left(\int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2.
\end{aligned} \tag{3.5}$$

Similarly, we obtain

$$\int_0^t h(t-s)\int_\Omega \nabla v[\nabla v(s)-\nabla v(t)]\,dxds \le \lambda(h\circ\nabla v)+\frac{1}{4\lambda}\left(\int_0^t h(s)\,ds\right)\|\nabla v(t)\|^2. \tag{3.6}$$

Inserting (3.5) and (3.6) into (3.4), we have

$$\begin{aligned} L'(t) &\ge H'(t)+\frac{\varepsilon}{j+1}\left(\|u_t\|_{j+2}^{j+2}+\|v_t\|_{j+2}^{j+2}\right)-\varepsilon\int_\Omega\left(u|u_t|^{m-1}u_t+v|v_t|^{r-1}v_t\right)dx \\ &+\varepsilon\left(\|\nabla u_t\|^2+\|\nabla v_t\|^2\right)-\varepsilon\left(\|\nabla u\|^2+\|\nabla v\|^2\right)-\varepsilon\left(\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\beta^\beta\right) \\ &+2\varepsilon(p+2)\int_\Omega F(u,v)dx+\varepsilon\left(\int_\Omega g(s)\,ds\right)\|\nabla u\|^2+\varepsilon\left(\int_\Omega h(s)\,ds\right)\|\nabla v\|^2 \\ &+\varepsilon\lambda(g\circ\nabla u+h\circ\nabla v)+\frac{\varepsilon}{4\lambda}\left[\left(\int_0^t g(s)\,ds\right)\|\nabla u\|^2+\left(\int_0^t h(s)\,ds\right)\|\nabla v\|^2\right]. \end{aligned} \tag{3.7}$$

By the definition of $E(t)$ and (3.1), we obtain

$$\begin{aligned} \int_\Omega F(u,v)\,dx &= H(t)-E_1+\frac{1}{j+2}\left(\|u_t\|_{j+2}^{j+2}+\|v_t\|_{j+2}^{j+2}\right)+\frac{1}{2}\left(\|\nabla u_t\|^2+\|\nabla v_t\|^2\right) \\ &+\frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|^2+\frac{1}{2}\left(1-\int_0^t h(s)\,ds\right)\|\nabla v\|^2 \\ &+\frac{1}{2}(g\circ\nabla u+h\circ\nabla v)+\frac{1}{\alpha}\|\nabla u\|_\alpha^\alpha+\frac{1}{\beta}\|\nabla v\|_\beta^\beta. \end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.7), we get

$$\begin{aligned} L'(t) &\ge H'(t)+\varepsilon\left(\frac{1}{j+1}+\frac{\gamma}{j+2}\right)\left(\|u_t\|_{j+2}^{j+2}+\|v_t\|_{j+2}^{j+2}\right)-\varepsilon\int_\Omega\left(u|u_t|^{m-1}u_t+v|v_t|^{r-1}v_t\right)dx \\ &+\varepsilon\left(1+\frac{\gamma}{2}\right)\left(\|\nabla u_t\|^2+\|\nabla v_t\|^2\right)+\gamma\varepsilon H(t)-\varepsilon\gamma E_1+\varepsilon(2(p+2)-\gamma)\int_\Omega F(u,v)\,dx \\ &+\varepsilon\left[\left(\frac{\gamma}{2}-1\right)-\left(\frac{\gamma}{2}-1+\frac{1}{4\lambda}\right)\int_0^\infty g(s)\,ds\right]\|\nabla u\|^2 \\ &+\varepsilon\left[\left(\frac{\gamma}{2}-1\right)-\left(\frac{\gamma}{2}-1+\frac{1}{4\lambda}\right)\int_0^\infty h(s)\,ds\right]\|\nabla v\|^2 \\ &+\varepsilon\left(\frac{\gamma}{2}-\lambda\right)(g\circ\nabla u+h\circ\nabla v)+\varepsilon\left(\frac{\gamma}{\alpha}-1\right)\|\nabla u\|_\alpha^\alpha+\varepsilon\left(\frac{\gamma}{\beta}-1\right)\|\nabla v\|_\beta^\beta. \end{aligned} \tag{3.9}$$

By using the Young's inequality, we get

$$\begin{aligned} \int_\Omega |u_t|^{m-1}u_t u\,dx &\le \frac{\delta_1^{m+1}}{m+1}\|u\|_{m+1}^{m+1}+\frac{m\delta_1^{-\frac{m+1}{m}}}{m+1}\|u_t\|_{m+1}^{m+1} \\ &\le \frac{\delta_1}{m+1}\|u\|_{m+1}^{m+1}+\frac{m\delta_1^{-\frac{m+1}{m}}}{m+1}H'(t) \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \int_\Omega |v_t|^{r-1}v_t v\,dx &\le \frac{\delta_2^{r+1}}{r+1}\|v\|_{r+1}^{r+1}+\frac{r\delta_2^{-\frac{r+1}{r}}}{r+1}\|v_t\|_{r+1}^{r+1} \\ &\le \frac{\delta_2^{r+1}}{r+1}\|v\|_{r+1}^{r+1}+\frac{r\delta_2^{-\frac{r+1}{r}}}{r+1}H'(t). \end{aligned} \tag{3.11}$$

Since $L^{2(p+2)}(\Omega)\hookrightarrow L^{m+1}(\Omega)$ and $L^{2(p+2)}(\Omega)\hookrightarrow L^{r+1}(\Omega)$, we have

$$\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\right)^m\|u\|_{m+1}^{m+1}\le C_2\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\right)^{m+\frac{m+1}{2(p+2)}} \tag{3.12}$$

and

$$\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\right)^r\|v\|_{r+1}^{r+1}\le C_3\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\right)^{r+\frac{r+1}{2(p+2)}}. \tag{3.13}$$

We use the following algebraic inequality

$$z^{v} \leq z+1 \leq \left(1+\frac{1}{a}\right)(z+a), \quad \forall z \geq 0,\ 0<v \leq 1, \tag{3.14}$$

we obtain, for $t \geq 0$,

$$\begin{aligned}\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\right)^{m+\frac{m+1}{2(p+2)}} &\leq d\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}+H(0)\right)\\ &\leq d\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}+H(t)\right)\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\right)^{r+\frac{r+1}{2(p+2)}} &\leq d\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}+H(0)\right)\\ &\leq d\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}+H(t)\right)\end{aligned} \tag{3.16}$$

for $d=1+\frac{1}{H(0)}$.
By (3.9)-(3.13),(3.15) and (3.16), we have

$$\begin{aligned}L'(t) \geq{}& \left(1+\frac{m\delta_1^{-\frac{m+1}{m}}}{m+1}+\frac{r\delta_2^{-\frac{r+1}{r}}}{r+1}\right)H'(t)+\varepsilon\left(\frac{1}{j+1}+\frac{\gamma}{j+2}\right)\left(\|u_t\|_{j+2}^{j+2}+\|v_t\|_{j+2}^{j+2}\right)\\ &-\left(\frac{\delta_1^{m+1}c_2 d}{m+1}+\frac{\delta_2^{r+1}c_3 d}{r+1}\right)\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\right)\\ &+\varepsilon\left(\gamma-\left(\frac{\delta_1^{m+1}c_2 d}{m+1}+\frac{\delta_2^{r+1}c_3 d}{r+1}\right)\right)H(t)\\ &+\varepsilon\left(1+\frac{\gamma}{2}\right)\left(\|\nabla u\|^2+\|\nabla v\|^2\right)+\varepsilon\left(2(p+2)-\gamma(p+2)\right)\int_\Omega F(u,v)\,dx\\ &+\varepsilon\left[\left(\frac{\gamma}{2}-1\right)-\left(\frac{\gamma}{2}-1+\frac{1}{4\lambda}\right)\int_0^\infty g(s)\,ds\right]\|\nabla u\|^2\\ &+\varepsilon\left(\frac{\gamma}{2}-\lambda\right)(g\circ\nabla u+h\circ\nabla v)\\ &+\varepsilon\left[\left(\frac{\gamma}{2}-1\right)-\left(\frac{\gamma}{2}-1+\frac{1}{4\lambda}\right)\int_0^\infty h(s)\,ds\right]\|\nabla v\|^2\\ &+\varepsilon\left(\frac{\gamma}{\alpha}-1\right)\|\nabla u\|_\alpha^\alpha+\varepsilon\left(\frac{\gamma}{\beta}-1\right)\|\nabla v\|_\beta^\beta.\end{aligned}$$

By use (3.2) and since

$$\min\left\{\frac{\gamma}{\alpha}-1,\ \frac{\gamma}{\beta}-1\right\}>0$$

and

$$1+\frac{\gamma}{2}>0$$

we obtain

$$\begin{aligned}L'(t) \geq{}& MH'(t)+\varepsilon\left(\frac{1}{j+1}+\frac{\gamma}{j+2}\right)\left(\|u_t\|_{j+2}^{j+2}+\|v_t\|_{j+2}^{j+2}\right)+\varepsilon(\gamma-K_1)H(t)\\ &+\varepsilon K_2\left(\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\beta^\beta\right)+\varepsilon K_3\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\\ &+\varepsilon\left(\frac{\gamma}{2}-\lambda\right)(g\circ\nabla u+h\circ\nabla v)+\varepsilon\left(1+\frac{\gamma}{2}\right)\left(\|\nabla u_t\|^2+\|\nabla v_t\|^2\right)\\ &+\varepsilon\left(\frac{(2(p+2)-\gamma(p+2))C_1}{2(p+2)}-K_1\right)\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\right),\end{aligned}$$

where

$$\begin{aligned}M &= 1+\frac{m\delta_1^{-\frac{m+1}{m}}}{m+1}+\frac{r\delta_2^{-\frac{r+1}{r}}}{r+1},\\ K_1 &= \frac{\delta_1^{m+1}c_2 d}{m+1}+\frac{\delta_2^{r+1}c_3 d}{r+1},\\ K_2 &= \min\left\{\frac{\gamma}{\alpha}-1,\frac{\gamma}{\beta}-1\right\}\end{aligned}$$

and

$$K_3=\left(\frac{\gamma}{2}-1\right)-\left(\frac{\gamma}{2}-1+\frac{1}{4\lambda}\right)\max\left(\int_0^\infty g(s)\,ds,\int_0^\infty h(s)\,ds\right).$$

Choose δ_1,δ_2 appropriate such that

$$b_1=\gamma-K_1>0,\;\; b_2=\frac{(2(p+2)-\gamma(p+2))C_1}{2(p+2)}-K_1>0 \text{ and } M>0.$$

Then, we can find positive constants b_1 and b_2 such that

$$\begin{aligned}L'(t)\ge{}& MH'(t)+\varepsilon\left(\frac{1}{j+1}+\frac{\gamma}{j+2}\right)\left(\|u_t\|_{j+2}^{j+2}+\|v_t\|_{j+2}^{j+2}\right)\\&+\varepsilon K_2\left(\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\beta^\beta\right)+\varepsilon\left(1+\frac{\gamma}{2}\right)\left(\|\nabla u_t\|^2+\|\nabla v_t\|^2\right)\\&+\varepsilon b_1H(t)+\varepsilon b_2\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\right)\ge 0.\end{aligned}$$

Because of $H'(t)\ge 0$, there exists constants $t>0$ such that

$$\begin{aligned}L'(t)\ge{}&\tilde{K}(H(t)+\|u_t\|_{j+2}^{j+2}+\|v_t\|_{j+2}^{j+2}+\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\beta^\beta\\&+\|\nabla u_t\|^2+\|\nabla v_t\|^2+\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)})\ge 0\end{aligned}\tag{3.17}$$

where $\tilde{K}=\min\left\{\varepsilon b_1,\varepsilon\left(\frac{1}{j+1}+\frac{\gamma}{j+2}\right),\varepsilon K_2,\varepsilon\left(1+\frac{\gamma}{2}\right),\varepsilon b_2\right\}$.
On the other hand, we can choose ε smaller so that

$$L(0)=H(0)+\varepsilon\int_\Omega(u_0u_1+v_0v_1)\,dx>0.\tag{3.18}$$

Furthermore, we have

$$L(t)\ge L(0),\;\; t\ge 0.\tag{3.19}$$

Next we estimate $L(t)$. Using Young's inequality, we obtain

$$\left|\int_\Omega|u_t|^{j+1}u\,dx\right|\le\frac{\mu_1^{j+2}}{j+2}\|u\|_{j+2}^{j+2}+\frac{(j+1)\,\mu_1^{-\frac{j+2}{j+1}}}{j+2}\|u_t\|_{j+2}^{j+2},\;\forall\mu_1>0.\tag{3.20}$$

Next, using the embedding $L^{2(p+2)}(\Omega)\hookrightarrow L^{j+2}(\Omega)$, the estimate (3.20) becomes

$$\begin{aligned}\left|\int_\Omega|u_t|^{j+1}u\,dx\right|&\le C\left(\|u\|_{2(p+2)}^{j+2}+\|u_t\|_{j+2}^{j+2}\right)\\&\le C\left(\left(\|u\|_{2(p+2)}^{2(p+2)}\right)^{\frac{j+2}{2(p+2)}}+\|u_t\|_{j+2}^{j+2}\right).\end{aligned}$$

Since $2(p+2)>j+2$ and $H(t)>H(0)$, use the inequality (3.14), we have

$$\begin{aligned}\left|\int_\Omega|u_t|^{j+1}u\,dx\right|&\le C\left[\left(1+\frac{1}{H(0)}\right)\left(\|u\|_{2(p+2)}^{2(p+2)}+H(0)\right)+\|u_t\|_{j+2}^{j+2}\right]\\&\le C\left[\left(1+\frac{1}{H(0)}\right)\left(\|u\|_{2(p+2)}^{2(p+2)}+H(t)\right)+\|u_t\|_{j+2}^{j+2}\right].\end{aligned}\tag{3.21}$$

Similarly, we have

$$\left|\int_\Omega|v_t|^{j+1}v\,dx\right|\le C\left[\left(1+\frac{1}{H(0)}\right)\left(\|v\|_{2(p+2)}^{2(p+2)}+H(t)\right)+\|v_t\|_{j+2}^{j+2}\right].\tag{3.22}$$

By Green identity and Hölder's inequality, we get

$$\begin{aligned}-\int_\Omega u_t\Delta u\,dx&=\int_\Omega\nabla u\nabla u_t\,dx\\&\le\left(\int_\Omega(\nabla u)^2dx\right)^{\frac12}\left(\int_\Omega(\nabla u_t)^2dx\right)^{\frac12}\\&=\|\nabla u\|\,\|\nabla u_t\|,\end{aligned}\tag{3.23}$$

similarly

$$-\int_{\Omega} v_t \Delta v dx \leq \|\nabla v\| \|\nabla v_t\|. \tag{3.24}$$

Next, using the embedding $L^{\alpha}(\Omega) \hookrightarrow L^2(\Omega)$ and $L^{\beta}(\Omega) \hookrightarrow L^2(\Omega)$ the estimate (3.23) and (3.24) becomes

$$\begin{cases} \|\nabla u\| \|\nabla u_t\| \leq C \|\nabla u\|_{\alpha} \|\nabla u_t\|, \\ \|\nabla v\| \|\nabla v_t\| \leq C \|\nabla v\|_{\beta} \|\nabla v_t\|. \end{cases} \tag{3.25}$$

By Young's inequality (3.25), we get

$$\begin{aligned} \|\nabla u\|_{\alpha} \|\nabla u_t\| &\leq \frac{1}{2}\left(\|\nabla u\|_{\alpha}^2 + \|\nabla u_t\|^2\right), \\ \|\nabla v\|_{\beta} \|\nabla v_t\| &\leq \frac{1}{2}\left(\|\nabla v\|_{\alpha}^2 + \|\nabla v_t\|^2\right). \end{aligned} \tag{3.26}$$

Since $\alpha \geq 2$, $\beta \geq 2$ and $H(t) > H(0)$, the inequality (3.14) yields

$$\begin{aligned} \|\nabla u\|_{\alpha}^2 &= \left(\|\nabla u\|_{\alpha}^{\alpha}\right)^{\frac{2}{\alpha}} \\ &\leq \left(1 + \frac{1}{H(0)}\right)\left(\|\nabla u\|_{\alpha}^{\alpha} + H(0)\right) \\ &\leq \left(1 + \frac{1}{H(0)}\right)\left(\|\nabla u\|_{\alpha}^{\alpha} + H(t)\right) \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} \|\nabla v\|_{\beta}^2 &= \left(\|\nabla v\|_{\beta}^{\beta}\right)^{\frac{2}{\beta}} \\ &\leq \left(1 + \frac{1}{H(0)}\right)\left(\|\nabla v\|_{\beta}^{\beta} + H(0)\right) \\ &\leq \left(1 + \frac{1}{H(0)}\right)\left(\|\nabla v\|_{\beta}^{\beta} + H(t)\right). \end{aligned} \tag{3.28}$$

Combining (3.20)-(3.28), we have

$$\begin{aligned} &\left| \frac{\varepsilon}{j+1} \int_{\Omega} \left(|u_t|^j u_t u + |v_t|^j v_t v\right) dx - \varepsilon \int_{\Omega} (\Delta u u_t + \Delta v v_t) dx \right| \\ &\leq \mu (H(t) + \|u_t\|_{j+2}^{j+2} + \|v_t\|_{j+2}^{j+2} + \|\nabla u\|_{\alpha}^{\alpha} + \|\nabla v\|_{\beta}^{\beta} + \|\nabla u_t\|^2 + \|\nabla v_t\|^2 \\ &+ \|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} L(t) \leq C^* (H(t) &+ \|u_t\|_{j+2}^{j+2} + \|v_t\|_{j+2}^{j+2} + \|\nabla u\|_{\alpha}^{\alpha} + \|\nabla v\|_{\beta}^{\beta} + \|\nabla u_t\|^2 + \|\nabla v_t\|^2 \\ &+ \|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}). \end{aligned} \tag{3.29}$$

A combination of (3.17) and (3.29) yields

$$L(t) \leq C^* L'(t) \text{ for all } t \geq 0, \tag{3.30}$$

where C^* is a some positive constants. Integrating the differential inequality (3.30) between 0 and t gives the following estimate for $L(t)$,

$$L(t) \geq L(0) e^{t/C^*}.$$

This completes the proof. □

4. Conclusion

In this paper, we obtained a exponential growth of solutions for a nonlinear coupled viscoelastic wave equations with nonlinear damping terms. This improves and extends many results in the literature such as (Houari [2], Pişkin [5]).

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