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A Bound for the Joint Spectral Radius of Operators in a Hilbert Space

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Abstract

We suggest a bound for the joint spectral radius of a finite set of operators in a Hilbert space. In appropriate situations that bound enables us to avoid complicated calculations and gives a new explicit stability test for the discrete time switched systems. The illustrative example is given. Our results are new even in the finite dimensional case.

1. Introduction and statement of the main result

Let $\mathscr{H}$ be a complex separable Hilbert space with a scalar product $(.,.)$, the norm $\|.\| = \sqrt{(.,.)}$ and unit operator I. By $\mathscr{B}(\mathscr{H})$ we denote the set of all bounded linear operators in $\mathscr{H}$. For an $A \in \mathscr{B}(\mathscr{H})$, $\sigma(A)$ is the spectrum, $r_s(A)$ is the spectral radius; A^* is adjoint to A, and $\|A\| = \sup_{h\in\mathscr{H}, h\neq 0} \|Ah\|/\|h\|$.

Let $\mathscr{M} = \{A_1, ..., A_\nu\}$ be a finite set of operators $A_j \in \mathscr{B}(\mathscr{H})$ $(j = 1, ..., \nu)$. Our main object is the joint spectral radius $\rho(\mathscr{M})$ of $\mathscr{M}$ defined by

$$\rho(\mathscr{M}) := \lim_{k\to\infty} \sup\{\|A_{i_k}\cdots A_{i_1}\|^{1/k} : A_i \in \mathscr{M}\},$$

cf. [1, 2]. The joint spectral radius arises naturally in a range of topics including the theory of difference equations [3], control and stability of discrete time switched systems [4, 5, 6, 7, 8, 9, 10, 11], wavelets [12], ergodic theory [13], etc.

The literature on the theory of the joint spectral radius is rather rich, cf. [14], [15], [16], [17, 18] and references therein. Mainly, the finite dimensional operators were considered and the numerical methods were developed.

In the present paper, under some restrictions, we suggest a bound for $\rho(\mathscr{M})$. In appropriate situations that bound enables us to avoid complicated calculations and gives an explicit stability test for the discrete time switched systems. The example characterizing the sharpness of our results is given. To the best of our knowledge, our results are new even in the finite dimensional case.

Let $A \in \mathscr{B}(\mathscr{H})$ with $r_s(A) < 1$. Then the discrete Lyapunov equation

$$X - A^*XA = I \tag{1.1}$$

has a positive definite self-adjoint solution $X(A)$ [19]. It can be represented by

$$X(A) = \sum_{j=0}^{\infty} (A^*)^j A^j \tag{1.2}$$

and

$$X(A) = \frac{1}{2\pi}\int_0^{2\pi} (Ie^{-i\omega} - A^*)^{-1}(Ie^{i\omega} - A)^{-1} d\omega, \tag{1.3}$$

cf. [20, Section 7.1]. We will say that $\mathscr{M}$ is Schur-Kohn stable, if $\rho(\mathscr{M}) < 1$. Now we are in a position to formulate our main result.

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Theorem 1.1. *Let there be an $A \in \mathscr{B}(\mathscr{H})$ with $r_s(A) < 1$, such that*

$$\|X(A)\|(2\|A - A_k\|\|A\| + \|A - A_k\|^2) < 1 \ \ (A_k \in \mathscr{M};\ k = 1, ..., \nu). \tag{1.4}$$

Then $\mathscr{M}$ is Schur-Kohn stable. Moreover,

$$\rho(\mathscr{M}) \leq \sqrt{1 - \frac{1}{\|X(A)\|}(1 - \max_{j=1,...,\nu} \|X(A)\|(2\|A - A_j\|\|A\| + \|A - A_j\|^2)}.$$

The proof of this theorem is presented in the next section. In Theorem 1.1, one can take $A = A_m$ for an $A_m \in \mathscr{M}$. Below we consider some concrete classes of operators. Note that from (1.2) and (1.3) it follows that

$$\|X(A)\| \leq \sum_{j=1}^{\infty} \|A^j\|^2 \tag{1.5}$$

and

$$\|X(A)\| \leq \frac{1}{2\pi} \int_0^{2\pi} \|(Ie^{i\omega} - A)^{-1}\|^2 d\omega. \tag{1.6}$$

If A is normal: $AA^* = A^*A$, then $\|A\| = r_s(A)$ and (1.5) implies

$$\|X(A)\| \leq \sum_{j=0}^{\infty} r_s^{2j}(A) = \frac{1}{1 - r_s^2(A)}. \tag{1.7}$$

2. Proof of Theorem 1.1

In this section for the simplicity we put $X(A) = X$.

Lemma 2.1. *Let $A, \tilde{A} \in \mathscr{B}(\mathscr{H})$, $r_s(A) < 1$ and X be a solution of (1.1). If*

$$\|X\|(2\|A - \tilde{A}\|\|A\| + \|A - \tilde{A}\|^2) < 1,$$

then

$$(X\tilde{A}x, \tilde{A}x) \leq (1 - \frac{c_0}{\|X\|})(Xx, x) \ \ (x \in \mathscr{H}),$$

where

$$c_0 := 1 - \|X\|(2\|A - \tilde{A}\|\|A\| + \|A - \tilde{A}\|^2).$$

Proof. Put $Y = \tilde{A} - A$. Then

$$X - \tilde{A}^* X\tilde{A} = X - (Y + A)^* X(Y + A) = X - A^*XA - Y^*XA - A^*XY - Y^*XY = I - Y^*XA - A^*XY - Y^*XY.$$

By (2.1)

$$\|I - Y^*XA - A^*XY - YXY\| \geq 1 - \|Y^*XA - A^*XY - Y^*XY\|, \geq 1 - \|X\|(2\|A - \tilde{A}\| + \|A - \tilde{A}\|^2) = c_0.$$

Thus,

$$X - \tilde{A}^* X\tilde{A} \geq c_0 I.$$

Hence,

$$(Xx, x) - (X\tilde{A}x, \tilde{A}x) \geq c_0(x, x) \geq c_0(\frac{X}{\|X\|}x, x) = \frac{c_0}{\|X\|}(Xx, x),$$

as claimed. □

Proof of Theorem 1.1: Define the norms

$$|x|_X = \sqrt{(Xx, x)} \ \ (x \in \mathscr{H}) \text{ and } |A|_X = \sup_{x \in \mathscr{H}} \frac{|Ax|_X}{|x|_X}.$$

Due to Lemma 2.1 and (1.4) we have

$$|A_j|_X^2 \leq 1 - \frac{c_j}{\|X\|}, \tag{2.1}$$

where

$$c_j := 1 - \|X\|(2\|A - A_j\|\|A\| + \|A - A_j\|^2).$$

Put

$$a_0 := \max_j \sqrt{1-\frac{c_j}{\|X\|}} = \sqrt{1-\frac{1}{\|X\|}(1-\max_j \|X\|(2\|A-A_j\|\|A\|+\|A-A_j\|^2)}$$

Then by (2.1)

$$\max_j |A_j|_X \le a_0. \tag{2.2}$$

Since X is positive definite, it is boundedly invertible. For any $T \in \mathscr{B}(\mathscr{H})$ one has

$$\frac{\|Tx\|^2}{\|x\|^2} = \frac{(X^{-1}XTx,Tx)}{(X^{-1}Xx,x)} \le \frac{(X^{-1}XTx,Tx)}{(\frac{1}{\|X\|}Xx,x)} \le \|X\|\|X^{-1}\|\frac{(XTx,Tx)}{(Xx,x)} \ (x \in \mathscr{H}).$$

So

$$\|T\|^2 \le \|X\|\|X^{-1}\|\|T|_X^2$$

Hence, according to (2.2),

$$\|A_{i_k}\cdots A_{i_1}\| \le (\|X\|\|X^{-1}\|)^{1/2}|A_{i_k}\cdots A_{i_1}|_X \le (\|X\|\|X^{-1}\|)^{1/2}a_0^k$$

and therefore,

$$\rho(\mathscr{M}) \le \overline{\lim}_{k\to\infty}(\|X\|\|X^{-1}\|)^{1/2k}a_0 = a_0,$$

as claimed. □

3. Concrete classes of operators

In this section we suggest estimates for $X(A)$ under various assumptions. From (1.6) it follows

$$\|X(A)\| \le \sup_{|z|=1} \|(Iz-A)^{-1}\|^2. \tag{3.1}$$

Let there be monotonically increasing non-negative continuous function $F(x)$ $(x \ge 0)$, such that $F(0)=0$, $F(\infty)=\infty$ and

$$\|(\lambda I-A)^{-1}\| \le F(1/\mathrm{dist}(A,\lambda)) \ \ (\lambda \notin \sigma(A)),$$

where $\mathrm{dist}(A,\lambda) = \inf_{s\in\sigma(A)}|s-\lambda|$. If $r_s(A) < 1$ and $|z|=1$, then obviously, $\mathrm{dist}(A,z) \ge 1-r_s(A)$ and therefore, $\|(Iz-A)^{-1}\| \le F(1/(1-r_s(A)))$. Now (3.1) implies

$$\|X(A)\| \le F^2\left(\frac{1}{1-r_s(A)}\right). \tag{3.2}$$

3.1. Operators in finite dimensional spaces

Let $\mathbb{C}^n$ $(n<\infty)$ be the complex n-dimensional Euclidean space with a scalar product $(.,.)$, the Euclidean norm $\|.\| = \sqrt{(.,.)}$ and unit matrix I, $\mathbb{C}^{n\times n}$ is the set of all $n\times n$ matrices. $\lambda_k(A), k=1,...,n$, are the eigenvalues of $A \in \mathbb{C}^{n\times n}$, counted with their multiplicities; $N_2(A) = (\text{trace } AA^*)^{1/2}$ is the Hilbert-Schmidt (Frobenius) norm of A. The quantity (the departure from normality of A)

$$g(A) = [N_2^2(A) - \sum_{k=1}^n |\lambda_k(A)|^2\,]^{1/2},$$

plays an essential role hereafter. The following relations are checked in [21, Section 3.1]:

$$g^2(A) \le N_2^2(A) - |\text{trace } A^2| \text{ and } g^2(A) \le \frac{N_2(A-A^*)}{2} = 2N_2^2(A_I),$$

where $A_I = (A-A^*)/2i$. If A is a normal matrix, then $g(A)=0$.
Due to Example 3.3 from [21],

$$\|A^m\| \le \sum_{k=0}^{n-1} \frac{m! r_s^{m-k}(A) g^k(A)}{(m-k)!(k!)^{3/2}} \ \ (m=1,2,...).$$

Now (1.5) implies

$$\|X(A)\| \le \xi_n(A) := \sum_{j=1}^{\infty}\left(\sum_{k=0}^{n-1} \frac{m! r_s^{m-k}(A) g^k(A)}{(m-k)!(k!)^{3/2}}\right)^2 \ \ (A \in \mathbb{C}^{n\times n}). \tag{3.3}$$

Note that if A is normal, then $g(A)=0$ and (3.3) gives us the sharp inequality (1.7).
Theorem 1.1 and (3.3) yield the following corollary.

Corollary 3.1. *Let $\mathscr{M}$ be a finite set of $n\times n$-matrices. Let there be an $n\times n$-matrix A with $r_s(A)<1$, such that*

$$\xi_n(A)\max_{B\in\mathscr{M}}(2\|A-B\|\|A\|+\|A-B\|^2)<1.$$

Then $\mathscr{M}$ is Schur-Kohn stable. Moreover,

$$\rho(\mathscr{M})\le\sqrt{1-\frac{1}{\xi_n(A)}(1-\xi_n(A)\max_{B\in\mathscr{M}}(2\|A-B\|\|A\|+\|A-B\|^2)}.$$

Let us point the more compact but less sharper estimate for $X(A)$. To this end put

$$\eta_n(A):=\sum_{k=0}^{n-1}\frac{g^k(A)}{\sqrt{k!}(1-r_s(A))^{k+1}}.$$

By Theorem 3.2 from [21]

$$\|(A-\lambda I)^{-1}\|\le\sum_{k=0}^{n-1}\frac{g^k(A)}{(\text{dist}(A,\lambda))^{k+1}\sqrt{k!}}\quad(A\in\mathbb{C}^{n\times n},\lambda\notin\sigma(A)).$$

Making use of (3.2) we can assert that

$$\|X(A)\|\le\eta_n^2(A)\ \ (A\in\mathbb{C}^{n\times n}).$$

So in Corollary 3.1 one can replace $\xi_n(A)$ by $\eta_n^2(A)$.

3.2. Hilbert-Schmidt operators

Denote by SN_2 the ideal of Hilbert-Schmidt operators in $\mathscr{H}$ with the finite norm $N_2(A)=(\text{trace }AA^*)^{1/2}$. In the infinite dimensional case we put

$$g(A)=[N_2^2(A)-\sum_{k=1}^{\infty}|\lambda_k(A)|^2\,]^{1/2},$$

where $\lambda_k(A),k=1,2,...$, are the eigenvalues of $A\in SN_2$, counted with their multiplicities and enumerated in the non-increasing order of their absolute values.
Since

$$\sum_{k=1}^{\infty}|\lambda_k(A)|^2\ge|\sum_{k=1}^{\infty}\lambda_k^2(A)|=|\text{trace }A^2|,$$

one can write

$$g^2(A)\le N_2^2(A)-|\text{trace }A^2|.$$

If A is a normal Hilbert-Schmidt operator, then $g(A)=0$, since

$$N_2^2(A)=\sum_{k=1}^{\infty}|\lambda_k(A)|^2$$

in this case. Moreover,

$$g^2(A)\le\frac{N_2^2(A-A^*)}{2}=2N_2^2(A_I),$$

cf. [21, Section 7.1]. Due to Corollary 7.4 from [21] for any $A\in SN_2$ we have

$$\|A^m\|\le\sum_{k=0}^{m}\frac{m!r_s^{m-k}(A)g^k(A)}{(m-k)!(k!)^{3/2}}\ \ (m=1,2,...).$$

Now (1.5) implies

$$\|X(A)\|\le\xi_\infty(A):=\sum_{j=1}^{\infty}\left(\sum_{k=0}^{m}\frac{m!r_s^{m-k}(A)g^k(A)}{(m-k)!(k!)^{3/2}}\right)^2\quad(A\in SN_2).\tag{3.4}$$

If A is normal, then (3.4) gives us inequality (1.7).
Furthermore, by Theorem 7.1 from [21] for any $A\in SN_2$ we have

$$\|R_\lambda(A)\|\le\sum_{k=0}^{\infty}\frac{g^k(A)}{(\text{dist}(A,\lambda))^{k+1}\sqrt{k!}}\quad(\lambda\notin\sigma(A)).$$

Inequality (3.2) gives us the more compact but less sharper estimate

$$\|X(A)\|\le\eta_\infty^2(A)\ \ (A\in SN_2),$$

where

$$\eta_\infty(A) := \sum_{j=0}^{\infty} \frac{g^j(A)}{\sqrt{j!}(1-r_s(A))^{j+1}}.$$

Now we can directly apply Theorem 1.1.
By the Schwarz inequality

$$(\sum_{j=0}^{\infty} \frac{(cg)^j(A)}{c^j\sqrt{j!}(1-r_s(A))^j})^2 \le \sum_{k=0}^{\infty} c^{2k} \sum_{j=0}^{\infty} \frac{g^{2j}(A)}{c^{2j}j!(1-r_s(A))^{2j}} =$$

$$\frac{1}{1-c^2} \exp\,[\frac{g^2(A)}{c^2(1-r_s(A))^2}] \;\; (c \in (0,1)).$$

Thus,

$$\|X(A)\| \le \frac{1}{(1-c^2)(1-r_s(A))^2} \exp\,[\frac{g^2(A)}{c^2(1-r_s(A))^2}] \;\; (A \in SN_2, c \in (0,1)).$$

In particular, taking $c^2 = 1/2$, we get

$$\|X(A)\| \le \hat{\eta}(A) := \frac{2}{(1-r_s(A))^2} \exp\,[\frac{2g^2(A)}{(1-r_s(A))^2}].$$

Now Theorem 1.1 implies the following corollary.

Corollary 3.2. *Let $\mathscr{M}$ be a finite set of bounded operators from $\mathscr{H}$ Let there be an $A \in SN_2$ with $r_s(A) < 1$, such that*

$$\hat{\eta}(A) \max_{B\in\mathscr{M}} (2\|A-B\|\|A\| + \|A-B\|^2) < 1.$$

Then $\mathscr{M}$ is Schur-Kohn stable. Moreover,

$$\rho(\mathscr{M}) \le \sqrt{1 - \frac{1}{\hat{\eta}(A)}(1-\hat{\eta}(A) \max_{B\in\mathscr{M}} (2\|A-B\|\|A\| + \|A-B\|^2)}.$$

Similarly, making use of Theorems 7.2, 7.3 from [21] one can apply Theorem 1.1 to Shatten-von Neumann operators.

3.3. Non-compact non-normal operators

In this subsection we suggest a norm estimate for the solution of (1.1) under the condition

$$A_I = (A - A^*)/(2i) \in SN_2. \tag{3.5}$$

To this end introduce the quantity

$$g_I(A) := \sqrt{2}\left[N_2^2(A_I) - \sum_{k=1}^{\infty} (\Im\, \lambda_k(A))^2\right]^{1/2}.$$

Obviously, $g_I(A) \le \sqrt{2}N_2(A_I)$. Due to Example 10.2 from [21],

$$\|A^m\| \le \sum_{k=0}^{m} \frac{m!r_s^{m-k}(A)g_I^k(A)}{(m-k)!(k!)^{3/2}} \;\; (m = 1,2,...).$$

Now (1.5) implies

$$\|X(A)\| \le \xi_I(A) := \sum_{j=0}^{\infty} \left(\sum_{k=0}^{m} \frac{m!r_s^{m-k}(A)g^k(A)}{(m-k)!(k!)^{3/2}}\right)^2 \;\; (A_I \in SN_2).$$

If A is normal from this inequality we get (1.7).
Furthermore, by Theorem 9.1 from [21] under condition (4.1) we have,

$$\|R_\lambda(A)\| \le \sum_{k=0}^{\infty} \frac{g_I^k(A)}{(\mathrm{dist}(A,\lambda))^{k+1}\sqrt{k!}}$$

and

$$\|R_\lambda(A)\| \le \frac{\sqrt{e}}{\mathrm{dist}(A,\lambda)} \exp\,[\frac{g_I^2(A)}{2(\mathrm{dist}(A,\lambda))^2}] \;\; (\lambda \notin \sigma(A)).$$

Inequality (3.2) implies

$$\|X(A)\| \le \eta_I^2(A) \text{ and } \|X(A)\| \le \hat{\eta}_I^2(A) \;\; (A_I \in SN_2),$$

where

$$\eta_I := \sum_{j=0}^{\infty} \frac{g_I^j(A)}{\sqrt{j!}(1-r_s(A))^{j+1}}$$

and

$$\hat{\eta}_I := \frac{\sqrt{e}}{1-r_s(A)} \exp\left[\frac{g_I^2(A)}{2(1-r_s(A))^2}\right].$$

Now we can directly apply Theorem 1.1.
Some other classes of operators can be considered, in particular, via norm estimates for operator functions from [21].

4. Example

The following example characterizes the sharpness of Theorem 1.1.
Let $\mathscr{H} = \mathbb{C}^n$, $\mathscr{M} = \{A_1, A_2\}$ with real positive matrices matrices $A_1 = \mathrm{diag}\,(a_k)_{k=1}^n, A_2 = \mathrm{diag}\,(b_k)_{k=1}^n$; $(a_k, b_k \geq 0)$ and $r_s(A_1) < 1$. So $a_m = \max_k a_k < 1$. Take $A = A_1$. Since A_1 is Hermitian, according to (1.7) condition (1.4) takes the form

$$\frac{1}{1-r_s^2(A_1)}(2r_s(A_1)\|A_1 - A_2\| + \|A_1 - A_2\|^2) < 1. \tag{4.1}$$

Besides, $\|A_1 - A_2\| = \max_k |a_k - b_k|$.
Assume that $r_s(A_2) \geq 1$. Namely, $b_m \geq 1$. So $\mathscr{M}$ is Schur-Kohn unstable. Then $|a_m - b_m| = b_m - a_m \geq 1 - a_m$ and

$$\frac{1}{1-a_m^2}(2a_m|a_m - b_m| + |a_m - b_m|^2) \geq \frac{1}{1-a_m^2}(2a_m(1-a_m) + (1-a_m)^2) \geq \frac{1}{1+a_m}(2a_m + 1 - a_m) \geq 1.$$

Therefore, condition (4.1) is not fulfilled. So condition (1.4) is necessary under consideration.

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