International Electronic Journal of Geometry
Volume 2 No. 1 pp. 25-33 (2009) ©IEJG

ON SOME PROBLEMS CONNECTED WITH THE TANGENCY OF SETS IN GENERALIZED METRIC SPACES

TADEUSZ KONIK

(Communicated by Josef MIKES)

Abstract. In this paper, some problems of the tangency of sets of the classes $\widetilde{M}_{p,k}$ having the Darboux property in the generalized metric spaces (E, l) and (E, L) are considered. Some sufficient conditions for the compatibility of the tangency relations of sets of the above classes have been given in Section 2 of this paper.

1. Introduction

Let E be an arbitrary non-empty set and let l be a non-negative real function defined on the Cartesian product $E_0 \times E_0$ of the family E_0 of all non-empty subsets of the set E.

Let l_0 be the function defined by the formula:

$$l_0(x,y) = l(\{x\},\{y\}) \quad \text{for} \quad x,y \in E. \tag{1.1}$$

By some conditions for the function l, the function l_0 defined by (1.1) will be the metric of the set E. For this reason the pair (E, l) can be treated as a certain generalization of a metric space and we shall call it (see [11]) the generalized metric space. Using (1.1) we may define in the space (E, l), similarly as in a metric space, the notions: the sphere $S_l(p, r)$ and the open ball $K_l(p, r)$ with the centre at the point p and the radius r.

Let $S_l(p, r)_u$ denotes (see [11]) the so-called u-neighbourhood of the sphere $S_l(p, r)$ in the space (E, l) defined by the formula:

$$S_l(p,r)_u = \begin{cases} \bigcup\limits_{q \in S_l(p,r)} K_l(q,u) & \text{for} \quad u > 0, \\ S_l(p,r) & \text{for} \quad u = 0. \end{cases} \tag{1.2}$$

Let k be any positive real number, and let a, b be arbitrary non-negative real functions defined in a certain right-hand side neighbourhood of 0 such that

$$a(r) \xrightarrow[r \to 0^+]{} 0 \quad \text{and} \quad b(r) \xrightarrow[r \to 0^+]{} 0. \tag{1.3}$$

2000 *Mathematics Subject Classification.* 53A99, 54E35.

Key words and phrases. generalized metric spaces, tangency of sets.

We say that the pair (A,B) of the sets $A,B\in E_0$ is (a,b)-clustered at the point p of the space (E,l), if 0 is the cluster point of the set of all real numbers $r>0$ such that the sets $A\cap S_l(p,r)_{a(r)}$ and $B\cap S_l(p,r)_{b(r)}$ are non-empty.

Let us denote (see [11])

$$T_l(a,b,k,p)=\{(A,B):\ A,B\in E_0,\ \text{the pair } (A,B) \text{ is } (a,b)\text{-clustered at the point } p \text{ of the space } (E,l) \text{ and}$$

$$\frac{1}{r^k}l(A\cap S_l(p,r)_{a(r)},B\cap S_l(p,r)_{b(r)})\xrightarrow[r\to 0^+]{}0\}. \tag{1.4}$$

If $(A,B)\in T_l(a,b,k,p)$, then we say that the set $A\in E_0$ is (a,b)-tangent of order k to the set $B\in E_0$ at the point p of the space (E,l).

The set $T_l(a,b,k,p)$ defined by (1.4) we call the (a,b)-tangency relation of order k at the point $p\in E$ (or shortly: the tangency relation) of sets in the generalized metric space (E,l).

Two tangency relations of sets $T_{l_1}(a_1,b_1,k,p)$ and $T_{l_2}(a_2,b_2,k,p)$ are called compatible in the set E, if $(A,B)\in T_{l_1}(a_1,b_1,k,p)$ if and only if $(A,B)\in T_{l_2}(a_2,b_2,k,p)$ for $A,B\in E_0$.

Let ρ be any metric of the set E.

We say that the set $A\in E_0$ has the Darboux property at the point p of the metric space (E,ρ), what we write: $A\in D_p(E,\rho)$ (see [4]), if there exists a number $\tau>0$ such that the set $A\cap S_\rho(p,r)$ is non-empty for $r\in(0,\tau)$.

We shall denote by $d_\rho A$ the diameter of the set A, and by $\rho(A,B)$ the distance of sets A,B in the metric space (E,ρ), i.e.

$$d_\rho A=\sup\{\rho(x,y):\ x,\,y\in A\}\quad\text{and}\quad \rho(A,B)=\inf\{\rho(x,y):\ x\in A,\ y\in B\} \tag{1.5}$$

for $A,B\in E_0$.

By $\mathfrak{F}_\rho$ we will denote the class of all functions l fulfilling the conditions:

1^0 $l:\ E_0\times E_0\longrightarrow\langle 0,\infty)$,

2^0 $\rho(A,B)\le l(A,B)\le d_\rho(A\cup B)$ for $A,B\in E_0$.

From the condition 2^0 it follows that

$$\rho(x,y)=\rho(\{x\},\{y\})\le l(\{x\},\{y\})\le d_\rho(\{x\}\cup\{y\})=\rho(x,y),$$

whence we get the equality

$$l(\{x\},\{y\})=\rho(x,y)\quad\text{for}\ \ x,y\in E. \tag{1.6}$$

Let f be an increasing subadditive and continuous function defined in a certain right-hand side neighbourhood of 0 such that $f(0)=0$.

We shall put from the definition:

$$L(A,B)=f(l(A,B))\quad\text{for}\ \ A,B\in E_0. \tag{1.7}$$

The class of all the functions L defined by the formula (1.7) we will denote by $\mathfrak{F}_{f,\rho}$. From this and from the conditions 1^0 and 2^0 it follows that every function $L\in\mathfrak{F}_{f,\rho}$ fulfils the conditions:

3^0 $L:\ E_0\times E_0\longrightarrow\langle 0,\infty)$,

4^0 $f(\rho(A,B))\le L(A,B)\le f(d_\rho(A\cup B))$ for $A,B\in E_0$.

Because

$$f(\rho(x,y))=f(\rho(\{x\},\{y\}))\le L(\{x\},\{y\})\le f(d_\rho(\{x\}\cup\{y\}))=f(\rho(x,y)),$$

then from here it follows that

$$L(\{x\},\{y\}) = f(\rho(x,y)) \quad \text{for} \quad l \in \mathfrak{F}_{f,\rho} \quad \text{and} \quad x,y \in E. \tag{1.8}$$

Let us denote

$$\rho'(x,y) = f(\rho(x,y)) \quad \text{for} \quad x,y \in E. \tag{1.9}$$

It is easy to prove, using the properties of the function f, that the function ρ' defined by the formula (1.8) is the metric of the set E. From here it follows that every function $L \in \mathfrak{F}_{f,\rho}$ generates on the set E the metric ρ' defined by (1.8).

If $f = id$ where id denotes the identity function defined in a right-hand side neighbourhood of 0, then from here and from the definitions of the functions l and L it follows that

$$L(A,B) = l(A,B) \quad \text{for } A,B \in E_0. \tag{1.10}$$

By A' we shall denote the set of all cluster points of the set $A \in E_0$, and let

$$\rho(x,A) = \inf\{\rho(x,y):\ y \in A\} \quad \text{for } x \in E. \tag{1.11}$$

Let us put from the definition (see [3]):

$$\begin{gathered}\widetilde{M}_{p,k} = \{A \in E_0 :\ p \in A' \text{ and there exists } \mu > 0 \text{ such that}\\ \text{for an arbitrary } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that}\\ \text{for every pair of points } (x,y) \in [A,p;\mu,k]\\ \text{if } \rho(p,x) < \delta \text{ and } \frac{\rho(x,A)}{\rho^k(p,x)} < \delta, \text{ then } \frac{\rho(x,y)}{\rho^k(p,x)} < \varepsilon\},\end{gathered} \tag{1.12}$$

where

$$[A,p;\mu,k] = \{(x,y):\ x \in E,\ y \in A \text{ and } \mu\rho(x,A) < \rho^k(p,x) = \rho^k(p,y)\}. \tag{1.13}$$

In the paper [7] (see also [8, 9]) I proved, among others, the following theorems concerning the compatibility of the tangency relations of sets:

Theorem 1.1. *If the functions a,b fulfil the condition*

$$\frac{a(r)}{r^k} \xrightarrow[r\to 0^+]{} 0 \quad \text{and} \quad \frac{b(r)}{r^k} \xrightarrow[r\to 0^+]{} 0, \tag{1.14}$$

then for arbitrary functions $L_1, L_2 \in \mathfrak{F}_{f,\rho}$ the tangency relations $T_{L_1}(a,b,k,p)$ and $T_{L_2}(a,b,k,p)$ are compatible in the classes of sets $\widetilde{M}_{p,k} \cap D_p(E,\rho')$.

Theorem 1.2. *If the functions a_i, b_i $(i = 1,2)$ fulfil the condition*

$$\frac{a_i(r)}{r^k} \xrightarrow[r\to 0^+]{} 0 \quad \text{and} \quad \frac{b_i(r)}{r^k} \xrightarrow[r\to 0^+]{} 0, \tag{1.15}$$

then for any function $L \in \mathfrak{F}_{f,\rho}$ the tangency relations $T_L(a_1,b_1,k,p)$ and $T_L(a_2,b_2,k,p)$ are compatible in the classes of sets $\widetilde{M}_{p,k} \cap D_p(E,\rho')$.

From these theorems it follows:

Corollary 1.1. *If the functions a_i, b_i $(i = 1,2)$ fulfil the condition* (1.15), *then for arbitrary functions $L_1, L_2 \in \mathfrak{F}_{f,\rho}$ the tangency relations $T_{L_1}(a_1,b_1,k,p)$ and $T_{L_2}(a_2,b_2,k,p)$ are compatible in the classes of sets $\widetilde{M}_{p,k} \cap D_p(E,\rho')$.*

We shall denote by $d_{\rho'}A$ the diameter of the set A, and by $\rho'(A,B)$ the distance of sets A,B in the metric space (E,ρ'). Hence and from Theorem 1.1 it follows the following:

Corollary 1.2. *If the sets $A,B \in \widetilde{M}_{p,k} \cap D_p(E,\rho')$, the functions a,b fulfil the condition* (1.14), *then for any function $L \in \mathfrak{F}_{f,\rho}$*

$$(A,B) \in T_{\rho'}(a,b,k,p) \Leftrightarrow (A,B) \in T_L(a,b,k,p) \Leftrightarrow (A,B) \in T_{d_{\rho'}}(a,b,k,p). \tag{1.16}$$

If $f = id$ then from this corollary and from the equality (1.9) immediately it follows:

Remark 1.1. For any function $l \in \mathfrak{F}_\rho$

$$(A,B) \in T_\rho(a,b,k,p) \Leftrightarrow (A,B) \in T_l(a,b,k,p) \Leftrightarrow (A,B) \in T_{d_\rho}(a,b,k,p), \tag{1.17}$$

when $A,B \in \widetilde{M}_{p,k} \cap D_p(E,\rho)$ and the functions a,b fulfil the condition (1.14).

In connection with these considerations the question arises: by which assumptions the tangency relations: $T_l(a,b,k,p)$ and $T_L(a,b,k,p)$ are compatible in the classes of sets $\widetilde{M}_{p,k}$?

The answer to this question will be given in Section 2 of this paper.

2. On the compatibility of the tangency relations of sets

Let ρ be any metric of the set E, and let $S_\rho(p,r)_u$ (see the formula (1.2)) denotes a u-neighbourhood of the sphere $S_\rho(p,r)$ in the metric space (E,ρ).

Lemma 2.1. *If the metric ρ' is defined by* (1.9), *then*

$$S_\rho(p,r)_u = S_{\rho'}(p,f(r))_{f(u)}. \tag{2.1}$$

Proof. From the properties of the function f and from the formula (1.9) we get

$$\begin{aligned} S_\rho(p,r) &= \{x \in E : \rho(p,x) = r\} = \{x \in E : f(\rho(p,x)) = f(r)\} \\ &= \{x \in E : \rho'(p,x) = f(r)\} = S_{\rho'}(p,f(r)), \end{aligned}$$

that is to say

$$S_\rho(p,r) = S_{\rho'}(p,f(r)). \tag{2.2}$$

Similarly

$$\begin{aligned} K_\rho(q,u) &= \{x \in E : \rho(q,x) < u\} = \{x \in E : f(\rho(q,x)) < f(u)\} \\ &= \{x \in E : \rho'(q,x) < f(u)\} = K_{\rho'}(q,f(u)), \end{aligned}$$

i.e.

$$K_\rho(q,u) = K_{\rho'}(q,f(u)). \tag{2.3}$$

From the equalities (2.2), (2.3) and from the formula (1.2) it follows the thesis (2.1) of this lemma.

Similarly we prove that

$$S_{\rho'}(p,r)_u = S_\rho(p,f^{-1}(r))_{f^{-1}(u)}. \tag{2.4}$$

where f^{-1} is the inverse function to the function f.

Let ρ' be the metric of the set E defined by the formula (1.9). Now we shall prove:

Lemma 2.2. *If $A \in D_p(E,\rho')$, then $A \in D_p(E,\rho)$ for any set $A \in E_0$.*

Proof. We assume that $A \in D_p(E,\rho')$ for $A \in E_0$. Hence it follows that there exists a number $\tau' > 0$ such that

$$A \cap S_{\rho'}(p,r') \neq \emptyset \quad \text{for} \quad r' \in (0,\tau'). \tag{2.5}$$

We shall put

$$\tau = f^{-1}(\tau') \quad \text{and} \quad r = f^{-1}(r'). \tag{2.6}$$

Let r be any number belonging to the interval $(0,\tau)$. Hence and from the equalities (2.6) it follows that $r' = f(r) \in (0,\tau')$.
From this and from (2.2), (2.5) we get

$$A \cap S_\rho(p,r) = A \cap S_{\rho'}(p,f(r)) = A \cap S_{\rho'}(p,r') \neq \emptyset$$

for $r \in (0,\tau)$, what means that $A \in D_p(E,\rho)$ when the set $A \in E_0$.

In the paper [5] I proved (see Lemma 1.1) the following implication:

$$A \in D_p(E,\rho) \Rightarrow A \in D_p(E,\rho') \quad \text{for} \quad A \in E_0, \tag{2.7}$$

when ρ' is the metric of the set E defined by the formula (1.9).

From here and from Lemma 2.2 of this paper it follows:

Corollary 2.1. *If ρ' is the metric of the set E defined by the formula* (1.9), *then $A \in D_p(E,\rho)$ if and only if $A \in D_p(E,\rho')$ for any set $A \in E_0$.*

Using the above results we shall prove some theorems concerning the tangency of sets in the generalized metric spaces (E,l) and (E,L).

Theorem 2.1. *If the sets $A, B \in D_p(E,\rho)$ and the functins a, b, f fulfil the coditions*:

$$a(f(r)) \leq f(a(r)) \quad \textit{and} \quad b(f(r)) \leq f(b(r)) \quad \textit{for} \quad r > 0, \tag{2.8}$$

$$f(r_1 r_2) \leq f(r_1) f(r_2) \quad \textit{for} \quad r_1, r_2 > 0, \tag{2.9}$$

then

$$(A,B) \in T_{d_\rho}(a,b,k,p) \Rightarrow (A,B) \in T_{d_{\rho'}}(a,b,k,p). \tag{2.10}$$

Proof. We assume that $(A,B) \in T_{d_\rho}(a,b,k,p)$ for $A, B \in E_0$. From here it follows:

$$\frac{1}{r^k} d_\rho((A \cap S_\rho(p,r)_{a(r)}) \cup (B \cap S_\rho(p,r)_{b(r)})) \xrightarrow[r \to 0^+]{} 0. \tag{2.11}$$

Let us put $r' = f(r)$. Hence, from Lemma 2.1 and from the assumption (2.8) of this theorem we get

$$S_{\rho'}(p,r')_{a(r')} = S_{\rho'}(p,f(r))_{a(f(r))} \subseteq S_{\rho'}(p,f(r))_{f(a(r))} = S_\rho(p,r)_{a(r)},$$

that is to say

$$S_{\rho'}(p,r')_{a(r')} \subseteq S_\rho(p,r)_{a(r)}. \tag{2.12}$$

Analogously

$$S_{\rho'}(p,r')_{b(r')} \subseteq S_\rho(p,r)_{b(r)}. \tag{2.13}$$

From (2.12) and (2.13) we have

$$\begin{aligned} &d_\rho((A \cap S_{\rho'}(p,r)_{a(r')}) \cup (B \cap S_{\rho'}(p,r)_{b(r')})) \\ &\leq d_\rho((A \cap S_\rho(p,r)_{a(r)}) \cup (B \cap S_\rho(p,r)_{b(r)})). \end{aligned}$$

Hence and from the properties of the function f it is evident

$$\begin{aligned}&f(d_\rho((A\cap S_{\rho'}(p,r)_{a(r')})\cup(B\cap S_{\rho'}(p,r)_{b(r')})))\\ &\leq f(d_\rho((A\cap S_\rho(p,r)_{a(r)})\cup(B\cap S_\rho(p,r)_{b(r)}))).\end{aligned}\tag{2.14}$$

Moreover for the continuous function f we get

$$\begin{aligned}f(d_\rho A)&=f(\sup\{\rho(x,y):\ x,y\in A\})=\sup\{f(\rho(x,y)):\ x,y\in A\}\\ &=\sup\{\rho'(x,y):\ x,y\in A\}=d_{\rho'}A,\end{aligned}$$

i.e.

$$f(d_\rho A)=d_{\rho'}A\quad\text{for}\quad A\in E_0.\tag{2.15}$$

Hence and from the inequality (2.14) it results that

$$\begin{aligned}&\frac{1}{(r')^k}d_{\rho'}((A\cap S_{\rho'}(p,r)_{a(r')})\cup(B\cap S_{\rho'}(p,r)_{b(r')}))\\ &\leq\frac{1}{(r')^k}f(d_\rho((A\cap S_\rho(p,r)_{a(r)})\cup(B\cap S_\rho(p,r)_{b(r)})))\\ &=\frac{1}{(f(r))^k}f(d_\rho((A\cap S_\rho(p,r)_{a(r)})\cup(B\cap S_\rho(p,r)_{b(r)}))).\end{aligned}\tag{2.16}$$

From the condition (2.9) the inequalities follow:

$$\frac{f(r_1)}{f(r_2)}\leq f(r_1/r_2)\quad\text{for}\quad r_1,r_2>0,\tag{2.17}$$

and

$$f(r^k)\leq(f(r))^k\quad\text{for}\quad r>0,$$

that is to say

$$\frac{1}{(f(r))^k}\leq\frac{1}{f(r^k)}\quad\text{for}\quad r>0.\tag{2.18}$$

From (2.18), (2.17) and from the inequality (2.16) we get

$$\begin{aligned}&\frac{1}{(r')^k}d_{\rho'}((A\cap S_{\rho'}(p,r)_{a(r')})\cup(B\cap S_{\rho'}(p,r)_{b(r')}))\\ &\leq\frac{1}{f(r^k)}f(d_\rho((A\cap S_\rho(p,r)_{a(r)})\cup(B\cap S_\rho(p,r)_{b(r)})))\\ &\leq f(\frac{1}{r^k}d_\rho((A\cap S_\rho(p,r)_{a(r)})\cup(B\cap S_\rho(p,r)_{b(r)}))).\end{aligned}\tag{2.19}$$

Hence, from (2.11) and from the properties of the function f it follows that

$$\frac{1}{(r')^k}d_{\rho'}((A\cap S_{\rho'}(p,r)_{a(r')})\cup(B\cap S_{\rho'}(p,r)_{b(r')}))\xrightarrow[r'\to0^+]{}0.\tag{2.20}$$

From the assumption that $A,B\in D_p(E,\rho)$ and from Corollary 1.2 of this paper it is evident that the pair of sets (A,B) is (a,b)-clustered at the point p of the space (E,ρ').

Hence and from the condition (2.20) it results that $(A,B)\in T_{d_{\rho'}}(a,b,k,p)$. This ends the proof.

Theorem 2.2. *If the sets* $A, B \in D_p(E, \rho)$ *and the functins* a, b, f *fulfil the inequalities* (2.8) *and*

$$f(r_1 r_2) = f(r_1) f(r_2) \quad \text{for} \;\; r_1, r_2 > 0, \tag{2.21}$$

then

$$(A, B) \in T_{\rho'}(a, b, k, p) \Rightarrow (A, B) \in T_\rho(a, b, k, p). \tag{2.22}$$

Proof. We assume that $(A, B) \in T_{\rho'}(a, b, k, p)$ for $A, B \in E_0$. From here it follows:

$$\frac{1}{(r')^k} \rho'(A \cap S_{\rho'}(p, r')_{a(r')}, B \cap S_{\rho'}(p, r')_{b(r')}) \xrightarrow[r' \to 0^+]{} 0. \tag{2.23}$$

Using the conditions (2.12) and (2.13) we get

$$\begin{aligned} &\rho(A \cap S_\rho(p, r)_{a(r)}, B \cap S_\rho(p, r)_{b(r)}) \\ &\leq \rho(A \cap S_{\rho'}(p, r')_{a(r')}, B \cap S_{\rho'}(p, r')_{b(r')}). \end{aligned} \tag{2.24}$$

From (1.9) and from the fact that f is the continuous function it appears that

$$\begin{aligned} f(\rho(A, B)) &= f(\inf\{\rho(x, y) : \; x \in A, \; y \in B\}) = \inf\{f(\rho(x, y)) : \; x \in A, \; y \in B\} \\ &= \inf\{\rho'(x, y) : \; x \in A, \; y \in B\} = \rho'(A, B), \end{aligned}$$

i.e.

$$f(\rho(A, B)) = \rho'(A, B) \quad \text{for} \quad A, B \in E_0. \tag{2.25}$$

From this and from the inequality (2.24) we get

$$\begin{aligned} &f(\rho(A \cap S_\rho(p, r)_{a(r)}, B \cap S_\rho(p, r)_{b(r)})) \\ &\leq f(\rho(A \cap S_{\rho'}(p, r')_{a(r')}, B \cap S_{\rho'}(p, r')_{b(r')})) \\ &= \rho'(A \cap S_{\rho'}(p, r')_{a(r')}, B \cap S_{\rho'}(p, r')_{b(r')}). \end{aligned}$$

Therefore

$$\begin{aligned} &\frac{1}{(r')^k} f(\rho(A \cap S_\rho(p, r)_{a(r)}, B \cap S_\rho(p, r)_{b(r)})) \\ &\leq \frac{1}{(r')^k} \rho'(A \cap S_{\rho'}(p, r')_{a(r')}, B \cap S_{\rho'}(p, r')_{b(r')}). \end{aligned}$$

Hence and from the condition (2.23) it results that

$$\frac{1}{(f(r))^k} f(\rho(A \cap S_\rho(p, r)_{a(r)}, B \cap S_\rho(p, r)_{b(r)})) \xrightarrow[r \to 0^+]{} 0, \tag{2.26}$$

where $f(r) = r'$ for $r > 0$.
From the assumption (2.21) we get the equalities:

$$\frac{1}{(f(r))^k} = \frac{1}{f(r^k)} \quad \text{for} \quad r > 0, \tag{2.27}$$

and

$$\frac{f(r_1)}{f(r_2)} = f(r_1/r_2) \quad \text{for} \quad r_1, r_2 > 0. \tag{2.28}$$

Using the conditions (2.26) - (2.28) we have

$$f(\frac{1}{r^k} \rho(A \cap S_\rho(p, r)_{a(r)}, B \cap S_\rho(p, r)_{b(r)})) \xrightarrow[r \to 0^+]{} 0.$$

Hence and from the properties of the function f it follows that

$$\frac{1}{r^k}\rho(A\cap S_\rho(p,r)_{a(r)}, B\cap S_\rho(p,r)_{b(r)}) \xrightarrow[r\to 0^+]{} 0. \tag{2.29}$$

From the assumption that $A, B \in D_p(E,\rho)$ it is evident that the pair of sets (A,B) is (a,b)-clustered at the point p of the space (E,ρ).

Hence and from the condition (2.29) it results that $(A,B) \in T_\rho(a,b,k,p)$. This ends the proof.

If the functions a, b, f fulfil the conditions (1.14), (2.8) and (2.21), the sets $A, B \in \widetilde{M}_{p,k} \cap D_p(E,\rho)$, then from Theorems 1.1, 2.1 and 2.2 it results the following diagram:

$$\begin{array}{ccc} (A,B)\in T_\rho(a,b,k,p) \Leftrightarrow (A,B)\in T_l(a,b,k,p) \Leftrightarrow (A,B)\in T_{d_\rho}(a,b,k,p) \\ \Uparrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Downarrow \\ (A,B)\in T_{\rho'}(a,b,k,p) \Leftrightarrow (A,B)\in T_L(a,b,k,p) \Leftrightarrow (A,B)\in T_{d_{\rho'}}(a,b,k,p). \end{array} \tag{2.30}$$

From this diagram it follows:

Corollary 2.2. *If the functions a, b, f fulfil the conditions* (1.14), (2.8) *and* (2.21), *then the tangency relations $T_l(a,b,k,p)$ and $T_L(a,b,k,p)$ are compatible in the classes of sets $\widetilde{M}_{p,k} \cap D_p(E,\rho)$ i.e.*

$$(A,B)\in T_l(a,b,k,p) \;\Leftrightarrow\; (A,B)\in T_L(a,b,k,p) \tag{2.31}$$

for $A, B \in \widetilde{M}_{p,k} \cap D_p(E,\rho)$.

From (2.30) and from Theorems 1.1, 1.2 (see Corollary 1.1) we get also:

Corollary 2.3. *If the functions a_i, b_i, f $(i = 1,2)$ fulfil the conditions* (1.15), (2.21) *and*

$$a_i(f(r)) \le f(a_i(r)) \quad \text{and} \quad b_i(f(r)) \le f(b_i(r)) \quad \text{for} \quad r > 0, \tag{2.32}$$

then the tangency relations $T_l(a_1,b_1,k,p)$ and $T_L(a_2,b_2,k,p)$ are compatible i.e.

$$(A,B)\in T_l(a_1,b_1,k,p) \;\Leftrightarrow\; (A,B)\in T_L(a_2,b_2,k,p) \tag{2.33}$$

for $A, B \in \widetilde{M}_{p,k} \cap D_p(E,\rho)$.

References

[1] S. Golkab and Z. Moszner, Sur le contact des courbes dans les espaces metriques généraux, Colloq. Math. 10(1963), 105-311.

[2] J. Grochulski, T. Konik and M. Tkacz, On the tangency of sets in metric spaces, Ann. Polon. Math. 38(1980), 121-131.

[3] T. Konik, Tangency relations for sets in some classes in generalized metric spaces, Math. Slovaca 48(4)(1998), 399-410.

[4] T. Konik, On the compatibility and the equivalence of the tangency relations of sets of the classes $A^*_{p,k}$, J. Geom. 63(1998), 124-133.

[5] T. Konik, On the compatibility of the tangency relations of sets of some classes, Bull. Acad. Sci. Rep. Moldova Matematica 3(28)(1998), 3-14.

[6] T. Konik, On some tangency relation of sets, Publ. Math. Debrecen 55/3-4(1999), 411-419.

[7] T. Konik, O styczności zbiorów w uogólnionych przestrzeniach metrycznych, Wydawnictwo Politechniki Czkestochowskiej, seria Monografie nr 77(2001), 1-71.

[8] T. Konik, Some cases of compatibility of the tangency relations of sets, Appl. Sci. (Balkan Soc. Geom.) 5(1)(2003), 41-47.

[9] T. Konik, Connections and applications of some tangency relation of sets, Balkan Journal of Geometry and Its Applications 8(2)(2003), 105-112.

[10] E. Pascali, Tangency and ortogonality in metric spaces, Demonstratio Math. 38(2)(2005), 437-449.

[11] W.Waliszewski, On the tangency of sets in generalized metric spaces, Ann. Polon. Math. 28(1973), 275-284.

Institute of Mathematics, Czestochowa University of Technology, Poland
E-mail address: konik@imi.pcz.pl