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CONSTANCY OF ϕ-HOLOMORPHIC SECTIONAL CURVATURE FOR INDEFINITE SASAKIAN MANIFOLDS

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Abstract. S. Tanno [6] provided an algebraic characterization for an almost Hermitian manifold to reduce to a space of constant holomorphic sectional curvature, which he later extended for the Sasakian manifolds as well. In the present paper, we generalize the same characterization for Indefinite Sasakian manifolds.

1. Introduction

For an almost Hermitian manifold (M^{2n}, g, J) where dimension of M is $2n \geq 4$, Tanno [6] has proved

Theorem 1.1. *Let dimension$(M) = 2n \geq 4$, assume that almost Hermitian manifold (M^{2n}, g, J) satisfies*

$$R(JX, JY, JZ, JX) = R(X, Y, Z, X) \tag{1.1}$$

for every tangent vectors X, Y and Z. Then (M^{2n}, g, J) is of constant holomorphic sectional curvature at x if and only if

$$R(X, JX)X \text{ is proportional to } JX \tag{1.2}$$

for every tangent vector X at $x \in M$.

Tanno [6] has also proved an analogous theorem for Sasakian manifolds as

Theorem 1.2. *A Sasakian manifold of dimension ≥ 5, is of constant ϕ-sectional curvature if and only if*

$$R(X, \phi X)X \text{ is proportional to } \phi X \tag{1.3}$$

for every vector field X such that $g(X, \xi) = 0$.

Nagaich [5] has proved generalized version of theorem (1.1), for indefinite almost Hermitian manifolds as

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Theorem 1.3. *Let (M^{2n}, g, J), $(n \geq 2)$ be an indefinite almost Hermitian manifold satisfying (1.1), then (M^{2n}, g, J) is of constant holomorphic sectional curvature if and only if*

$$R(X, JX)X \text{ is proportional to } JX \tag{1.4}$$

for every tangent vector X at $x \in M$.

The aim of this paper is to generalize the theorem (1.2), for an indefinite Sasakian manifold by proving the following

Theorem 1.4. *Let $(M^{2n+1}, \phi, \eta, \xi, g)(n \geq 2)$ be an indefinite Sasakian manifold. Then M^{2n+1} is of constant ϕ-sectional curvature if and only if*

$$R(X, \phi X)X \text{ is proportional to } \phi X \tag{1.5}$$

for every vector field X such that $g(X, \xi) = 0$.

2. Preliminaries

2.1. **Sasakian Manifold.** Let M be a $(2n+1)$-dimensional Riemannian manifold endowed with a $(1,1)$-type tensor field ϕ, a vector field ξ, a 1-form η and a metric g such that

$$\phi(\xi) = 0, \tag{2.1}$$

$$\eta(\phi X) = 0, \tag{2.2}$$

$$\eta(\xi) = 1, \tag{2.3}$$

$$\phi^2(X) = -X + \eta(X)\xi, \tag{2.4}$$

$$\eta(X) = g(X, \xi), \tag{2.5}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.6}$$

for all vector fields $X, Y \in \chi(M)$.

If $d\phi(X, Y) = g(X, \phi Y)$ then M is said to have a contact Riemannian structure (ϕ, ξ, η, g). If, moreover, the structure is normal then the contact Riemannian structure is called a Sasakian structure and M is called a Sasakian manifold.

Let ∇ be the covariant derivative with respect to g, then the curvature tensor field R of M satisfies

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y \tag{2.7}$$

$$\begin{aligned} R(X, Y)\phi Z &= \phi R(X, Y)Z + g(\phi X, Z)Y \\ &\quad - g(Y, Z)\phi X + g(X, Z)\phi Y - g(\phi Y, Z)X \end{aligned} \tag{2.8}$$

Let σ be a plane section in tangent space $T_p(M)$ at a point p of M be spanned by vectors X and Y then the sectional curvature of σ is given by

$$K(X, Y) = \frac{R(X, Y, X, Y)}{g(X, X)g(Y, Y) - g(X, Y)^2} \tag{2.9}$$

A plane section $X, \phi X$ where X is orthonormal to ξ is called ϕ-section and curvature associated with this is called ϕ-sectional curvature and is denoted by $H(X)$, where

$$H(X) = K(X, \phi X) = R(X, \phi X, X, \phi X). \tag{2.10}$$

If a Sasakian manifold M has constant ϕ-sectional curvature c then it is called a Sasakian space form and is denoted by $M^{2n+1}(c)$. The curvature tensor R of a Sasakian space form $M^{2n+1}(c)$ is given by

$$\begin{aligned} R(X,Y,Z,W) &= \frac{(c+3)}{4}\{g(X,Z)g(Y,W)-g(Y,Z)g(X,W)\} \\ &+\frac{(c-1)}{4}\{\eta(Y)\eta(Z)g(X,W)-\eta(X)\eta(Z)g(Y,W) \\ &+\eta(X)\eta(W)g(Y,Z)-\eta(Y)\eta(W)g(X,Z)+g(\phi X,Z)g(\phi Y,W) \\ &-g(\phi Y,Z)g(\phi X,W)-2g(X,\phi Y)g(\phi Z,W)\}. \end{aligned} \tag{2.11}$$

2.2. **Indefinite Sasakian Manifold.** Let M be a $(2n+1)$-dimensional Riemannian manifold endowed with a $(1,1)$-type tensor field ϕ, a vector field ξ, a 1-form η and an indefinite metric g such that

$$\phi(\xi)=0, \tag{2.12}$$

$$\eta(\phi X)=0, \tag{2.13}$$

$$\eta(\xi)=-1, \tag{2.14}$$

$$\phi^2(X)=-X+\eta(X)\xi, \tag{2.15}$$

$$\eta(X)=g(X,\xi), \tag{2.16}$$

$$g(\phi X,\phi Y)=g(X,Y)+\eta(X)\eta(Y), \tag{2.17}$$

for all vector fields $X,Y\in\chi(M)$.

If indefinite contact Riemannian structure (ϕ,ξ,η,g) is normal then the indefinite contact Riemannian structure is called an indefinite Sasakian structure and M is called an indefinite Sasakian manifold.

A non-zero vector is called space-like, time-like or null if it satisfies $g(X,X)>,<or=0$ if $X\neq 0$ respectively.

Let ∇ be the covariant derivative with respect to g, then the curvature tensor field R of M satisfies

$$R(X,Y)\xi=\eta(Y)X-\eta(X)Y \tag{2.18}$$

$$\begin{aligned} R(X,Y)\phi Z &= \phi R(X,Y)Z-g(\phi X,Z)Y+g(Y,Z)\phi X \\ &-g(X,Z)\phi Y+g(\phi Y,Z)X \end{aligned} \tag{2.19}$$

And the curvature tensor R of indefinite Sasakian space form is given as, [4]

$$\begin{aligned} R(X,Y,Z,W) &= \frac{(c-3)}{4}\{g(X,Z)g(Y,W)-g(Y,Z)g(X,W)\} \\ &+\frac{(c+1)}{4}\{\eta(Y)\eta(Z)g(X,W)-\eta(X)\eta(Z)g(Y,W) \\ &+\eta(X)\eta(W)g(Y,Z)-\eta(Y)\eta(W)g(X,Z)+g(\phi X,Z)g(\phi Y,W) \\ &-g(\phi Y,Z)g(\phi X,W)-2g(X,\phi Y)g(\phi Z,W)\}. \end{aligned} \tag{2.20}$$

3. **Proof of the main Theorem**

Let $(M^{2n+1}, \phi, \eta, \xi, g)(n \geq 2)$ be an indefinite Sasakian manifold. To prove the theorem, we shall consider two different cases.

Also, to prove the theorem for dimension$(M) \geq 5$, we shall consider cases when dimension$(M) = 5$ and when dimension$(M) > 5$, that is, when dimension$(M) \geq 7$.

3.1. **Case I.** When the metric is space-like, that is, when $g(X,X) = g(Y,Y)$.
Let M be of constant ϕ-sectional curvature then (2.11) gives

$$R(X, \phi X)X = c \quad \phi X \tag{3.1}$$

Conversely, let $\{X, Y\}$ be an orthonormal pair of tangent vectors such that $g(\phi X, Y) = g(X, Y) = g(Y, \xi) = 0$ and dimension$(M) \geq 7$ then $\acute{X} = (X + Y)/\sqrt{2}$ and $\acute{Y} = (\phi X - \phi Y)/\sqrt{2}$ are also form an orthonormal pair of tangent vectors such that $g(\phi \acute{X}, \acute{Y}) = 0$ then (3.1) gives

$$R(\acute{X}, \phi\acute{X}, \acute{Y}, \acute{X}) = 0 \tag{3.2}$$

This gives

$$H(X) - H(Y) + 2R(X, \phi X, Y, \phi X) - 2R(X, \phi Y, Y, \phi Y) = 0. \tag{3.3}$$

The hypothesis implies

$$H(X) = H(Y). \tag{3.4}$$

Now, if $sp\{U, V\}$ is holomorphic then for $\phi U = pU + qV$ where p and q are constant, we have

$$sp\{U, \phi U\} = sp\{U, pU + qV\} = sp\{U, V\} \tag{3.5}$$

Similarly

$$sp\{V, \phi V\} = sp\{U, V\} \tag{3.6}$$

$$sp\{U, \phi U\} = sp\{V, \phi V\} \tag{3.7}$$

This implies

$$R(U, \phi U, U, \phi U) = R(V, \phi V, V, \phi V) \tag{3.8}$$

Or

$$H(U) = H(V) \tag{3.9}$$

If $sp\{U, V\}$ is not holomorphic section then we can choose unit vectors $X \in sp\{U, \phi U\}^{\perp}$ and $Y \in sp\{V, \phi V\}^{\perp}$ such that $sp\{X, Y\}$ is holomorphic. Thus, we get

$$H(U) = H(X) = H(Y) = H(V) \tag{3.10}$$

This shows that any holomorphic section has the same ϕ-sectional curvature.

Now, let the dimension$(M) = 5$ and let $\{X, Y\}$ be a set of orthonormal vectors such that $g(X, \phi Y) = 0$, we have $H(X) = H(Y)$ as before, then using property (1.5), we have

$$R(X, \phi X)X = H(X)\phi X$$
$$R(X, \phi X)Y = R(X, \phi X, Y, \phi Y)\phi Y$$
$$R(X, \phi Y)X = R(X, \phi Y, X, Y)Y + R(X, \phi Y, X, \phi Y)\phi Y$$
$$R(X, \phi Y)Y = R(X, \phi Y, Y, X)X + R(X, \phi Y, Y, \phi X)\phi X$$

$$R(Y, \phi X)Y = R(Y, \phi X, Y, X)X + R(Y, \phi X, Y, \phi X)\phi X$$
$$R(Y, \phi X)X = R(Y, \phi X, X, Y)Y + R(Y, \phi X, X, \phi Y)\phi Y$$
$$R(Y, \phi Y)X = R(Y, \phi Y, X, \phi X)\phi X$$
$$R(Y, \phi Y)Y = H(Y)\phi Y = H(X)\phi Y$$

Now, define $X^* = aX + bY$ such that $a^2 + b^2 = 1$ and $a^2 \neq b^2$. Then, using above relations, we get

$$R(X^*, \phi X^*)X^* = C_1X + C_2Y + C_3\phi X + C_4\phi Y \tag{3.11}$$

Where C_1 and C_2 are not necessary for argument and

$$C_3 = a^3H(X) + ab^2C_5 \tag{3.12}$$

$$C_4 = b^3H(X) + a^2bC_5 \tag{3.13}$$

Where $C_5 = R(X, \phi X, Y, \phi Y) + R(X, \phi Y, X, \phi Y) + R(X, \phi Y, Y, \phi X)$. On the other hand

$$\begin{aligned} R(X^*, \phi X^*)X^* &= H(X^*)\phi X^* \\ &= H(X^*)\{a\phi X + b\phi Y\} \end{aligned} \tag{3.14}$$

Comparing (3.11) and (3.14) we get

$$a^2H(X) + b^2C_5 = H(X^*) \tag{3.15}$$

$$b^2H(X) + a^2C_5 = H(X^*) \tag{3.16}$$

On solving (3.15) and (3.16) we get

$$H(X) = H(X^*)$$

Similarly, we can prove

$$H(Y) = H(Y^*)$$

Thus, M has constant ϕ-sectional curvature.

3.2. **Case II.** When the metric is time-like, that is, when $g(X, X) = -g(Y, Y)$. Here if X is space-like then Y is a time-like vector or vice versa.

Initially assume that M be of constant ϕ-sectional curvature then (2.20) gives

$$R(X, \phi X)X = c\phi X \tag{3.17}$$

Conversely, let $\{X, Y\}$ be an orthonormal pair of tangent vectors such that $g(\phi X, Y) = g(X, Y) = g(Y, \xi) = 0$ and dimension$(M) \geq 7$ then $\acute{\acute{X}} = (X + iY)/\sqrt{2}$ and $\acute{\acute{Y}} = (i\phi X + \phi Y)/\sqrt{2}$ also form an orthonormal pair of tangent vectors such that $g(\phi\acute{\acute{X}}, \acute{\acute{Y}}) = 0$ then (3.17) gives

$$R(\acute{\acute{X}}, \phi\acute{\acute{X}}, \acute{\acute{Y}}, \acute{\acute{X}}) = 0 \tag{3.18}$$

This gives

$$H(X) = H(Y) \tag{3.19}$$

Then using the same argument as in **Case I**, we get any holomorphic section has same ϕ-sectional curvature.

Now, let the dimension$(M) = 5$ and $g(X,X) = -g(Y,Y)$ such that $g(X,\phi Y) = 0$ then we have $H(X) = H(Y)$ as before then using property (1.5), we have

$$R(X,\phi X)X = H(X)\phi X$$

$$R(X,\phi X)Y = -R(X,\phi X,Y,\phi Y)\phi Y$$

$$R(X,\phi Y)X = -R(X,\phi Y,X,Y)Y - R(X,\phi Y,X,\phi Y)\phi Y$$

$$R(X,\phi Y)Y = R(X,\phi Y,Y,X)X + R(X,\phi Y,Y,\phi X)\phi X$$

$$R(Y,\phi X)Y = R(Y,\phi X,Y,X)X + R(Y,\phi X,Y,\phi X)\phi X$$

$$R(Y,\phi X)X = -R(Y,\phi X,X,Y)Y - R(Y,\phi X,X,\phi Y)\phi Y$$

$$R(Y,\phi Y)X = R(Y,\phi Y,X,\phi X)\phi X$$

$$R(Y,\phi Y)Y = -H(Y)\phi Y = -H(X)\phi Y$$

Now, define $X^{**} = aX + bY$ such that $a^2 - b^2 = 1$ and $a^2 \neq b^2$. Then, using above relations, we get

$$R(X^{**},\phi X^{**})X^{**} = \acute{C_1}X + \acute{C_2}Y + \acute{C_3}\phi X + \acute{C_4}\phi Y \tag{3.20}$$

Where $\acute{C_1}$ and $\acute{C_2}$ are not necessary for argument and

$$\acute{C_3} = a^3 H(X) + ab^2\acute{C_5} \tag{3.21}$$

$$\acute{C_4} = -b^3 H(X) - a^2 b\acute{C_5} \tag{3.22}$$

Where $\acute{C_5} = R(X,\phi X,Y,\phi Y) + R(X,\phi Y,X,\phi Y) + R(X,\phi Y,Y,\phi X)$. On the other hand

$$\begin{aligned} R(X^{**},\phi X^{**})X^{**} &= H(X^{**})\phi X^{**} \\ &= H(X^{**})\{a\phi X + b\phi Y\} \end{aligned} \tag{3.23}$$

Comparing (3.20) and (3.23) we get

$$a^2 H(X) + b^2\acute{C_5} = H(X^{**}) \tag{3.24}$$

$$-b^2 H(X) - a^2\acute{C_5} = H(X^{**}) \tag{3.25}$$

On solving (3.24) and (3.25) we get

$$H(X) = H(X^{**})$$

Similarly, we can prove

$$H(Y) = H(Y^{**})$$

Thus, M has constant ϕ-sectional curvature.

Remark 3.1. In the present paper, we have considered the cases of space-like and time-like vectors only. However, we are still investigating the same results for light-like (or null) vectors.

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