

POLYNOMIAL POLY-VECTOR FIELDS

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ABSTRACT. In this text we give a decomposition result on polynomial poly-vector fields generalizing a result on the decomposition of homogeneous Poisson structures. We discuss consequences of this decomposition result in particular for low dimension and low polynomial degree. We provide the tools to calculate simple cubic Poisson structures in dimension three and quadratic Poisson structures in dimension four. Our decomposition result has a nice effect on the relation between Poisson structures and Jacobi structures.

1. INTRODUCTION

We motivate our discussion on poly-vector fields by Poisson structures. For this we recall: A Poisson structure on the n -dimensional manifold M is a skew symmetric bi-linear map $\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$ which obeys

P1: The map $\{f, \cdot\} : C^\infty(M) \rightarrow C^\infty(M)$ is a derivation for all $f \in C^\infty(M)$.

P2: The Jacobi identity $\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$ holds for all $f, g, h \in C^\infty(M)$.

Due to P1 a Poisson structure is equivalent to a skew symmetric tensor field $\Pi \in \Gamma\Lambda^2 TM$. The Poisson bracket and the field are related by $\{f, g\} = \Pi(df, dg)$. This may be generalized to brackets of degree higher than two, i.e. $\{\cdot, \dots, \cdot\} : \times^{2l} C^\infty(M) \rightarrow C^\infty(M)$. If we impose conditions similar to P1 and P2 we arrive at the so called generalized Poisson structures due to [1] and [2].

To translate condition P2 to the tensor Π , we have to introduce poly-vector fields. We denote the vector fields on M by $\mathfrak{X}^1(M) = \mathfrak{X}(M) = \Gamma TM$ and the k -vector fields by $\mathfrak{X}^k(M) := \Lambda^k \mathfrak{X}(M)$. A natural grading is given by $\mathfrak{X}_{\text{nat}}^k(M) := \mathfrak{X}^k(M)[-k]$. The Lie bracket of vector fields extends to the Schouten bracket $[\cdot, \cdot] : \mathfrak{X}^k(M) \times \mathfrak{X}^l(M) \rightarrow \mathfrak{X}^{k+l-1}(M)$ with $[X, U] = L_X U$ for $X \in \mathfrak{X}(M)$. This turns the shifted algebra $\mathfrak{T}_{\text{poly}}(M) = \left(\bigoplus_{k \geq 0} \mathfrak{X}^k(M)[-k]\right)[1]$ into a Gerstenhaber algebra. This means, the

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bracket obeys the graded Jacobi identity

$$(-1)^{\tilde{U}\tilde{W}}[U, [V, W]] + (-1)^{\tilde{V}\tilde{W}}[W, [U, V]] + (-1)^{\tilde{U}\tilde{V}}[V, [W, U]] = 0 \quad (1)$$

such that $\mathfrak{T}_{\text{poly}}(M)$ has a graded Lie algebra structure. Furthermore, the bracket is compatible with the wedge product $\mathfrak{X}^k(M) \times \mathfrak{X}^l(M) \rightarrow \mathfrak{X}^{k+l}(M)$ on $\mathfrak{T}_{\text{poly}}(M)$ in the sense that

$$[U, V \wedge W] = [U, V] \wedge W + (-1)^{\tilde{U}(\tilde{V}+1)} V \wedge [U, W]. \quad (2)$$

The symmetry of the wedge product is given by $U \wedge V = (-1)^{(\tilde{U}+1)(\tilde{V}+1)} V \wedge U$. To distinguish the shifted grading of $\mathfrak{T}_{\text{poly}}(M)$ from the natural grading we add a tilde in the first case. If we express the Poisson structures by the associated bi-vector Π , condition P2 is connected to the Schouten bracket. More precisely, a bi-vector field yields a Poisson structure if and only if

$$\text{P2}' : [\Pi, \Pi] = 0.$$

In local coordinates, $\Pi = \Pi^{ij} \partial_i \wedge \partial_j$, the latter is written as $\Pi^{[ij} (\partial_\ell \Pi^{kl]} = 0$. Here we use the convention that we sum over corresponding upper and lower indices. Moreover, the bracket indicates the skew symmetrization of the enclosed indices.

In the following we deal with polynomial poly-vector fields, i.e. poly-vector fields which coefficients are polynomials. For a bi-vectors this means we have a decomposition $\Pi = \sum \Pi^{ij} \partial_i \wedge \partial_j$ with polynomial coefficients Π^{ij} . For two poly-vectors σ_1 and σ_2 which are homogeneous of degree k and ℓ the bracket $[\sigma_1, \sigma_2]$ is homogeneous of degree $k + \ell - 1$. This yields that for a polynomial bi-vector field to obey [P2], it is necessary that its highest homogeneous part is a Poisson structure itself. For this reason, the restriction to homogeneous polynomial poly-vector fields is natural.

In the case of Poisson structures, the following holds. For degree zero all constant bi-vectors provide Poisson structures. For degree one we write $\Pi^{ij} = f_k^{ij} x^k$ and condition [P2] is given by $f_l^{[ij} f_n^{kl]} = 0$, i.e. f_k^{ij} are the structure constants of a Lie algebra. For a discussion of the connection between Lie algebras and Poisson structures see [4] and for the classification of linear Poisson structures in four dimensions see [17]. For the discussion of quadratic Poisson structures we consider the injective map $\iota_0 : \mathfrak{gl}_n \rightarrow \mathfrak{X}(\mathbb{R}^n)$ with $\iota_0(A) = A^i_j x^j \partial_i \in \mathfrak{X}(\mathbb{R}^n)$ which gives the linear vector fields on \mathbb{R}^n . The Lie bracket on $\mathfrak{g} := \mathfrak{gl}_n$ naturally extends to a Gerstenhaber bracket on $\hat{\mathfrak{g}} = (\bigoplus_{k \geq 0} \Lambda^k \mathfrak{g}[-k])[1]$ and ι_0 to a morphism $\iota : \hat{\mathfrak{g}} \rightarrow \mathfrak{T}_{\text{poly}}$ of Gerstenhaber algebras. A classical R-matrix is an element $r \in \Lambda^2 \mathfrak{gl}_n$ which obey the Yang-Baxter equation $[r, r] = 0$. A classical R-matrices yields a Poisson structure by $r = r^{ijkl} E_{ij} \wedge E_{kl} \mapsto \iota_1(r) = r^{ijkl} x^i x^k \partial_j \wedge \partial_l$. In [3] this connection between quadratic Poisson structures and classical R-matrices is recognized and a 1-1 correspondence is shown to be true at least in dimension $n = 2$. In higher dimensions this correspondence is not true, because the graded components ι_k have nontrivial kernels, in particular for $k = 2, 3$. For $n = 3$ the authors in [14] give an explicit example $r \in \hat{\mathfrak{g}}^{(1)} = \Lambda^2 \mathfrak{gl}_n$ such that $[r, r] \neq 0$ and $\iota_1(r) \neq 0$ but $\iota_2[r, r] = [\iota_1(r), \iota_1(r)] = 0$. Up to now the nonlinear Poisson structures resist descriptions as nice as in the linear case. Therefore, the only chance to describe them is to find an explicit way of construction. This has been done for quadratic Poisson structures in [5] and [12] and used in [14].

In this text we give a decomposition formula for polynomial poly-vector fields into irreducible components, see Theorem 3.4. For any poly-vector fields the Schouten bracket can be decomposed with respect to this decomposition, see Theorem 3.5 and Corollary 3.6, and for even poly-vector fields the generalized Poisson condition P2' translates into compatibility conditions on the irreducible components, see Theorem 3.8 and Proposition 4.2. The decomposition result yields a connection between polynomial Poisson structures and polynomial Jacobi structures which is collected in Theorem 4.5. The case where the polynomial degree and the vector field degree coincide turns out to be an exceptional combination. In the last part of this text we provide some tools to calculate polynomial Poisson structures for low dimensions and low orders and give a lot of examples.

2. THE TRACE OPERATOR

We consider the volume form Ψ on the oriented manifold M . Ψ induces an isomorphism of poly-vector fields and differential forms. A k -vector field $U \in \mathfrak{X}^k(M)$ is mapped to an $(n-k)$ -form $\Psi(U) \in \Omega^{n-k}(M)$ by $\Psi(U)(W) := \Psi(U \wedge W)$. In local coordinates with $\Psi = dx^1 \wedge \dots \wedge dx^n = \frac{1}{n!} \epsilon_{i_1 \dots i_n} dx^{i_1} \wedge \dots \wedge dx^{i_n}$ and $U = \frac{1}{k!} U^{j_1 \dots j_k} \partial_{j_1} \wedge \dots \wedge \partial_{j_k}$ this reads

$$\Psi(U) = \frac{1}{k!(n-k)!} \epsilon_{j_1 \dots j_k i_1 \dots i_{n-k}} U^{j_1 \dots j_k} dx^{i_1} \wedge \dots \wedge dx^{i_{n-k}}. \quad (3)$$

We use this isomorphism to define a derivation $D : \mathfrak{T}_{\text{poly}}(M) \rightarrow \mathfrak{T}_{\text{poly}}(M)$ of degree -1 by

$$D : \mathfrak{X}^k(M) \rightarrow \mathfrak{X}^{k-1}(M), \quad D = \Psi^{-1} \circ d \circ \Psi. \quad (4)$$

This is in fact a differential, i.e. $D^2 = 0$. We note that if we restrict D to vector fields we recover the divergence with respect to the volume form Ψ . We will recall this soon. In local coordinates the action of D is given by

$$DU = \frac{1}{(k-1)!} \partial_m U^{i_1 \dots i_{k-1} m} \partial_{i_1} \wedge \dots \wedge \partial_{i_{k-1}}, \quad (5)$$

where we used $\epsilon^{i_1 \dots i_\ell a_1 \dots a_{n-\ell}} \epsilon_{j_1 \dots j_\ell a_1 \dots a_{n-\ell}} = \ell!(n-\ell)! \delta_{j_1 \dots j_\ell}^{i_1 \dots i_\ell}$.

With respect to the non-shifted grading of poly-vector fields, D is compatible with the wedge product and the Schouten bracket in the following sense.

Proposition 2.1. *For all poly-vector fields U and V we have*

$$D(U \wedge V) = (-1)^V DU \wedge V + U \wedge DV + (-1)^V [U, V], \quad (6)$$

$$[DU, U] = -[U, DU], \quad DU \wedge U = U \wedge DU, \quad (7)$$

and

$$D[U, V] = [U, DV] - (-1)^V [DU, V]. \quad (8)$$

In particular, for U even $D[U, U] = -2[DU, U]$, and so $[DU, U] = 0$ if $[U, U] = 0$.

Proof. Because both sides of (6) are \mathbb{R} -linear we only have to check the identity for homogeneous U and V . I.e. if we consider local coordinates with $\Psi = dx^1 \wedge \dots \wedge dx^n$ we may assume $U = f \partial_{i_1} \wedge \dots \wedge \partial_{i_k}$ and $V = g \partial_{i_1} \wedge \dots \wedge \partial_{i_{k'}}$. Then (6) follows from a careful calculation.

To prove (7) we recall the commutation properties of the wedge product and the Schouten bracket and get $[DU, U] = -(-1)^{(\tilde{U}-1)\tilde{U}}[U, DU] = -[U, DU]$ as well as $DU \wedge U = (-1)^{(U-1)U}U \wedge DU = U \wedge DU$.

$$\begin{aligned} D^2(U \wedge V) &= D((-1)^V DU \wedge V + U \wedge DV + (-1)^V [U, V]) \\ &= (-1)^V DU \wedge DV + (-1)^V (-1)^V [DU, V] + (-1)^{V-1} DU \wedge DV \\ &\quad + (-1)^{V-1} [U, DV] + (-1)^V D[U, V] \\ &= (-1)^V (D[U, V] - [U, DV] + (-1)^V [DU, V]). \end{aligned}$$

$D^2 = 0$ yields

$$D[U, V] = [U, DV] - (-1)^V [DU, V] = [U, DV] - (-1)^{(U-1)(V-1)} [V, DU]$$

which proves (8). With $U = V$ even this is $D[U, U] = -2[DU, U]$. \square

Remark 2.2. The relations from Proposition 2.1 can for example be found in [10], but we emphasize on the slightly different sign convention used in our paper. In [7] relation (6) is the starting point for a discussion of Poisson structures in particular for the investigation of Poisson structures in dimension three and four.

We mentioned before that if we restrict D to vector fields we recover the divergence with respect to the volume form Ψ . In this case (6) reads as $D(X \wedge Y) = (\operatorname{div} Y)X - (\operatorname{div} X)Y - [X, Y]$. A connection to the notion of divergence induced by a connection ∇ on the manifold M is given by the next Proposition.

Proposition 2.3. [9] *Let Ψ be a volume form on the manifold M . Consider a connection ∇ on M with vanishing torsion $T = 0$ which is compatible with the volume form, i.e. $\nabla \Psi = 0$. Then the two notions of divergence of vector fields given by*

$$\operatorname{div}_{\Psi} X := \Psi^{-1} \circ d \circ \Psi(X) \quad \text{and} \quad \operatorname{div}_{\nabla} X := \operatorname{tr}(\nabla X) \quad (9)$$

coincide.

The existence of a differential D does not depend on whether the manifold is orientable or not, although in our case it will be. This is due to

Proposition 2.4. [10] *Let M be a manifold with torsion free connection ∇ . Then there exists a differential $D_{\nabla} : \mathfrak{X}^k(M) \rightarrow \mathfrak{X}^{k-1}(M)$ with (6) and which coincides for vector fields X with the divergence associated to ∇ .*

We will next concentrate on polynomial poly-vector fields $\mathfrak{P} \subset \mathfrak{T}_{\text{poly}}$. These are poly-vector fields whose coefficients in a local coordinate system are polynomials in the coordinates. The space \mathfrak{P} is bi-graded, because every subspace $\mathfrak{P}^{(k)} \subset \mathfrak{T}_{\text{poly}}^{(k-1)}$ admits a further \mathbb{Z} -grading due to the degree of its coefficient polynomial. We denote these spaces by $\mathfrak{P}^{(\ell)} = \bigoplus_k \mathfrak{P}^{(k, \ell)}$. An element $A \in \mathfrak{P}^{(k, \ell)}$ can be written as $A = \frac{1}{k! \ell!} A_{i_1 \dots i_k}{}^{j_1 \dots j_{\ell}} x^{i_1} \dots x^{i_k} \partial_{j_1} \wedge \dots \wedge \partial_{j_{\ell}}$ with $A_{\bullet \bullet}$ totally symmetric in its lower indices and totally skew symmetric in its upper indices.

Definition 2.5. A polynomial ℓ -vector field A is called k -homogeneous if $A \in \mathfrak{P}^{(k, \ell)}$.

A_{\bullet}^{\bullet} transforms like a tensor with the given index picture, if we restrict to linear coordinate transformations. In this case we identify

$$\mathfrak{P}^{(k,\ell)} = S^k(V^*) \otimes \Lambda^\ell V.$$

where V is an n -dimensional real vector space. The restriction of the differential on $\mathfrak{T}_{\text{poly}}$ to the polynomial poly-vector fields is part of the next Proposition.

Proposition 2.6. *The differential D on $\mathfrak{T}_{\text{poly}}$, see (4), restricted to the polynomial poly-vector fields \mathfrak{P} is the trace operator. In particular, with respect to the bi-grading of \mathfrak{P} the differential D is of degree $(-1, -1)$. I.e.*

$$\begin{aligned} D : \mathfrak{P}^{(k,\ell)} &\rightarrow \mathfrak{P}^{(k-1,\ell-1)}, \\ (DA)_{i_1 \dots i_{k-1}}^{j_1 \dots j_{\ell-1}} &= A_{i_1 \dots i_{k-1} m}^{j_1 \dots j_{\ell-1} m}. \end{aligned} \tag{10}$$

Corollary 2.7. For linear vector fields A , i.e. for elements in $V^* \otimes V = \text{End}(V)$, the differential D is the usual trace for endomorphisms, i.e. $DA = \text{tr} A$.

Proof. We write the poly-vector field in the form

$$A = \frac{1}{k!\ell!} A_{i_1 \dots i_k}^{j_1 \dots j_\ell} x^{i_1} \dots x^{i_k} \partial_{j_1} \wedge \dots \wedge \partial_{j_\ell}.$$

We recall (5) which yields

$$\begin{aligned} DA &= \frac{1}{(\ell-1)!k!} \partial_m (A_{i_1 \dots i_k}^{j_1 \dots j_{\ell-1} m} x^{i_1} \dots x^{i_k}) \partial_{j_1} \wedge \dots \wedge \partial_{j_{\ell-1}} \\ &= \frac{1}{(\ell-1)!k!} A_{i_1 \dots i_k}^{j_1 \dots j_{\ell-1} m} \partial_m (x^{i_1} \dots x^{i_k}) \partial_{j_1} \wedge \dots \wedge \partial_{j_{\ell-1}} \\ &= \frac{1}{(\ell-1)!k!} A_{i_1 \dots i_k}^{j_1 \dots j_{\ell-1} m} \sum_{\alpha=1}^k \delta_m^{i_\alpha} x^{i_1} \dots \widehat{x^{i_\alpha}} \dots x^{i_k} \partial_{j_1} \wedge \dots \wedge \partial_{j_{\ell-1}} \\ &= \frac{1}{(\ell-1)!(k-1)!} A_{i_1 \dots i_{k-1} m}^{j_1 \dots j_{\ell-1} m} x^{i_1} \dots x^{i_{k-1}} \partial_{j_1} \wedge \dots \wedge \partial_{j_{\ell-1}}. \end{aligned}$$

□

3. THE DECOMPOSITION OF POLY-VECTOR FIELDS

As before we consider $V = \mathbb{R}^n$. We decompose the space of polynomial poly-vector fields, $\mathfrak{P}^{(k,\ell)} = S^k(V^*) \otimes \Lambda^\ell V$, into its irreducible representation spaces with respect to $\mathfrak{sl}_n \mathbb{R}$ and formulate the decomposition in terms of the differential D . For some basic facts on representations and weights of the special linear algebra we cordially refer the reader to [6, §§15.1-15.3].

Definition 3.1. We denote the representation space of $\mathfrak{sl}_n \mathbb{R}$ with weight λ by V_λ . Furthermore, we denote by λ_m the weight $(\underbrace{0, \dots, 0}_m, 1)$ and for the representation space with weight $\lambda = k\lambda_{n-1} + \lambda_\ell$ we use the short notation $V_{k,\ell}$.

We recall the isomorphism of representations

$$V^* \simeq \Lambda^{n-1} V = V_{\lambda_{n-1}}.$$

The space $S^k(V^*) \simeq S^k(\Lambda^{n-1}V)$ is itself irreducible and its weight is given by $\lambda = k\lambda_{n-1} = (0, \dots, 0, k)$. To get the decomposition of $\mathfrak{P}^{(k,\ell)}$ we calculate

$$\begin{aligned} \mathfrak{P}^{(k,\ell)} &= S^k(V^*) \otimes \Lambda^\ell V \\ &\simeq V_{k\lambda_{n-1}} \otimes V_{\lambda_\ell} \\ &= V_{k\lambda_{n-1} + \lambda_\ell} \oplus V_{\lambda_n + (k-1)\lambda_{n-1} + \lambda_{\ell-1}} \\ &\simeq V_{k\lambda_{n-1} + \lambda_\ell} \oplus V_{(k-1)\lambda_{n-1} + \lambda_{\ell-1}} \\ &= V_{k,\ell} \oplus V_{k-1,\ell-1}. \end{aligned} \quad (11)$$

The dimension of the representation space $V_{k,\ell}$ is

$$\dim V_{k,\ell} = \frac{(n+k)!}{(n+k-\ell)k!l!(n-\ell-1)!},$$

in particular, $\dim V_{k,k} = \frac{1}{(k!)^2} \prod_{j=1}^k (n^2 - j^2)$.

We note that $V_{1,1} = \mathfrak{sl}_n \mathbb{R}$, $\mathfrak{P}^{(k,0)} = S^k(V^*) = V_{k\lambda_{n-1}} = V_{k,0}$, $\mathfrak{P}^{(0,\ell)} = \Lambda^\ell V = V_{\lambda_\ell} = V_{0,\ell}$ and $\mathfrak{P}^{k,n} \simeq S^k(V^*) = V_{k-1,n-1}$.

The trace operator $D : \mathfrak{P}^{(k,\ell)} \rightarrow \mathfrak{P}^{(k-1,\ell-1)}$ gives rise to an exact sequence. For $m > 0$ it is

$$\begin{aligned} S^{n+m}(V^*) &\xrightarrow{D} S^{n+m}(V^*) \oplus V_{n+m-2,n-2} \xrightarrow{D} \dots \\ \dots &\xrightarrow{D} V_{m+2,2} \oplus V_{m+1,1} \xrightarrow{D} V_{m+1,1} \oplus S^m(V^*) \xrightarrow{D} S^m(V^*), \end{aligned} \quad (12)$$

for $0 < m < n$

$$\begin{aligned} S^{n-m}(V^*) &\xrightarrow{D} S^{n-m}(V^*) \oplus V_{n-m-2,n-2} \xrightarrow{D} \dots \\ \dots &\xrightarrow{D} V_{2,m+2} \oplus V_{1,m+1} \xrightarrow{D} V_{1,m+1} \oplus \Lambda^m V \xrightarrow{D} \Lambda^m V, \end{aligned} \quad (13)$$

and for $m = 0$

$$\begin{aligned} S^n(V^*) &\xrightarrow{D} S^n(V^*) \oplus V_{n-2,n-2} \xrightarrow{D} \dots \\ \dots &\xrightarrow{D} V_{3,3} \oplus V_{2,2} \xrightarrow{D} V_{2,2} \oplus \mathfrak{sl}_n \mathbb{R} \xrightarrow{D} \mathfrak{sl}_n \mathbb{R} \oplus \mathbb{R} \xrightarrow{D} \mathbb{R}. \end{aligned} \quad (14)$$

The kernel and the image of the $\mathfrak{sl}_n \mathbb{R}$ -invariant trace are representation spaces, too. Due to the irreducibility of the decomposition (11) this yields

$$\ker(D|_{\mathfrak{P}^{(k,\ell)}}) = V_{k,\ell}, \quad \text{im}(D|_{\mathfrak{P}^{(k,\ell)}}) = V_{k-1,\ell-1}. \quad (15)$$

We recall the wedge product

$$\wedge : \mathfrak{P}^{(k,\ell)} \otimes \mathfrak{P}^{(k',\ell')} \longrightarrow \mathfrak{P}^{(k+k',\ell+\ell')}$$

to describe the inclusion

$$S^{k-1}(V^*) \otimes \Lambda^{\ell-1}V \supset V_{k-1,\ell-1} \hookrightarrow V_{k-1,\ell-1} \subset S^k(V^*) \otimes \Lambda^\ell V. \quad (16)$$

Consider the element $e^{(k,\ell)} \in \mathfrak{P}^{(1,1)}$ defined by

$$e^{(k,\ell)} = \frac{1}{n+k-\ell} e_0, \quad \text{with } e_0 = x^m \partial_m. \quad (17)$$

This vector field has certain properties collected in the next lemma.

Lemma 3.2. For $A \in \mathfrak{P}^{(k',\ell')}$ we have

$$De^{(k,\ell)} = \frac{n}{n+k-\ell}, \quad (18)$$

$$[e^{(k,\ell)}, A] = \frac{k'-\ell'}{n+k-\ell}A. \quad (19)$$

If furthermore $DA = 0$, we have

$$D(A \wedge e^{(k,\ell)}) = \frac{n+k'-\ell'}{n+k-\ell}A. \quad (20)$$

Proof. The trace of e_0 is n , so the first part is obvious. The second part follows from $[e_0, fY] = [e_0, f]Y + f[e_0, Y]$ and

$$\begin{aligned} [e_0, x^{i_1} \cdots x^{i_{k'}}] &= x^m \partial_m (x^{i_1} \cdots x^{i_{k'}}) \\ &= x^m \sum_{\alpha=1}^{k'} \delta_m^{i_\alpha} x^{i_1} \cdots \widehat{x^{i_\alpha}} \cdots x^{i_{k'}} \\ &= k' x^{i_1} \cdots x^{i_{k'}}, \end{aligned}$$

as well as

$$\begin{aligned} [e_0, \partial_{j_1} \wedge \cdots \wedge \partial_{j_{\ell'}}] &= \partial_m \wedge [x^m, \partial_{j_1} \wedge \cdots \wedge \partial_{j_{\ell'}}] \\ &= \partial_m \wedge \sum_{\alpha=1}^{\ell'} (-1)^\alpha \delta_m^{j_\alpha} \partial_{j_1} \wedge \cdots \wedge \widehat{\partial_{j_\alpha}} \wedge \cdots \wedge \partial_{j_{\ell'}} \\ &= -\ell' \partial_{j_1} \wedge \cdots \wedge \partial_{j_{\ell'}}. \end{aligned}$$

We plug A and $e^{(k,\ell)}$ into (6) and with $DA = 0$ as well as (19) we get

$$\begin{aligned} D(A \wedge e^{(k,\ell)}) &= A \wedge De^{(k,\ell)} - [A, e^{(k,\ell)}] \\ &= \frac{n}{n+k-\ell}A + [e^{(k,\ell)}, A] \\ &= \frac{n+k'-\ell'}{n+k-\ell}A. \end{aligned}$$

□

Corollary 3.3. The preimage of an element $A \in V_{k-1,\ell-1} \subset S^{k-1}(V^*) \otimes \Lambda^{\ell-1}V$ in $V_{k-1,\ell-1} \subset S^k(V^*) \otimes \Lambda^\ell V$ with respect to D is given by

$$A \wedge e^{(k,\ell)}.$$

The vector $e^{(k,\ell)}$ does only depend on the difference $(k-\ell)$ and so is fixed for the sequences (12)-(14).

This discussion proves the following theorem.

Theorem 3.4. Each polynomial poly-vector field $A \in \mathfrak{P}^{(k,\ell)}$ admits a unique decomposition $A = A_0 + A_1$ with $A_0 \in V_{k,\ell} \subset \mathfrak{P}^{(k,\ell)}$ and $A_1 \in V_{k-1,\ell-1} \subset \mathfrak{P}^{(k,\ell)}$. Explicitly the fields are connected by $DA = DA_1$, $DA_0 = 0$, $A_1 = DA \wedge e^{(k,\ell)}$, i.e.

$$A = A_0 + DA \wedge e^{(k,\ell)}. \quad (21)$$

In the next step we write the bracket of two polynomial poly-vector field in terms of the poly-vector fields we get by applying the decomposition with respect to (21).

Theorem 3.5. *Let $A \in \mathfrak{P}^{(k,\ell)}$ and $B \in \mathfrak{P}^{(k',\ell')}$ polynomial poly-vector fields with decompositions $A = A_0 + DA \wedge e^{(k,\ell)}$ and $B = B_0 + DB \wedge e^{(k',\ell')}$, see (21). Then*

$$\begin{aligned} & [A, B]_0 \\ &= [A_0, B_0] + \frac{\Delta'}{n + \Delta} DA \wedge B_0 + (-1)^{\ell'} \frac{\Delta}{n + \Delta'} A_0 \wedge DB \\ & \quad + \frac{\Delta'}{n + \Delta} [A_0, DB] \wedge e^{(k'',\ell'')} - (-1)^{\ell'} \frac{\Delta}{n + \Delta'} [DA, B_0] \wedge e^{(k'',\ell'')}, \end{aligned} \quad (22)$$

and

$$\begin{aligned} & D[A, B] \\ &= [A_0, DB] - (-1)^{\ell'} [DA, B_0] \\ & \quad - \frac{(n + \Delta'')(\Delta - \Delta')}{(n + \Delta)(n + \Delta')} \left(DA \wedge DB + (-1)^{\ell'} [DA, DB] \wedge e^{(k'',\ell'')} \right), \end{aligned} \quad (23)$$

with $\Delta = k - \ell$, $k'' = k + k'$, $\ell'' = \ell + \ell'$, $\Delta' = k' - \ell'$ and $\Delta'' = \Delta + \Delta' = k'' - \ell''$.

Proof. Let A and B as above as stated. We write $e = e^{(k,\ell)}$, $e' = e^{(k',\ell')}$, and $e'' = e^{(k'',\ell'')}$. From the symmetry properties of the wedge product and the bracket and from the compatibility (6) we get

$$\begin{aligned} & [A \wedge V, B \wedge Y] \\ &= (-1)^{(\tilde{A}+1)(\tilde{V}+\tilde{B}+1)} [V, B] \wedge A \wedge Y + (-1)^{\tilde{V}\tilde{Y}+\tilde{B}(\tilde{Y}+1)+1} [A, Y] \wedge V \wedge B \\ & \quad + (-1)^{\tilde{B}(\tilde{V}+1)} [A, B] \wedge V \wedge Y + (-1)^{\tilde{A}(\tilde{Y}+\tilde{V})+\tilde{B}(\tilde{Y}+1)+\tilde{V}+\tilde{A}} [V, Y] \wedge A \wedge B. \end{aligned} \quad (24)$$

For $V = e$ and $Y = e'$ this is

$$[A \wedge e, B \wedge e'] = A \wedge [e, B] \wedge e' - [e', A] \wedge B \wedge e. \quad (25)$$

We make use of the decompositions of A and B and formulas (25) as well as (19) and get

$$\begin{aligned} [A, B] &= [A_0 + DA \wedge e, B_0 + DB \wedge e'] \\ &= [A_0, B_0] + [A_0, DB \wedge e'] + [DA \wedge e, B_0] + [DA \wedge e, DB \wedge e'] \\ &= [A_0, B_0] + [A_0, DB] \wedge e' + (-1)^{(\ell-1)(\ell'-1)} DB \wedge [A_0, e'] \\ & \quad + DA \wedge [e, B_0] + (-1)^{\ell'-1} [DA, B_0] \wedge e \\ & \quad + DA \wedge [e, DB] \wedge e' - [e', DA] \wedge DB \wedge e \\ &= [A_0, B_0] + (-1)^{\ell'} \frac{\Delta}{n + \Delta'} A_0 \wedge DB + \frac{\Delta'}{n + \Delta} DA \wedge B_0 \\ & \quad + \frac{1}{n + \Delta'} [A_0, DB] \wedge e_0 - (-1)^{\ell'} \frac{1}{n + \Delta} [DA, B_0] \wedge e_0 \\ & \quad + \frac{\Delta' - \Delta}{(n + \Delta)(n + \Delta')} DA \wedge DB \wedge e_0. \end{aligned}$$

To get the trace of this expression we use Proposition 2.1.

$$\begin{aligned}
D[A, B] &= [A, DB] - (-1)^{\ell'} [DA, B] \\
&= [A_0, DB] - (-1)^{\ell'} [DA, B_0] + [DA \wedge e, DB] - (-1)^{\ell'} [DA, DB \wedge e'] \\
&= [A_0, DB] - (-1)^{\ell'} [DA, B_0] + DA \wedge [e, DB] + (-1)^{\ell'} [DA, DB] \wedge e \\
&\quad - (-1)^{\ell'} [DA, DB] \wedge e' - (-1)^{\ell' + \ell(\ell' - 1)} DB \wedge [DA, e'] \\
&= [A_0, DB] - (-1)^{\ell'} [DA, B_0] + \left(\frac{\Delta'}{n + \Delta} - \frac{\Delta}{n + \Delta'} \right) DA \wedge DB \\
&\quad + (-1)^{\ell'} \left(\frac{1}{n + \Delta} - \frac{1}{n + \Delta'} \right) [DA, DB] \wedge e_0.
\end{aligned}$$

Using $\Delta(n + \Delta) - \Delta'(n + \Delta') = (n + \Delta'')(\Delta - \Delta')$ proves the second part of Theorem 3.5. The first part is obtained by rearranging the summands in $[A, B]_0 = [A, B] - D[A, B] \wedge e''$. \square

If we restrict to the case $A = B$ we get the following consequences.

Corollary 3.6. Let $A \in \mathfrak{P}^{(k, \ell)}$ with ℓ even. The bracket of A with itself in terms of its decomposition with respect to (21) is given by

$$[A, A]_0 = [A_0, A_0] + \frac{2(k - \ell)}{n + k - \ell} (DA \wedge A_0 - [DA, A_0] \wedge e^{(2k, 2\ell)}), \quad (26)$$

$$D[A, A] = -2[DA, A_0]. \quad (27)$$

In $[DA, A_0]$ and $DA \wedge A_0$ we may replace A_0 by A .

Corollary 3.7. All polynomial poly-vector fields of type $A \wedge e^{(k, \ell)}$ have vanishing bracket $[A \wedge e^{(k, \ell)}, A \wedge e^{(k, \ell)}]$.

From Corollary 3.6 we immediately conclude the following result on even homogeneous polynomial poly-vector fields, e.g. (generalized) Poisson structures.

Theorem 3.8. Let $A \in \mathfrak{P}^{(k, \ell)}$ and ℓ even with $A = A_0 + DA \wedge e^{(k, \ell)}$ as in Theorem 3.4.

(1) Suppose $k \neq \ell$. Then $[A, A] = 0$ if and only if

$$[A_0, A_0] = \frac{2(\ell - k)}{n + k - \ell} DA \wedge A_0. \quad (28)$$

(2) For $k = \ell$ we have $[A, A] = 0$ if and only if

$$[A_0, A_0] = [DA, A_0] = 0. \quad (29)$$

Remark 3.9. We emphasize on the fact that in the first point, the vanishing of $[DA, A_0]$ is hidden in the stated equation (28). This is seen by taking the trace on both sides of (28) and then using (8) as well as (6).

The further specialization to $k = \ell = 2$ or $\ell = 2$ gives the decomposition result on quadratic Poisson structures as stated in [12] or [13].

For $k = 1$ and $\ell = 2$, i.e. for linear Poisson structures, the above result has been used in [11] for the classification of deformations of Lie-Poisson structures in dimension three.

At the end of this section we will give a remark on diffeomorphisms and their effect on the decomposition given in Theorem 3.4. A diffeomorphism which respects the polynomial structure as well as the degree has to be linear. For a linear diffeomorphism $L(x)^j = L_i^j x^i$ and a polynomial poly-vector field we write L_*A for the induced action. E.g. for a linear vector field $A \in \mathfrak{gl}_n$ we have $L_*A = L^{-1}AL$, in particular $L_*e^{(k,\ell)} = e^{(k,\ell)}$. The isomorphism (3) is compatible with this diffeomorphism in the sense that for poly-vector fields A and B we have

$$L^* \circ \Psi \circ L_*A(B) = \Psi(L_*A, L_*B) = \det L \Psi(A, B) = \det L \Psi(A)(B), \quad (30)$$

or $L^* \circ \Psi \circ L_* = \det L \Psi$. Furthermore, d commutes with L^* so that we get

$$DL_*A = \Psi^{-1}d\Psi L_*A = \det L L_*\Psi^{-1}d\Psi A = \det L L_*DA. \quad (31)$$

This yields

Proposition 3.10. *Let $A, \hat{A} \in \mathfrak{P}^{(k,\ell)}$ be polynomial poly-vector fields with trace decomposition $A = A_0 + B \wedge e$ and $\hat{A} = \hat{A}_0 + \hat{B} \wedge e$, respectively. Then A and \hat{A} are diffeomorphic if and only if there exist a linear diffeomorphism L such that $A_0 = L_*\hat{A}_0$ and $B = L_*\hat{B}$.*

4. APPLICATIONS TO POISSON AND JACOBI STRUCTURES

4.1. Poisson structures. Theorem 3.8 has a nice consequence in the case $k \geq 3$. Let $A \in \mathfrak{P}^{(k,2)}$ be a Poisson structure with non vanishing trace. Furthermore suppose that its trace free part A_0 is a Poisson structure, too. Due to the vanishing of the expression $DA \wedge A_0$, the trace free part has the form $A_0 = DA \wedge C$ for a vector field C . This motivates the next definition.

Definition 4.1. A (generalized) Poisson structure is called simple if its trace-free part is itself a (generalized) Poisson structure.

In the case of Poisson structures we may summarize

Proposition 4.2. *A simple homogeneous polynomial Poisson structure of degree ≥ 3 has rank two.*

Bi-vector fields in dimension two. For $V = \mathbb{R}^2$ we have $V^* \simeq V$ as representation spaces and $V_{m-1,1} \simeq S^m(V)$. In particular, $[A, A] = 0$ for all $A \in \mathfrak{P}^{(m,2)}(V)$. Sequence (12) is written as

$$S^m(V) \xrightarrow{D} S^m(V) \oplus S^{m-2}(V) \longrightarrow S^m(V)$$

and we have $\ker D = \{0\}$.

Let us consider $m = 2$. We have $\mathfrak{sl}_2\mathbb{R} \simeq S^2(V)$ via

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mapsto \begin{pmatrix} -b & a \\ a & c \end{pmatrix}$$

which is the $(\partial_1 \wedge \partial_2)$ -component of $\frac{1}{2} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \wedge e^{(1,1)}$. This yields the result mentioned in the introduction, namely that every quadratic Poisson structure is obtained by an element in $\Lambda^2 \mathfrak{gl}_2$. This, of course, may be generalized to arbitrary $m \geq 2$.

Proposition 4.3. *Every homogeneous polynomial Poisson structure of degree m in dimension two is obtained by*

$$S^{m-1}(V^*) \otimes V \supset S^m(V) \ni A \mapsto A \wedge e^{(m,2)} \in S^m(V^*) \otimes \Lambda^2(V).$$

Linear bi-vector fields. Consider linear bi-vectors and the splitting $\mathfrak{P}^{(1,2)} = V_{1,2} \oplus \Lambda^1 V$.

- (i) The bi-vector with trace $v \in \Lambda^1 V \simeq V$ is $A = \frac{1}{n-1} v \wedge \mathbb{1}$, i.e. $A^{ij}_k = \frac{2}{n-1} v^{[i} \delta_k^{j]}$. These coefficients give rise to a Lie algebra ($\hat{=}$ Poisson structure in the linear case) due to Corollary 3.7.
- (ii) In dimension $n = 3$ this may be expressed by the help of the cross product. Let $E = (E_1, E_2, E_3)$ be a basis of the Lie algebra. The bracket can be expressed as follows

$$\begin{aligned} [E_i, E_j] &= v^{[i} \delta_k^{j]} E_k = \frac{1}{2} (v^i \delta_k^j E_k - v^j \delta_k^i E_k) \\ &= \frac{1}{2} (v^i E_j - v^j E_i) = \frac{1}{4} \epsilon^{ijk} (v \times E)_k. \end{aligned}$$

E.g. $v = 2e_1$ gives

$$[E_1, E_2] = E_2, \quad [E_1, E_3] = E_3, \quad [E_2, E_3] = 0,$$

which is the Lie algebra $\mathfrak{g}_{(1,1)} := \mathbb{R} \times \mathbb{R}^2$ of rank 2. Furthermore, consider the Lie algebra $\mathfrak{g}_{(\alpha,\beta)}$ given by

$$[E_1, E_2] = \alpha E_2, \quad [E_1, E_3] = \beta E_3, \quad [E_2, E_3] = 0.$$

This algebra has structure constants $A^{12}_2 = \alpha$, $A^{13}_3 = \beta$ and trace $DA = (\alpha + \beta)e_1$. The decomposition (21) together with the discussion above yields

$$\begin{aligned} (A_0)^{12}_2 &= \frac{\alpha - \beta}{2}, & (A_0)^{13}_3 &= \frac{\beta - \alpha}{2}, \\ (A_1)^{12}_2 &= (A_1)^{13}_3 = \frac{\alpha + \beta}{2}. \end{aligned}$$

In particular $[A_0, A_0] = DA \wedge A_0 = 0$, i.e. we have a decomposition of $\mathfrak{g}_{(\alpha,\beta)}$ into a pair of Lie algebras $(\mathfrak{g}_{(\frac{\alpha-\beta}{2}, \frac{\beta-\alpha}{2})}, \mathfrak{g}_{(\frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2})})$.

- (iii) Consider a semisimple Lie algebra and its decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha} \mathfrak{g}_{\alpha}$. The roots $\alpha \in \mathfrak{h}^*$ obey $[H, X] = \alpha(H)X$ for all H in the Cartan algebra \mathfrak{h} and X in the one dimensional subspace \mathfrak{g}_{α} . The $\alpha(H)$ are the only non-vanishing diagonal elements of the structure constants and with $\alpha(H)$ the element $-\alpha(H)$ is present, too. So the sum over these elements vanish and so does the trace of \mathfrak{g} which is given by

$$D\mathfrak{g} = \sum_i \left(\sum_{\alpha} \alpha(H_i) \right) H_i$$

for a basis $\{H_i\}$ of \mathfrak{h} .

- (iv) Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be the vector space decomposition of the Lie algebra \mathfrak{g} into its semisimple and solvable parts with basis $\{E_i\}$ and $\{F_j\}$, respectively. We have $[\mathfrak{g}_0, \mathfrak{g}_1] \subset \mathfrak{g}_1$ so that the trace of \mathfrak{g} decomposes like

$$D\mathfrak{g} = D\mathfrak{g}_1 + \sum_i \text{tr}(\text{ad}E_i|_{\mathfrak{g}_1}) E_i. \quad (32)$$

A similar but not so strong decomposition of the trace may be obtained by the decomposition $\mathfrak{g} = \mathfrak{g}^{(1)} \oplus \mathfrak{g}/\mathfrak{g}^{(1)}$ with $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$ and $[\mathfrak{g}/\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] \subset \mathfrak{g}^{(1)}$. It is not so strong in the sense that $\mathfrak{g}_1 \subset \mathfrak{g}/\mathfrak{g}^{(1)}$ and $\mathfrak{g}_0 \supset \mathfrak{g}^{(1)}$.

4.2. Jacobi structures. A (generalized) Jacobi structure on the manifold M is by definition a pair $(\Lambda, E) \in \mathfrak{X}^{2\ell}(M) \times \mathfrak{X}^{2\ell-1}(M)$ which obeys

$$[\Lambda, E] = 0 \quad \text{and} \quad [\Lambda, \Lambda] = 2E \wedge \Lambda. \quad (33)$$

This is due to [15] but with the field E rescaled.

Definition 4.4. A Jacobi structure (Λ, E) is called k -homogeneous if $\Lambda \in \mathfrak{P}^{(k, 2\ell)}$ and $E \in \mathfrak{P}^{(k-1, 2\ell-1)}$ are homogeneous of degree k and $k-1$, respectively.

A k -homogeneous (generalized) Poisson structure Π and a k -homogeneous (generalized) Jacobi structure (Λ, E) are called associated, if

$$\Pi_0 = \Lambda_0 \quad \text{and} \quad E_0 = \frac{k-2\ell}{n+k-2\ell} (D\Lambda - D\Pi). \quad (34)$$

Theorem 4.5. (1) For every homogeneous Jacobi structure (Λ, E) of degree $k \neq 2\ell$ a k -homogeneous Poisson structure is defined via its decomposition with respect to Theorem 3.4

$$\Pi_0 = \Lambda_0 \quad \text{and} \quad D\Pi = D\Lambda + \frac{n+k-2\ell}{2\ell-k} E_0.$$

(2) Let $\Pi = \Pi_0 + D\Pi \wedge e^{(k, 2\ell)}$ be a k -homogeneous Poisson structure, $k \neq 2\ell$, which admits a splitting $D\Pi = F_0 + \tilde{E}_0$ such that there exists $\xi \in \mathfrak{P}^{(k-2, 2\ell-2)}$ with

$$\xi \wedge \Pi_0 = [\Pi_0, F_0] + F_0 \wedge \tilde{E}_0.$$

This defines a Jacobi structure given by

$$\Lambda = \Pi_0 + F_0 \wedge e^{(k, 2\ell)} \quad \text{and} \quad E = \frac{2\ell-k}{n+k-2\ell} (\tilde{E}_0 + \xi \wedge e^{(k, 2\ell)}).$$

(3) In the case $k = 2\ell$ a pair of associated Jacobi- and Poisson structure obeys $E_0 = 0$ and $D\Lambda = D\Pi + \eta$ for some η with $[\eta, \Lambda_0] = -DE \wedge \Lambda_0$.

Remark 4.6. The second item of the preceding theorem includes two special cases, namely two structures coming from $\xi = 0$. The first is the obvious one with $E = 0$, i.e. $F_0 = D\Lambda$, which yields the Jacobi structure

$$\Lambda = \Pi, \quad E = 0.$$

The second special case is $F_0 = 0$ with Jacobi structure given by

$$\Lambda = \Pi_0, \quad E = \frac{2\ell-k}{n+k-2\ell} D\Pi.$$

For $\ell = 1$ the latter is the structure also given in [16].

Proof. Let $\Lambda = \Lambda_0 + D\Lambda \wedge e$ and $E = E_0 + DE \wedge e$ form a k -homogeneous Jacobi structure with decompositions with respect to Theorem 3.4. We use the notations $e = e^{(k, 2\ell)}$, $e'' = e^{(2k, 4\ell)}$, and $\Delta = k - 2\ell$. We reformulate the Jacobi conditions

(33) in terms of the fields Λ_0 , E_0 , $D\Lambda$ and DE by using Theorem 3.5. We plug in $A = E$, $B = \Lambda$, $\ell' = 2\ell$, and $\Delta = \Delta' = k - 2\ell$. This yields

$$[E, \Lambda]_0 = [E_0, \Lambda_0] + \frac{\Delta}{n + \Delta} \left(E_0 \wedge D\Lambda + DE \wedge \Lambda_0 + ([E_0, D\Lambda] - [DE, \Lambda_0]) \wedge e'' \right)$$

and $D[E, \Lambda] = [E_0, D\Lambda] - [DE, \Lambda_0] = -D(E_0 \wedge D\Lambda + DE \wedge \Lambda_0)$. So the first Jacobi condition is written as

$$[E, \Lambda] = 0 \iff \begin{cases} [E_0, \Lambda_0] = -\frac{\Delta}{n + \Delta} (E_0 \wedge D\Lambda + DE \wedge \Lambda_0) & \text{if } \Delta \neq 0, \\ [E_0, \Lambda_0] = 0 \text{ and } [E_0, D\Lambda] - [DE, \Lambda_0] = 0 & \text{if } \Delta = 0. \end{cases}$$

We use (3.6) for $A = \Lambda$ as well as $E \wedge \Lambda = E_0 \wedge \Lambda_0 + (DE \wedge \Lambda_0 + E_0 \wedge D\Lambda) \wedge e$ and $D(E \wedge \Lambda) = DE \wedge \Lambda_0 + E_0 \wedge D\Lambda + [E, \Lambda]$. This yields for the second Jacobi condition

$$[\Lambda, \Lambda] = 2E \wedge \Lambda \iff \begin{cases} [\Lambda_0, \Lambda_0] + \frac{2\Delta}{n + \Delta} D\Lambda \wedge \Lambda_0 - \frac{2\Delta}{n + \Delta} [D\Lambda, \Lambda_0] \wedge e'' \\ = 2E_0 \wedge \Lambda_0 + \left(\frac{2\Delta}{n + \Delta} (DE \wedge \Lambda_0 + E_0 \wedge D\Lambda) - 2[E, \Lambda] \right) \wedge e'' \\ \text{and} \\ [D\Lambda, \Lambda_0] = -DE \wedge \Lambda_0 - E_0 \wedge D\Lambda - [E, \Lambda] \end{cases} \iff \begin{cases} [\Lambda_0, \Lambda_0] = 2(E_0 - \frac{\Delta}{n + \Delta} D\Lambda) \wedge \Lambda_0 - 2\frac{n + 2\Delta}{n + \Delta} [E, \Lambda] \wedge e'' \\ \text{and} \\ [D\Lambda, \Lambda_0] = -DE \wedge \Lambda_0 - E_0 \wedge D\Lambda - [E, \Lambda]. \end{cases}$$

So the two Jacobi conditions together translate as follows

$$[E, \Lambda] = 0 \quad \text{and} \quad [\Lambda, \Lambda] = 2E \wedge \Lambda \iff \begin{cases} [D\Lambda, \Lambda_0] = -DE \wedge \Lambda_0 - E_0 \wedge D\Lambda & (*) \\ \text{and} \\ [\Lambda_0, \Lambda_0] = 2(E_0 - \frac{\Delta}{n + \Delta} D\Lambda) \wedge \Lambda_0. & (**) \end{cases}$$

Suppose $\Delta \neq 0$, i.e. $k \neq 2\ell$. Then for $\Pi_0 := \Lambda_0$ and $D\Pi = D\Lambda - \frac{n + \Delta}{\Delta} E_0$ we have

$$[\Pi_0, \Pi_0] = -\frac{2\Delta}{n + \Delta} D\Pi \wedge \Pi_0,$$

i.e. (28). This proves the first item. The second item is almost the same up to the fact that we have to make sure the compatibility (*) of Λ and E_0 with the trace $DE = \xi$. In the third item the vanishing of E_0 follows from the definition. Therefore, $D\Pi$ and $D\Lambda$ are not related via E . Then the Jacobi structure $(\Lambda = \Lambda_0 + D\Lambda \wedge e, E = \xi \wedge e)$ and the Poisson structure $\Pi = \Lambda_0 + D\Pi \wedge e$ are associated only if the difference $\eta = D\Lambda - D\Pi$ obeys $[\eta, \Lambda_0] = -\xi \wedge \Lambda_0$, i.e. $D\Pi$ must coincide with some part of $D\Lambda$ which leaves Λ_0 invariant. \square

5. TOOLS FOR EXPLICIT CALCULATIONS

In this section we provide the tools to give the lists of simple cubic Poisson structures in dimension three and quadratic Poisson structures in dimension four. The latter supplements the list of quadratic Poisson structures in dimension three which may be found e.g. in [12] or [5].

5.1. Cubic Poisson structures in dimension three. Let Π be a cubic Poisson structure. Theorem 3.4 provides a decomposition of the bi-vector field Π into a pair (Π_0, A_0) . Here Π_0 denotes the trace free part of Π and depends on a bi-quadratic polynomial f , via the isomorphism (3). The vector field $A_0 = A - DA \wedge e^{(3,2)}$ is the trace free part of a quadratic vector field A and encodes the trace of Π .

The Poisson condition $[\Pi, \Pi] = 0$ may be translated to the function f and the vector field A_0 . We do this for the equivalent conditions $[\Pi_0, \Pi_0] = A_0 \wedge \Pi_0$ and $[A_0, \Pi_0] = 0$, compare Theorem 3.8. We recall the calculations in (37) and that $[\Pi_0, \Pi_0] = D(\Pi_0 \wedge \Pi_0)$ vanish in dimension three, i.e. Π is simple. So we are left with the conditions

$$A_0(f) = 0 \quad \text{and} \quad A_0 \wedge \Pi_0 = 0. \quad (35)$$

For Π_0 being a two-form, the last equality forces it to be of the form $\Pi_0 = A_0 \wedge C$ for a vector field C (compare Proposition 4.2) and the Poisson structure is of course of rank ≤ 2 . If we translate the Poisson condition with respect to these data, we get

$$L_C A_0 = 0, \quad (36)$$

where we used $[A_0, A_0] = 0$. For a trace free vector field C the operators D and L_C commute so that the condition may be read $L_C A = 0$.

To gain a list of all simple Poisson structures with trace free vector field $C \in \mathfrak{P}^{(1,1)}$ we restrict to its Jordan form due to Proposition 3.10. We write the components of the vector field $A = A^k \partial_k$ with respect to the basis of quadratic polynomials, i.e. $A^k = A_{11}^k x^2 + A_{22}^k y^2 + A_{33}^k z^2 + A_{12}^k xy + A_{13}^k xz + A_{23}^k yz$ or

$$A^k = (A_{11}^k, A_{22}^k, A_{33}^k, A_{12}^k, A_{13}^k, A_{23}^k)^T.$$

The bracket of A with the vector field $C = C_i^j x^i \partial_j$ is given by

$$[C, A]^k = C_i^j x^i \frac{\partial A^k}{\partial x^j} - C_j^k A^j.$$

Because of the restriction to the Jordan form of C , we will only need the following vectors for the calculations

$$(x \partial_x A^k, y \partial_y A^k, z \partial_z A^k, x \partial_y A^k, y \partial_z A^k, y \partial_x A^k) = \begin{pmatrix} 2A_{11}^k & 0 & 0 & A_{12}^k & 0 & 0 \\ 0 & 2A_{22}^k & 0 & 0 & A_{23}^k & A_{12}^k \\ 0 & 0 & 2A_{33}^k & 0 & 0 & 0 \\ A_{12}^k & A_{12}^k & 0 & 2A_{22}^k & A_{13}^k & 2A_{11}^k \\ A_{13}^k & 0 & A_{13}^k & A_{23}^k & 0 & 0 \\ 0 & A_{23}^k & A_{23}^k & 0 & 2A_{33}^k & A_{13}^k \end{pmatrix}.$$

The next list gives all pairs of trace less $(C, A_0) \in V_{1,1} \times V_{2,1} \subset \mathfrak{P}^{(1,1)} \times \mathfrak{P}^{(2,1)}$ which yield a simple Poisson structure

$$\Pi = A_0 \wedge (C + e^{(3,2)})$$

in the described way. This list is complete up to a permuting the variables.

$$[A] \quad C = \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} \quad \text{with } a_1 + a_2 + a_3 = 0.$$

$$[A.1] \quad a_1 a_2 a_3 \neq 0.$$

[A.1.1] $a_i \neq 2a_j$ for all $1 \leq i, j \leq 3$. In this case $A = 0$ is the only solution of $[C, A] = 0$.

[A.1.2] $a_2 = 2a_1$, i.e. $C = a \cdot \text{diag}(1, 2, -3)$. We have one non-zero solution of $[C, A] = 0$ and this solution is trace free. It is given by

$$A_0 = \alpha x^2 \partial_y.$$

[A.2] $a_1 = 0, a_2 \neq 0$. The space of solutions A is six-dimensional and we find a trace free basis

$$A_0 \in \text{span} \{ y^2 \partial_x, z^2 \partial_x, yz \partial_x, x^2 \partial_x - xy \partial_y - xz \partial_z, \\ x^2 \partial_x - 3xy \partial_y + xz \partial_z, x^2 \partial_x + xy \partial_y - 3xz \partial_z \}.$$

[A.3] $a_1 = a_2 = 0$. In this the case the Poisson structure is of the form $\Pi = A_0 \wedge e^{(3,2)}$ with an arbitrary trace free quadratic vector field A_0 .

$$[B] \quad C = \begin{pmatrix} a & 1 \\ & a \\ & & -2a \end{pmatrix}.$$

[B.1] $a \neq 0$. There is no non-zero solution of $[C, A] = 0$.

[B.2] $a = 0$. The space of solutions is eight dimensional and the projection onto the trace free part is six dimensional:

$$A_0 \in \text{span} \{ z^2 \partial_z - (xz \partial_x + yz \partial_y), 3xz \partial_z - (x^2 \partial_x + xy \partial_y), \\ x^2 \partial_y, z^2 \partial_y, xz \partial_y, x^2 \partial_z \}.$$

[C] $C = \begin{pmatrix} 0 & 1 \\ & 0 & 1 \\ & & & 0 \end{pmatrix}$. The dimension of the space of solutions of $[C, A] = 0$ is four. The projection on the trace free part is three dimensional and given by

$$A_0 \in \text{span} \{ x^2 \partial_z, xy \partial_z - x^2 \partial_y, xy \partial_y + x^2 \partial_x + 2y^2 \partial_z - 3xz \partial_z \}$$

$$[D] \quad C = \begin{pmatrix} a & b \\ -b & a \\ & & -2a \end{pmatrix} \quad \text{with } b \neq 0.$$

[D.1] $a \neq 0$. As before we have no solution in this generic case.

[D.2] $a = 0$. The solutions of $[C, A] = 0$ span a three dimensional subspace of $\mathfrak{P}^{(2,1)}$ which has a trace free basis. So the Poisson structures are built up by

$$A_0 \in \text{span} \{ (y^2 + x^2) \partial_z, z(y \partial_x - x \partial_y), z(x \partial_x + y \partial_y) \}.$$

5.2. Quadratic Poisson structures in dimension four. In this particular dimension we handle with quadratic bi-vectors and their trace given by linear vector fields. A Poisson structure Π is described by a pair (Π_θ, A) where Π_θ is a trace free Poisson structure and A is linear vector field which is compatible with Π_θ in the way that $L_A \Pi_\theta = [A, \Pi_\theta] = 0$ holds, see Theorem 3.8. The linear vector field encodes the trace of Π and so is trace free, i.e. $A \in \mathfrak{sl}_4 \mathbb{R}$. Furthermore, because Π_θ is trace free, there exists a cubic tri-vector field L such that $\Pi_\theta = DL$. If we consider the isomorphism (3) we may write $L = \Psi^{-1} \theta$ with a cubic one-form θ . The trace free bi-vector is then given by $\Pi_\theta = \Psi^{-1} d\theta$, see (4). This explains the notation for the trace free part of Π . The compatibility condition may be reformulated by

$$\begin{aligned} [A, \Pi_\theta] &= D(A \wedge \Pi_\theta) = \Psi^{-1} \circ d \circ \Psi(A \wedge \Pi_\theta) = \Psi^{-1} \circ d \circ \iota_A \circ \iota_{\Pi_\theta} \Psi \\ &= \Psi^{-1} \circ d \circ \iota_A d\theta = \Psi^{-1} (-\iota_A \circ d^2 \theta + L_A d\theta) = \Psi^{-1} \circ L_A d\theta \\ &= \Psi^{-1} \circ dL_A \theta = \Pi_{L_A \theta} \end{aligned} \quad (37)$$

and so reads as

$$(i) \quad L_A \theta = 0.$$

The Poisson condition on Π_θ translates to θ in the following way. The commutator of Π_θ with itself is a tri-vector field and via Ψ a one-form. We look at its components

$$(\Psi \circ [\Pi_\theta, \Pi_\theta])_m = (\Psi \circ D(\Pi_\theta \wedge \Pi_\theta))_m = \partial_m \Psi(\Pi_\theta \wedge \Pi_\theta).$$

The function $\Psi(\Pi_\theta \wedge \Pi_\theta)$ which is the only coefficient in $d\theta \wedge d\theta$ is a homogeneous bi-quadratic polynomial. Its derivative vanishes if and only if it vanishes itself. So the Poisson condition on Π_θ can be written as

$$(ii) \quad d\theta \wedge d\theta = 0.$$

To characterize all quadratic Poisson structures in the four-dimensional case we have to look for pairs of one-forms θ with cubic coefficients and trace free matrices A such that (i) and (ii) are satisfied. We write $\Pi_{\theta, A}$ for the resulting Poisson structure. We call a one-form and a matrix which fulfills condition (i) compatible. This is a linear condition on the coefficients of the 1-form. Because of remark 5.1 below, which is an easy consequence of Proposition 3.10 and (30) as well as (i) and (ii), we may restrict ourself to matrices A which are in Jordan form. Condition (ii) is non-linear and yields algebraic relations of degree two for the coefficients.

Remark 5.1. $\Pi_{\theta, A}$ and $\Pi_{\eta, B}$ are Poisson isomorphic if and only if there is a linear isomorphism L such that

$$\eta = \det L L^* \theta, \quad \text{and } B = LAL^{-1}.$$

For the explicit calculations in (i) we expand the cubic one-form in the form $\theta = \theta_k dx^k$ and

$$\theta_k = \sum_{0 \leq m \leq n \leq o \leq 3} \theta_{k; mno} x^m x^n x^o \quad (38)$$

with $(x^0, x^1, x^2, x^3) := (t, x, y, z)$. For $A \in \mathfrak{sl}_4 \mathbb{R}$ we have

$$L_A \theta_k = A^i_k \theta_i + A^i_j x^j \frac{\partial \theta_k}{\partial x^i} \quad (39)$$

and $L_A\theta = 0$ is a system on the 80 coefficients $\theta_{k;mno}$. We write the coefficients of the respective 1-forms as vectors in the basis of the cubic polynomials as given in (38). In view of (39) and the restriction to the Jordan form of the matrix A we only need the following vectors:

$$(\theta_k, t\partial_t\theta_k, x\partial_x\theta_k, y\partial_y\theta_k, z\partial_z\theta_k, t\partial_x\theta_k, x\partial_y\theta_k, y\partial_z\theta_k, x\partial_t\theta_k, z\partial_y\theta_k) =$$

$$\begin{pmatrix} \theta_{k;012} & \theta_{k;012} & \theta_{k;012} & \theta_{k;012} & 0 & 2\theta_{k;112} & 2\theta_{k;220} & \theta_{k;013} & 2\theta_{k;002} & 0 \\ \theta_{k;013} & \theta_{k;013} & \theta_{k;013} & 0 & \theta_{k;013} & 2\theta_{k;113} & \theta_{k;023} & 0 & 2\theta_{k;003} & \theta_{k;012} \\ \theta_{k;023} & \theta_{k;023} & 0 & \theta_{k;023} & \theta_{k;023} & \theta_{k;123} & 0 & 2\theta_{k;330} & 0 & 2\theta_{k;220} \\ \theta_{k;123} & 0 & \theta_{k;123} & \theta_{k;123} & \theta_{k;123} & 0 & 2\theta_{k;223} & 2\theta_{k;331} & \theta_{k;023} & 2\theta_{k;221} \\ \theta_{k;001} & 2\theta_{k;001} & \theta_{k;001} & 0 & 0 & 2\theta_{k;110} & \theta_{k;002} & 0 & 3\theta_{k;000} & 0 \\ \theta_{k;002} & 2\theta_{k;002} & 0 & \theta_{k;002} & 0 & \theta_{k;012} & 0 & \theta_{k;003} & 0 & 0 \\ \theta_{k;003} & 2\theta_{k;003} & 0 & 0 & \theta_{k;003} & \theta_{k;013} & 0 & 0 & 0 & \theta_{k;002} \\ \theta_{k;110} & \theta_{k;110} & 2\theta_{k;110} & 0 & 0 & 3\theta_{k;111} & \theta_{k;012} & 0 & 2\theta_{k;001} & 0 \\ \theta_{k;112} & 0 & 2\theta_{k;112} & \theta_{k;112} & 0 & 0 & 2\theta_{k;221} & \theta_{k;113} & \theta_{k;012} & 0 \\ \theta_{k;113} & 0 & 2\theta_{k;113} & 0 & \theta_{k;113} & 0 & \theta_{k;123} & 0 & \theta_{k;013} & \theta_{k;112} \\ \theta_{k;220} & \theta_{k;220} & 0 & 2\theta_{k;220} & 0 & \theta_{k;221} & 0 & \theta_{k;023} & 0 & 0 \\ \theta_{k;221} & 0 & \theta_{k;221} & 2\theta_{k;221} & 0 & 0 & 3\theta_{k;222} & \theta_{k;123} & \theta_{k;220} & 0 \\ \theta_{k;223} & 0 & 0 & 2\theta_{k;223} & \theta_{k;223} & 0 & 0 & 2\theta_{k;332} & 0 & 3\theta_{k;222} \\ \theta_{k;330} & \theta_{k;330} & 0 & 0 & 2\theta_{k;330} & \theta_{k;331} & 0 & 0 & 0 & \theta_{k;023} \\ \theta_{k;331} & 0 & \theta_{k;331} & 0 & 2\theta_{k;331} & 0 & \theta_{k;332} & 0 & \theta_{k;330} & \theta_{k;123} \\ \theta_{k;332} & 0 & 0 & \theta_{k;332} & 2\theta_{k;332} & 0 & 0 & 3\theta_{k;333} & 0 & 2\theta_{k;223} \\ \theta_{k;000} & 3\theta_{k;000} & 0 & 0 & 0 & \theta_{k;001} & 0 & 0 & 0 & 0 \\ \theta_{k;111} & 0 & 3\theta_{k;111} & 0 & 0 & 0 & \theta_{k;112} & 0 & \theta_{k;110} & 0 \\ \theta_{k;222} & 0 & 0 & 3\theta_{k;222} & 0 & 0 & 0 & \theta_{k;223} & 0 & 0 \\ \theta_{k;333} & 0 & 0 & 0 & 3\theta_{k;333} & 0 & 0 & 0 & 0 & \theta_{k;332} \end{pmatrix}$$

A careful examination yields 8 cases with 43 subcases in total. We will restrict here to three examples. The whole list may be found in [8].

- $A = \text{diag}(a_1, a_2, a_3, a_4)$, $|a_i|$ distinct, and no relation of the form $a_i = -3a_j$ holds for any i, j . The only 1-form θ for which (i) holds is

$$\theta = \alpha_0xyz dt + \alpha_1tyz dx + \alpha_2txz dy + \alpha_3txy dz.$$

In particular, (ii) is also satisfied. The associated Poisson structure Π_θ is

$$\begin{aligned} \Pi_\theta &= \alpha_{01}tx\partial_t \wedge \partial_x + \alpha_{02}ty\partial_t \wedge \partial_y + \alpha_{03}tz\partial_t \wedge \partial_z \\ &\quad + \alpha_{12}xy\partial_x \wedge \partial_y + \alpha_{13}xz\partial_x \wedge \partial_z + \alpha_{23}yz\partial_y \wedge \partial_z. \end{aligned}$$

- $A = \begin{pmatrix} a & 1 & & \\ & a & & \\ & & -a & 1 \\ & & & -a \end{pmatrix}$, $a \neq 0$. The compatible 1-form is

$$\begin{aligned} \theta &= ty(\alpha_1y dt + \alpha_2t dy) + (tz - xy)(\delta_1(z dt - y dx) + \delta_2(t dz - x dy)) \\ &\quad + ty(\beta_1(z dt - y dx) + \beta_2(t dz - x dy)) + (tz - xy)(\gamma_1y dt + \gamma_2t dy) \end{aligned}$$

and condition (ii) yields

$$\delta_1 - \delta_2 = 0, \quad (\beta_1 - \beta_2)((\beta_1 - \beta_2) + (\gamma_1 + \gamma_2)) = 0.$$

The first equation makes the second summand of θ a total derivative, such that it does not enter into the Poisson structure. A sample structure is

given by the further choice $\beta_1 = \beta_2$, $\gamma_1 = \gamma_2$, i.e.

$$\Pi_\theta = \alpha ty \partial_x \wedge \partial_z + \beta \Psi(d(ty) \wedge d(tz - xy)).$$

$$\bullet A = \begin{pmatrix} a & d & & & & \\ -d & a & & & & \\ & & -a & e & & \\ & & -e & -a & & \end{pmatrix}, de \neq 0, d^2 \neq e^2. \text{ The compatible 1-form is}$$

$$\theta = (y^2 + z^2)(\alpha_1(x dt - t dx) + \alpha_2(t dt + x dx)) \\ + (t^2 + x^2)(\beta_1(z dy - y dz) + \beta_2(y dy + z dz))$$

which also satisfies (ii). A sample structure is given by the choice $\alpha_2 = \beta_2$ which combines the respective summands to a total derivative. The further choice $\alpha_1 = \beta_1$ yields

$$\Pi_\theta = (t^2 + x^2)\partial_t \wedge \partial_x + (y^2 + z^2)\partial_y \wedge \partial_z \\ + (ty + xz)(\partial_x \wedge \partial_y - \partial_t \wedge \partial_z) + (tz - xy)(\partial_t \wedge \partial_y + \partial_x \wedge \partial_z).$$

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