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CR-SUBMANIFOLDS OF LORENTZIAN MANIFOLDS

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Abstract. This paper is about the talk given in the VIth Geometry Symposium in Bursa, Turkey, on July 2008. We present the notion of CR-submanifolds of a Lorentzian almost contact manifold, study their principal characteristics and the particular cases in which the manifold is a Lorentzian Sasakian manifold or a Lorentzian Sasakian space form. We also present some new results for CR-submanifolds of a generalized Lorentzian Sasakian space form.

1. Introduction

CR-submanifolds were introduced by A. Bejancu in [5]. Given a submanifold, M, of a Kaehler manifold, $\widetilde{N}(J,g)$, for every tangent vector, $X \in TM$, let us write $JX = TX + NX$, where TX and NX are the tangent and normal component respectively. M is called a CR-submanifold if TM admits a decomposition in two distributions, $TM = D \oplus D^{\perp}$, where D is invariant, $JD \subseteq D$, and $D^{\perp}$ is anti-invariant, $JD^{\perp} \subseteq T^{\perp}M$. These submanifolds generalized both invariant and anti-invariant ones.

D.E. Blair and B.Y. Chen introduced the notion of CR-submanifold of an Hermitian manifold, and many other authors have amplified this study by considering different structures on the ambient manifold, as M. Barros and F. Urbano dealing with generalized complex space forms, [4], or considering other decompositions of the tangent bundle as generic and skew CR submanifolds, [17].

For a contact Riemannian manifold, $\widetilde{M}(\phi,\eta,\xi,g)$, A. Bejancu and N. Papaghiuc, [6], consider a submanifold M tangent to ξ, they called it semi-invariant if $TM = D \oplus D^{\perp} \oplus <\xi>$, where the first distribution is invariant and the second anti-invariant. M. Kobayashi defined a CR-submanifold of a Sasakian manifold if $TM = D \oplus D^{\perp}$, [14]. Obviously both definitions are the same for submanifolds tangent to ξ. From now on we only consider submanifolds tangent to ξ.

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Also A. Bejancu and N. Papaghiuc studied CR-submanifolds of a Sasakian space form, [7] while other authors have leaded with other decompositions of the tangent bundle like semi-slant submanifolds and generalized CR-submanifolds, [9] and [15].

On semi-defined geometry, Kalpana and G. Guha initiated the study of semi-invariant submanifolds of a Lorentzian para Sasakian manifold, [12]. Furthermore, H. Gill and K.K. Dube have recently introduced generalized CR-submanifolds of a trans Lorentzian Sasakian manifold,[11], and of a trans hyperbolic Sasakian manifold, [10]. Let us remember this structures.

First, an odd dimensional manifold $\widetilde{M}^{2n+1}$ is called a Lorentzian almost para contact manifold if it is doted with a structure (ϕ, ξ, η, g) with

$$\begin{aligned} &\phi^2 X = X + \eta(X)\xi, && \eta(\xi) = -1,\\ &g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y,) && \eta(X) = g(X,\xi), \end{aligned}$$

and it is called Lorentzian para Sasakian if $(\widetilde{\nabla}_X \phi)Y = g(X,Y) + \eta(Y)X + 2\eta(X)\eta(Y)\xi$.

$(\widetilde{M}^{2n+1}, \phi, \xi, \eta, g)$ is called a hyperbolic almost contact manifold if

$$\begin{aligned} &\phi^2 X = X + \eta(X)\xi, && \eta(\xi) = -1,\\ &g(\phi X, \phi Y) = -g(X,Y) - \eta(X)\eta(Y), && \eta(X) = g(X,\xi). \end{aligned}$$

Our purpose is to define CR-submanifolds of a Lorentzian almost contact manifold, that is a manifold with an almost contact structure and a compatible Lorentzian metric, $(\widetilde{M}^{2n+1}, \phi, \xi, \eta, g)$,

$$\begin{aligned} &\phi^2 X = -X + \eta(X)\xi, && \eta(\xi) = 1,\\ &g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), && \eta(X) = -g(X,\xi). \end{aligned} \tag{1.1}$$

It is Lorentzian Sasakian if

$$(\widetilde{\nabla}_X \phi)Y = -g(X,Y)\xi - \eta(Y)X. \tag{1.2}$$

After the introduction, Section 2 contains some preliminaries results, in Section 3, we define CR-submanifolds of a Lorentzian almost contact manifold, and study the integrability of some distributions. In Section 4, we characterized CR-submanifolds of Lorentzian Sasakian space forms, and we do the same with generalized Lorentzian Sasakian space forms in Section 5. Most of the results of Section 3 and 4 appear in an earlier paper [1].

2. **Preliminaries**

Let us consider a submanifold M of a Lorentzian almost contact metric manifold $(\widetilde{M}, \phi, \xi, \eta, g)$, tangent to the structure vector field ξ. Put $\phi X = TX + NX$ for any tangent vector field X, where TX (resp. NX) denotes the tangential (resp. normal) component of ϕX. Similarly, $\phi V = tV + nV$ for any normal vector field V with tV tangent and nV normal to M.

Given a submanifold M of a Lorentzian almost contact manifold $(\widetilde{M}, \phi, \xi, \eta, g)$, we also use g for the induced metric on M.

We denote by $\widetilde{\nabla}$ the Levi-Civita connection on $\widetilde{M}$ and by ∇ the induced Levi-Civita connection on M. Thus, the Gauss and Weingarten formulas are respectively given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.1}$$

$$\widetilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{2.2}$$

for vector fields X, Y tangent to M and a vector field V normal to M, where h denotes the second fundamental form, $\nabla^\perp$ the normal connection and A_V the shape operator in the direction of V. The second fundamental form and the shape operator are related by

$$g(h(X,Y),V) = g(A_V X, Y). \tag{2.3}$$

The submanifold M is said to be *totally geodesic* if h vanishes identically.

3. **CR-submanifolds, example and integrability conditions**

In this section, we introduce the notion of CR-submanifold of a Lorentzian almost contact manifold. This definition generalizes the notions of both invariant and anti-invariant submanifolds and it is the equivalent one to CR-submanifolds on the Riemannian setting. We characterize the anti-invariant case and present different examples of CR-submanifolds in a Lorentzian almost contact manifold. Finally, we study the integrability of all the distributions involved in the definition of CR-submanifolds.

A submanifold, M, of a Lorentzian almost contact manifold, $(\widetilde{M}^{2n+1}, \phi, \eta, \xi, g)$, is called a *CR-submanifold* if $TM = D \oplus D^\perp \oplus < \xi >$, with D an invariant distribution and $D^\perp$ an anti-invariant distribution.

Note that ξ is a timelike vector field and all vector fields in $D \oplus D^\perp$ are spacelike.

A CR-submanifold is also characterized by the decomposition of the normal bundle.

Proposition 3.1. [1] *Let M be a submanifold of a Lorentzian almost contact manifold $(\widetilde{M}, \phi, \xi, \eta, g)$. Then, M is a CR-submanifold if and only if $T^\perp M = \overline{D} \oplus \overline{D}^\perp$; where $\overline{D}$ is the maximal anti-invariant space in $T^\perp M$ and $\overline{D}^\perp$ is its orthogonal complement on $T^\perp M$.*

A CR-submanifold is known to be *invariant*, *anti-invariant* and *proper* if $D^\perp = 0$, $D = 0$ and $D \neq 0 \neq D^\perp$ respectively. And it is called *vertical proper* if $\overline{D} \neq 0 \neq \phi D^\perp$.

We can prove the next proposition about the distribution D.

Proposition 3.2. [1] *Let M be a CR-submanifold of a normal Lorentzian Sasakian manifold. Then, D is even dimensional.*

The anti-invariant case is characterized in the next result.

Theorem 3.1. [1] *Let M be a submanifold of a normal Lorentzian Sasakian manifold. M is anti-invariant if and only if $\nabla T = 0$.*

The next theorem, a Lorentzian version of the ones offered in [9], helps us to construct a number of examples of CR-submanifolds.

Theorem 3.2. [1] *Let M be a CR-submanifold of an almost Hermitian manifold $(\widetilde{N}^{2n}, G, J)$. Then, $M \times \mathbf{R}$ is a CR-submanifold of $(N^{2n} \times \mathbf{R}, \phi, \xi, \eta, g)$.*

But we can also present some other direct examples, they are based on semi-slant submanifolds of a Riemannian Sasakian manifold given on [9]. Let us consider on $\mathbf{R}^{2m+1}$ the following normal Lorentzian Sasakian structure (ϕ_0, ξ, η, g), given by

$$\eta = \frac{1}{2}\left(dz - \sum_{i=1}^{m} y^i dx^i\right), \qquad \xi = \frac{\partial}{\partial z},$$

$$g = -\eta \otimes \eta + \frac{1}{4} \sum_{i=1}^{m} (dx^i \otimes dx^i + dy^i \otimes dy^i),$$

$$\phi_0 \left(X_i \frac{\partial}{\partial x^i} + Y_i \frac{\partial}{\partial y^i} + Z \frac{\partial}{\partial z} \right) = \sum_{i=1}^{m} \left(Y_i \frac{\partial}{\partial x^i} - X_i \frac{\partial}{\partial y^i} \right) + \sum_{i=1}^{m} Y_i y^i \frac{\partial}{\partial z},$$

where $\{x^i, y^i, z\}$, $i = 1, ..., m$ are the cartesian coordinates.

The first example is with odd dimensional $D^{\perp}$:

Example 3.1. [1] The equation $x(u, v, w, s, t) = 2(u, 0, w, 0, v, 0, 0, s, t)$, defines a CR-submanifold in $\mathbf{R}^9$ with its normal Lorentzian Sasakian structure, (ϕ_0, ξ, η, g). To prove this fact, we take the orthogonal basis

$$e_1 = 2\left(\frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial z} \right), \quad e_2 = 2\frac{\partial}{\partial y_1}, \quad e_3 = 2\left(\frac{\partial}{\partial x_3} + y_3 \frac{\partial}{\partial z} \right),$$

$$e_4 = 2\frac{\partial}{\partial y_4}, \quad e_5 = 2\frac{\partial}{\partial z} = \xi,$$

and define the distributions $D = < e_1, e_2 >$, and $D^{\perp} = < e_3, e_4 >$. It is clear that $TM = D \oplus D^{\perp} \oplus < \xi >$.

And the second one with even dim$D^{\perp}$:

Example 3.2. [1] The equation $x(u, v, s, t) = 2(u, v, s, 0, t)$, defines a CR-submanifold in $\mathbf{R}^5$ with its normal Lorentzian Sasakian structure (ϕ_0, ξ, η, g). In this case, $TM = D \oplus D^{\perp} \oplus < \xi >$, just taking

$$D = \text{Span}\left\{ 2\left(\frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial z} \right), 2\frac{\partial}{\partial y_1} \right\} \quad \text{and} \quad D^{\perp} = \text{Span}\left\{ 2\left(\frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial z} \right) \right\}.$$

We can state the following theorems about the integrability of the distributions involved in the definition of a CR-submanifold.

Theorem 3.3. [1] *Let M be a CR-submanifold of a Lorentzian almost contact manifold with $D \neq 0$. Then D and $D \oplus D^{\perp}$ are never integrable.*

Theorem 3.4. [1] *Let M be a CR-submanifold of a normal contact Lorentzian manifold. Then,*

i) *$D^{\perp}$ is always integrable,*
ii) *$D^{\perp} \oplus < \xi >$ is always integrable,*
iii) *$D \oplus < \xi >$ is integrable if and only if $h(X, TY) - h(Y, TX) = 0$ for all $X, Y \in D \oplus < \xi >$.*

Studying the last condition on the above theorem we get this result, which was suggested by Prof. B.Y. Chen during the talk.

Theorem 3.5. *Let M be a CR-submanifold of a normal Lorentzian Sasakian manifold. If $h(X, TY) - h(Y, TX) = 0$ for all $X, Y \in D \oplus < \xi >$, then $D \oplus < \xi >$ is minimal.*

Proof. As we prove in Theorem 3.2, D is even dimensional. For every $X \in D$, $\phi X = TX \in D$, so we can choose an orthonormal basis on $D \oplus < \xi >$: $\{e_1, ..., e_{2r+1}\}$ with $e_{r+i} = Te_i$ for $i = 1, ..., r$ and $e_{2r+1} = \xi$.

Because of the condition, $h(X, TY) = h(Y, TX)$, so

$$h(TX, TY) = h(Y, T^2 X) = -h(Y, X).$$

And for a Lorentzian Sasakian manifold, $\widetilde{\nabla}_X \xi = \phi X$, so $h(X,\xi) = NX$ and $h(\xi,\xi) = N\xi = 0$.

So we have prove that $h(Te_i, Te_i) = -h(e_i, e_i)$ and $h(\xi,\xi) = 0$. So $\sum_{i=1}^{2r+1} h(e_i, e_i) = 0$ and $D \oplus < \xi >$ is minimal. □

4. CR-submanifolds of Lorentzian Sasakian space forms

A *Lorentzian Sasakian space form* is a Lorentzian Sasakian manifold with constant ϕ-sectional curvature. We can characterize proper CR-submanifolds of a normal contact Lorentzian manifold with constant ϕ-sectional curvature. The following theorems are equivalent for normal Lorentzian Sasakian space forms to the ones proved by M. M. Tripathi, [18], for generalized complex space forms and [16] for Sasakian space foms.

We use the expression of the Riemann curvature tensor of a Lorentzian contact manifold with constant ϕ-sectional curvature k given by T. Ikawa in [13]:

$$\begin{aligned} \widetilde{R}(X,Y)Z &= \frac{k-3}{4}\{g(Y,Z)X - g(X,Z)Y\}+ \\ &+\frac{k+1}{4}\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\}+ \\ &+\frac{k+1}{4}\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}. \end{aligned} \tag{4.1}$$

Theorem 4.1. [1] *Let M be a submanifold tangent to ξ of a Lorentzian Sasakian space form $\widetilde{M}(k)$ with $k \neq -1$. Then, M is a vertical proper CR-submanifold if and only if the maximal invariant subspaces $D_x = T_xM \cap \phi(T_xM)$, $x \in M$, define a non-trivial subbundle D of TM such that*

$$\widetilde{R}(D, D, D^\perp, D^\perp) = 0,$$

where $D^\perp$ denotes the non-trivial orthogonal complementary subbundle of D in TM.

Theorem 4.2. [1] *Let M be a submanifold tangent to ξ of a Lorentzian Sasakian space form $\widetilde{M}(k)$ with $k \neq 0$. Then, M is a proper CR-submanifold if and only if the maximal anti-invariant subspaces $D_x^\perp$ of T_xM, $x \in M$, define a non-trivial subbundle $D^\perp$ of TM such that*

$$\widetilde{R}(D, \phi D, \overline{D}, D) = 0,$$

where D denotes the orthogonal complementary subbundle of $D^\perp \oplus < \xi >$ in TM, and $\overline{D}$ denotes the orthogonal complementary of $\phi D^\perp$ in $T^\perp M$.

Using the same reasoning but for distributions on the normal bundle we get some results.

Theorem 4.3. [1] *Let M be a submanifold tangent to ξ of a Lorentzian Sasakian space form $\widetilde{M}(k)$ with $k \neq -1$. Then M is a vertical proper CR-submanifold if and only if the maximal invariant subspaces $\overline{D}_x = T_x^\perp M \cap \phi(T_x^\perp M)$, $x \in M$, of $T_x^\perp M$ define a non-trivial subbundle $\overline{D}$ of $T^\perp M$ such that*

$$\widetilde{R}(\overline{D}, \overline{D}, \overline{D}^\perp, \overline{D}^\perp) = 0,$$

where $\overline{D}^{\perp}$ *denotes the non-trivial orthogonal complementary subbundle of* $\overline{D}$ *in* $T^{\perp}M$.

And also,

Theorem 4.4. [1] *Let* M *be a submanifold tangent to* ξ *of a Lorentzian Sasakian space form* $\widetilde{M}(k)$ *with* $k \neq 0$. *Then,* M *is a proper CR-submanifold if and only if the maximal anti-invariant subspaces* $\overline{D}_x^{\perp}$ *of* $T_x^{\perp}M$, $x \in M$, *define a non-trivial subbundle* $\overline{D}^{\perp}$ *of* $T^{\perp}M$ *such that*

$$\widetilde{R}(\overline{D}, \phi\overline{D}, D, \overline{D}) = 0,$$

where D *denotes the non-trivial orthogonal complementary subbundle of* $\phi D^{\perp} \oplus <\xi>$ *in* TM, *and* $\overline{D}$ *denotes the non-trivial orthogonal complementary of* $\overline{D}^{\perp}$ *in* $T^{\perp}M$.

5. CR-submanifolds of generalized Lorentzian Sasakian space forms

A Lorentzian almost contact manifold is called *generalized Lorentzian Sasakian space form* if it has got pointwise constant ϕ-sectional curvature, [2]. Such a submanifold is characterized by the expression of the Riemann curvature tensor:

$$\begin{aligned} \widetilde{R}(X,Y)Z \ = \ & f_1\{g(Y,Z)X - g(X,Z)Y\}+ \\ & +f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\}+ \\ & +f_3\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{5.1}$$

for certain functions f_1, f_2 and f_3. We denote this submanifolds by $\widetilde{M}(f_1, f_2, f_3)$.

As we did for Lorentzian Sasakian space form we characterized CR-submanifolds of generalized Lorentzian Sasakian space forms because of the Riemann curvature tensor and its action over some distributions.

Theorem 5.1. *Let* M *be a submanifold tangent to* ξ *of a generalized normal Lorentzian Sasakian space form* $\widetilde{M}(f_1, f_2, f_3)$ *with* $f_2 \neq 0$. *Then,* M *is a vertical proper CR-submanifold if and only if the maximal invariant subspaces* $D_x = T_xM \cap \phi(T_xM)$, $x \in M$, *define a non-trivial subbundle* D *of* TM *such that*

$$\widetilde{R}(D, D, D^{\perp}, D^{\perp}) = 0,$$

where $D^{\perp}$ *denotes the non-trivial orthogonal complementary subbundle of* D *in* TM.

Proof. Let M be a CR-submanifold, in virtue of (5.1), for every $X, Y \in D$ and $Z, V \in D^{\perp}$ we get $R(X, Y, Z, W) = 0$.

For the converse, let suppose that the maximal invariant subspace and its complementary subbundle in TM verify the equation above. For all $X \in D$ and $Z, W \in D^{\perp}$,

$$\widetilde{R}(X, \phi X, Z, W) = -2f_2 g(\phi X, \phi X)g(\phi Z, W) = -2f_2 g(X, X)g(\phi Z, W) = 0.$$

Since X is space-like and $f_2 \neq 0$, $g(\phi Z, W) = 0$ and $\phi D^{\perp}$ is orthogonal to $D^{\perp}$.

Moreover $\phi D^{\perp}$ is orthogonal to D, $g(X, \phi Z) = -g(\phi X, Z) = 0$ because D is an invariant distribution, and of course $\phi D^{\perp}$ is orthogonal to $<\xi>$. Therefore, $\phi D^{\perp} \subseteq TM^{\perp}$, and then $D^{\perp}$ is anti-invariant. So, TM admits a decomposition $D \oplus D^{\perp} \oplus <\xi>$, with both an invariant and an anti-invariant distribution, and M is a CR-submanifold. □

Theorem 5.2. *Let M be a submanifold tangent to ξ of a generalized normal Lorentzian Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$ with $f_1 \neq -3f_2$. Then, M is a proper CR-submanifold if and only if the maximal anti-invariant subspaces $D_x^\perp$ of T_xM, $x \in M$, define a non-trivial subbundle $D^\perp$ of TM such that*

$$\widetilde{R}(D, \phi D, \overline{D}, D) = 0,$$

where D denotes the orthogonal complementary subbundle of $D^\perp \oplus < \xi >$ in TM, and $\overline{D}$ denotes the orthogonal complementary of $\phi D^\perp$ in $T^\perp M$.

Proof. The direct assertion it is a straightforward computation. For the converse, let suppose that the maximal anti-invariant subbundle of T_xM and its complementary verify the equation above. Then,

$$0 = \widetilde{R}(X, \phi X, V, X) = (3f_2 + f_1)g(X, X)g(\phi X, V),$$

for $X \in D$, $V \in \overline{D}$. First, as $3f_2 + f_1 \neq 0$, $g(\phi X, V) = 0$ for all $X \in D$ and $V \in \overline{D}$, that is ϕD is orthogonal to $\overline{D}$. Since $D^\perp$ is anti-invariant, for every $X \in D$, $Z \in D^\perp$, $g(\phi X, Z) = -g(X, \phi Z) = 0$, so ϕD is orthogonal to $D^\perp$. And by hypothesis, D is orthogonal to ξ. Therefore, we deduce that $\phi D \subset TM$, and then $\phi D = D$. That is, D is invariant and M is a CR-submanifold. □

In virtue of Proposition 3.1 a CR-submanifold is characterized by the decomposition of the normal bundle. So again studying the relation between $\widetilde{R}$ and the distributions of the normal bundle we get some results.

Theorem 5.3. *Let M be a submanifold tangent to ξ of a generalized normal Lorentzian Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$ with $f_2 \neq 0$. Then M is a vertical proper CR-submanifold if and only if the maximal invariant subspaces $\overline{D}_x = T_x^\perp M \cap \phi(T_x^\perp M)$, $x \in M$, of $T_x^\perp M$ define a non-trivial subbundle $\overline{D}$ of $T^\perp M$ such that*

$$\widetilde{R}(\overline{D}, \overline{D}, \overline{D}^\perp, \overline{D}^\perp) = 0,$$

where $\overline{D}^\perp$ denotes the non-trivial orthogonal complementary subbundle of $\overline{D}$ in $T^\perp M$.

Proof. If M is a CR-submanifold, let be $N, U \in \overline{D}$ and $V, W \in \overline{D}^\perp$. Then

$$R(N, U, V, W) =$$

$$= f_2(g(\phi U, V)g(\phi N, W) - g(N, \phi V)g(\phi U, W) + 2g(N, \phi U)g(\phi V, W)) = 0,$$

because $g(\phi U, V) = g(\phi N, V) = g(\phi V, W) = 0$.

Conversely, let be $\overline{D}$ and $\overline{D}^\perp$ be on the hypothesis. For $N \in \overline{D}$ and $U, V \in \overline{D}^\perp$ using (5.1):

$$0 = \widetilde{R}(N, \phi N, U, V) = -f_2 g(N, N)g(\phi U, V).$$

Since $f_2 \neq 0$ and N is space-like, it must be $g(\phi U, V) = 0$, so $\phi\overline{D}^\perp$ is orthogonal to $\overline{D}^\perp$. But always $\phi\overline{D}^\perp$ is orthogonal to $\overline{D}$, because $g(\phi U, N) = -g(U, \phi N) = 0$. Then $\phi\overline{D}^\perp \subset TM$ and it is anti-invariant. Therefore, $T^\perp M$ admits a decomposition $\overline{D} \oplus \overline{D}^\perp$ where one is an invariant distribution and the other is an anti-invariant one, and that implies, by virtue of Proposition 3.1, that M is a CR-submanifold. □

Theorem 5.4. *Let M be a submanifold tangent to ξ of a generalized normal Lorentzian Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$ with $f_1 \neq -3f_2$. Then, M is a proper CR-submanifold if and only if the maximal anti-invariant subspaces $\overline{D}_x^{\perp}$ of $T_x^{\perp}M$, $x \in M$, define a non-trivial subbundle $\overline{D}^{\perp}$ of $T^{\perp}M$ such that*

$$\widetilde{R}(\overline{D}, \phi\overline{D}, D, \overline{D}) = 0,$$

where D denotes the non-trivial orthogonal complementary subbundle of $\phi D^{\perp} \oplus <\xi>$ in TM, and $\overline{D}$ denotes the non-trivial orthogonal complementary of $\overline{D}^{\perp}$ in $T^{\perp}M$.

Proof. It is just a simple computation to prove the direct assertion. Conversely, suppose that both distributions, $\overline{D}$ and $\overline{D}^{\perp}$, satisfy the equation above, let us prove that $\overline{D}^{\perp}$ is anti-invariant. Then, in virtue of Proposition 3.1, M is a CR-submanifold.

Let be $X \in D$ and $V \in \overline{D}$,

$$0 = \widetilde{R}(V, \phi V, X, V) = (f_1 + 3f_2)g(V, V)g(\phi V, X).$$

As $f_1 \neq -3f_2$ and V is a space-like vector field, this implies $g(\phi V, X) = 0$ and then $\phi\overline{D} \perp D$. Of course, $\phi\overline{D}$ is orthogonal to $\overline{D}$ and ξ. Finally, it is also orthogonal to $\phi\overline{D}^{\perp}$, because, given $W \in \overline{D}^{\perp}$, it is $g(\phi V, \phi W) = g(V, W) = 0$. Thus, $\phi\overline{D} \subseteq \overline{D}$, and then invariant. Therefore, $T^{\perp}M = \overline{D} \oplus \overline{D}^{\perp}$, one is invariant and the other anti-invariant, which the proof finishes. □

Finally, we present an example of a CR-submanifold of a generalized Lorentzian Sasakian space form. The construction is similar to the one made in Theorem 3.2 but now using warped products. First we remember how to obtain an example of generalized Lorentzian Sasakian space form.

Theorem 5.5. [2] *Let $\widetilde{N}^{2n}(F_1, F_2)$ be a generalized complex space form. Then the warped product $\widetilde{M}^{2n+1} = \mathbf{R} \times_f \widetilde{N}$, endowed with the Lorentzian almost contact structure (ϕ, ξ, η, g_f), is a generalized Lorentzian Sasakian space form, $\widetilde{M}(f_1, f_2, f_3)$, with functions*

$$f_1 = \frac{(F_1 \circ \pi) + f'^2}{f^2}, \quad f_2 = \frac{(F_2 \circ \pi)}{f^2}, \quad f_3 = \frac{(F_1 \circ \pi) + f'^2}{f^2} - \frac{f''}{f}.$$

Theorem 5.6. *Let M be a CR-submanifold of an almost Hermitian manifold $(\widetilde{N}^{2n}, G, J)$. Then, $M \times_f \mathbf{R}$ is a CR-submanifold of $(\widetilde{N}^{2n} \times_f \mathbf{R}, \phi, \xi, \eta, g)$.*

Proof. TM can be decompose in $D \oplus D^{\perp}$ with D invariant and $D^{\perp}$ anti-invariant. Therefore, $T(M \times_f \mathbf{R}) = D \oplus D^{\perp} \oplus <\xi>$ with $D = \{(U, 0)/U \in D\}$ and $D^{\perp} = \{(V, 0)/V \in D^{\perp}\}$. It only rests proving that D is invariant and $D^{\perp}$ is anti-invariant.

First, for any $(U, V) \in D$

$$\phi(U, 0) = (JU, 0) \in D$$

so D is invariant. Moreover, $\phi(V, 0) = (JV, 0)$ is normal to $T(M \times_f \mathbf{R})$ so $D^{\perp}$ is anti-invariant. □

References

[1] Alegre, P., CR-submanifolds of normal Lorentzian Sasakian manifolds, Submitted.
[2] Alegre, P. and Carriazo, A., Lorentzian Sasakian manifolds with constant pointwise ϕ-sectional curvature, Submitted.
[3] Alegre, P., Blair, D.E. and Carriazo, A., Generalized Sasakian-space-forms, Israel J. Math., 141(2004), 157-183.
[4] Barros, M. and Urbano, F., CR-submanifolds of generalized complex space forms, An. Şti. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat., 25(1979), no. 2, 355-363.
[5] Bejancu, A., CR-submanifolds of a Kaehler manifold. I., Proc. Amer. Math. Soc., 69(1978), no. 1, 135-142.
[6] Bejancu, A. and Papaghiuc, N., Semi-invariant submanifolds of a Sasakian manifold, An. Şti. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat., 27, s.1 (1981), 163-170.
[7] Bejancu, A. and Papaghiuc, N., Semi-invariant submanifolds of a Sasakian space form, Colloq. Math., 48(1984), 77-88.
[8] Blair, D.E. and Chen, B.Y., On CR-submanifolds of Hermitian manifolds, Israel J. Math., 34(1979), no. 4, 353-363.
[9] Cabrerizo, J.L., Carriazo, A., Fernández, L.M. and Fernández, M., Semi-Slant submanifolds in Sasakian manifolds, Geometriae Dedicata, 78(1999), 183-199.
[10] Gill, H. and Dube, K.K., Generalized CR-submanifolds of the trans hyperbolic Sasakian manifold, Demonstratio Math., 38, no.4, (2005), 953-960.
[11] Gill, H. and Dube, K.K., Generalized CR-submanifolds of a trans Lorentzian para Sasakian manifold, Proc. Natl. Acad. Sci. India Sect. A, 76(A), II, (2006), 119-124.
[12] Kalpana and Guha, G., Semi-invariant submanifold of a Lorentzian para-Sasakian manifold, Ganit, 13(1993), no. 1-2, 71-76.
[13] Ikawa, T., Spacelike maximal surfaces with constant scalar normal curvature in a normal contact Lorentzian manifold, Bull. Malaysian Math. Soc., 21(1998), 31-36.
[14] Kobayashi, M., CR submanifolds of a Sasakian manifold, Tensor (N.S.), 35, no. 3, 297-307.
[15] Papaghiuc, N., Almost semi-invariant submanifolds in Sasakian Space forms, An. şt. Univ. Iaşi., 29(1983), 5-10.
[16] Papaghiuc, N., Some theorems on semi-invariant submanifolds of a Sasakian manifold, An. Şti. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat, 32(1986), 73-76.
[17] Ronsse, G.S., Generic and skew CR submanifolds of a Kaehler manifold, Bull. Inst. Math. Acad. Sinica, 18(1990), no. 2, 127-141.
[18] Tripathi, M.M., Some characterizations of CR-submanifolds of generalized complex space forms, Kuwait J. Sci. Engrg., 23(1996), no. 2, 133-138.

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