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Affine Translation Surfaces in the Isotropic 3-Space

Muhittin Evren Aydin* and Mahmut Ergut

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ABSTRACT

In this paper, we describe (linear) Weingarten affine translation surfaces of first kind in the isotropic 3-space. In addition, we obtain such surfaces that satisfy certain equations in terms of the position vector and the Laplace operator.

Keywords: *Isotropic space; affine translation surface; Weingarten surface.*
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1. Introduction

It is well-known that a *translation surface* in a Euclidean 3-space $\mathbb{R}^3$ formed by translating two curves lying in orthogonal planes is the graph of a function $z(x,y) = f(x) + g(y)$ for the standard coordinate system of $\mathbb{R}^3$. One of the famous minimal surfaces of $\mathbb{R}^3$ is the *Scherk's translation surface* which is the graph of ([26])

$$z(x,y) = \frac{1}{c}\log\left|\frac{\cos(cx)}{\cos(cy)}\right|, \ c \in \mathbb{R}^* := \mathbb{R} - \{0\}.$$

The recent results relating to such surfaces in $\mathbb{R}^3$ and $\mathbb{R}^3_1$ (Minkowskian 3-space) of constant Gaussian and mean curvature were well-structured in [17]. In order for their generalizations in various ambient spaces, see [4, 5, 7, 12, 14, 20, 21, 25, 28, 29].

In 2013, Liu and Yu [15] defined the *affine translation surfaces* in $\mathbb{R}^3$ as the graph of the function

$$z(x,y) = f(x) + g(y+ax), \ a \in \mathbb{R}^*$$

and described the minimal affine translation surfaces (so-called *affine Scherk surface*) given in explicit form

$$z(x,y) = \frac{1}{c}\log\left|\frac{\cos\left(c\sqrt{1+a^2}x\right)}{\cos\left(c\left[y+ax\right]\right)}\right|, \ a, c \in \mathbb{R}^*.$$

Those are indeed the translation surfaces whose the translating curves lie in non-orthogonal planes. Then, Liu and Jung [16] obtained the affine translation surfaces in $\mathbb{R}^3$ of arbitrary constant mean curvature. Further, Yang and Fu [30] classified these surfaces in an affine 3-space of constant mean and Gaussian curvature.

In the isotropic 3-space $\mathbb{I}^3$ that is one of real-Cayley-Klein spaces, up to the absolute figure, there exist three different types of translation surfaces formed by translating two curves lying in orthogonal planes (see [19, 27]):

Type 1 Two translating curves lie in the isotropic planes $x = 0$ and $y = 0$,

$$z(x,y) = f(x) + g(y);$$

Type 2 one translating curve lies in the non-isotropic plane $z = 0$ and another one in the isotropic plane $x = 0$,

$$y(x,z) = f(x) + g(z);$$

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** **Corresponding author***

Type 3 two translating curves lie in the non-isotropic planes $y-z=\pi$ and $y+z=\pi$,

$$x\left(y,z\right)=\frac{1}{2}\left(f\left(\frac{y+z-\pi}{2}\right)+g\left(\frac{\pi-y+z}{2}\right)\right),$$

where x,y,z are the standart coordinates in $\mathbb{I}^3$. A surface of one type cannot be carried into that of another type by the isometries of $\mathbb{I}^3$. Such surfaces of constant isotropic Gaussian and mean curvature were obtained in [19] as well as Weingarten ones. In addition, the translation surfaces of Type 1 in $\mathbb{I}^3$ that satisfy the condition

$$\triangle^{I,II}r_i=\lambda_i r_i,\ \lambda_i\in\mathbb{R},\ i=1,2,3,$$

were presented in [13], where r_i is the coordinate function of the position vector and $\triangle^{I,II}$ the Laplace operator with respect to the first and second fundamental forms, respectively. This condition is natural, being related to the so-called *submanifolds of finite type*, introduced by B.-Y. Chen in the late 1970's (see [8, 9, 11]). More details for isotropic counterparts of translation surfaces can be found in [2, 3, 6].

In this paper, we investigate the translation surfaces in $\mathbb{I}^3$ formed by translating of two curves lying in the isotropic planes, not necessary orthogonal. We call such surfaces *affine translation surfaces of first kind* and classify ones of Weingarten type. Morever, we describe the affine translation surfaces of first kind that satisfy the condition $\triangle^{I,II}r_i=\lambda_i r_i$.

2. Preliminaries

The isotropic 3-space $\mathbb{I}^3$ is defined from the projective 3-space $P\left(\mathbb{R}^3\right)$ with an absolute figure consisting of a plane ω and two complex-conjugate straight lines f_1,f_2 in ω (see [1, 10, 18], [22]-[24]). Denote the projective coordinates by $(X_0:X_1:X_2:X_3)$ in $P\left(\mathbb{R}^3\right)$. Then the *absolute plane* ω is given by $X_0=0$ and the *absolute lines* f_1,f_2 by $X_0=X_1+iX_2=0$, $X_0=X_1-iX_2=0$. The intersection point $F(0:0:0:1)$ of these two lines is called the *absolute point*. The group of motions of $\mathbb{I}^3$ is a six-parameter group given in the affine coordinates $x=\frac{X_1}{X_0}$, $y=\frac{X_2}{X_0}$, $z=\frac{X_3}{X_0}$, $X_0\neq 0$, by

$$(x,y,z)\longmapsto(x',y',z'):\begin{cases}x'=a_1+x\cos\phi-y\sin\phi,\\ y'=a_2+x\sin\phi+y\cos\phi,\\ z'=a_3+a_4x+a_5y+z,\end{cases}$$

where $a_1,...,a_5,\phi\in\mathbb{R}$. The metric of $\mathbb{I}^3$ is induced by the absolute figure, i.e. $ds^2=dx^2+dy^2$. In the affine model of $\mathbb{I}^3$, the lines in $z-$direction correspond to *isotropic lines*. The plane containing an isotropic line is said to be *isotropic*. Other planes are *non-isotropic*.

Let M^2 be a surface immersed in $\mathbb{I}^3$. We call the surface M^2 *admissible* if it has no isotropic tangent planes. Such a surface can get the form

$$r:D\subseteq\mathbb{R}^2\longrightarrow\mathbb{I}^3,\ \ (x,y)\longmapsto\left(r_1\left(x,y\right),r_2\left(x,y\right),r_3\left(x,y\right)\right).$$

The components E,F,G of the first fundamental form I of M^2 can be calculated via the metric induced from $\mathbb{I}^3$. Denote the Laplace operator of M^2 with respect to I by $\triangle^I$. Then it is defined as

$$\triangle^I\phi=\frac{1}{\sqrt{W}}\left\{\frac{\partial}{\partial x}\left(\frac{G\phi_x-F\phi_y}{\sqrt{W}}\right)-\frac{\partial}{\partial y}\left(\frac{F\phi_x-E\phi_y}{\sqrt{W}}\right)\right\},\ \phi_x=\frac{\partial\phi}{\partial x},\tag{2.1}$$

where ϕ is a smooth function on M^2 and $W=EG-F^2$. The unit normal vector field of M^2 is completely isotropic, i.e. $(0,0,1)$. Morever, the components of the second fundamental form II are

$$L=\frac{\det\left(r_{xx},r_x,r_y\right)}{\sqrt{W}},\ M=\frac{\det\left(r_{xy},r_x,r_y\right)}{\sqrt{W}},\ N=\frac{\det\left(r_{yy},r_x,r_y\right)}{\sqrt{W}},\tag{2.2}$$

where $r_{xy}=\frac{\partial^2 r}{\partial x\partial y}$, etc. The *relative curvature* (so-called the *isotropic curvature* or *isotropic Gaussian curvature*) and the *isotropic mean curvature* are respectively defined by

$$K=\frac{LN-M^2}{EG-F^2},\ H=\frac{EN-2FM+LG}{2(EG-F^2)}.\tag{2.3}$$

Assume that nowhere M^2 has parabolic points, i.e. $K \neq 0$. Then the Laplace operator with respect to II is given by

$$\triangle^{II} \phi = -\frac{1}{\sqrt{|w|}} \left\{ \frac{\partial}{\partial x} \left(\frac{N\phi_x - M\phi_y}{\sqrt{|w|}} \right) - \frac{\partial}{\partial y} \left(\frac{M\phi_x - L\phi_y}{\sqrt{|w|}} \right) \right\} \tag{2.4}$$

for a smooth function ϕ on M^2 and $w = \det(II)$.

In particular; if M^2 is a graph surface in $\mathbb{I}^3$ of a smooth function $z = z(x, y)$, then the metric on M^2 induced from $\mathbb{I}^3$ is given by $dx^2 + dy^2$. Thus its Laplacian turns to

$$\triangle^{I} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{2.5}$$

Further, the matrix of second fundamental form II of M^2 corresponds to the Hessian matrix $\mathcal{H}(z)$, i.e.,

$$\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} z_{xx} & z_{xy} \\ z_{xy} & z_{yy} \end{pmatrix}.$$

Accordingly, the formulas in (2.3) reduce to

$$K = \det(\mathcal{H}(z)), \; H = \frac{\text{trace}(\mathcal{H}(z))}{2}. \tag{2.6}$$

3. Weingarten affine translation surfaces

Let M^2 be the graph surface in $\mathbb{I}^3$ of the function $z(x, y) = f(u) + g(v)$, where

$$u = ax + by, \; v = cx + dy. \tag{3.1}$$

If $ad - bc \neq 0$, we call the surface M^2 an *affine translation surface of first kind* in $\mathbb{I}^3$ and the pair (u, v) *affine parameter coordinates*. Especially; if the matrix of coefficients in (3.1) is orthogonal, then such a surface reduces to the translation surface of Type 1 in $\mathbb{I}^3$. Henceforth, let us fix some notations as below:

$$\frac{\partial f}{\partial x} = a\frac{df}{du} = af', \; \frac{\partial f}{\partial y} = bf', \; \frac{\partial g}{\partial x} = c\frac{dg}{dv} = cg', \; \frac{\partial g}{\partial y} = dg',$$

and so on. By (2.6), the relative curvature K and the isotropic mean curvature H of M^2 turn to

$$K = (ad - bc)^2 f''g'' \text{ and } 2H = \left(a^2 + b^2\right) f'' + \left(c^2 + d^2\right) g''. \tag{3.2}$$

Now we can state the following result to describe the Weingarten affine translation surfaces of first kind in $\mathbb{I}^3$ that satisfy the condition

$$K_x H_y - K_y H_x = 0, \tag{3.3}$$

where the subscript means the partial derivative.

Theorem 3.1. *Let M^2 be a Weingarten affine translation surface of first kind in $\mathbb{I}^3$. Then one of the following occurs:*

(i) *M^2 is the graph of*

$$z(x, y) = c_1 u^2 + \frac{c_1 \left(a^2 + b^2\right)}{(c^2 + d^2)} v^2 + c_2 u + c_3 v + c_4, \; c_1, ..., c_4 \in \mathbb{R};$$

(ii) *M^2 is the graph of either*

$$z(x, y) = f(u) + c_1 v^2 + c_2 v + c_3, \; f''' \neq 0, \; c_1, c_2, c_3 \in \mathbb{R}$$

or

$$z(x, y) = g(v) + c_1 u^2 + c_2 u + c_3, \; g''' \neq 0, \; c_1, c_2, c_3 \in \mathbb{R},$$

where (u, v) is the affine parameter coordinates given by (3.1).

Proof. It follows from (3.2) and (3.3) that

$$\left[\left(a^2+b^2\right)f''-\left(c^2+d^2\right)g''\right]f'''g'''=0. \tag{3.4}$$

To solve (3.4) , we have several cases:

Case (a) $\left(a^2+b^2\right)f''=\left(c^2+d^2\right)g''$. Then we derive

$$z\left(x,y\right)=c_1u^2+\frac{c_1\left(a^2+b^2\right)}{\left(c^2+d^2\right)}v^2+c_2u+c_3v+c_4,\ c_1,...,c_4\in\mathbb{R},$$

which gives the statement (i) of the theorem.

Case (b) $\left(a^2+b^2\right)f''\neq\left(c^2+d^2\right)g''$. Then, by (3.4), the surface has the form either

$$z\left(x,y\right)=g\left(v\right)+c_1u^2+c_2u+c_3,\ g'''\neq 0$$

or

$$z\left(x,y\right)=f\left(u\right)+c_4v^2+c_5v+c_6,\ f'''\neq 0,\ c_1,...,c_6\in\mathbb{R}.$$

This implies the second statement of the theorem. Therefore the proof is completed. □

Now we intend to find the linear Weingarten affine translation surfaces of first kind in $\mathbb{I}^3$ that satisfy

$$\alpha K+\beta H=\gamma,\ \alpha,\beta,\gamma\in\mathbb{R},\ \left(\alpha,\beta,\gamma\right)\neq\left(0,0,0\right). \tag{3.5}$$

Without lose of generality, we may assume $\alpha\neq 0$ in (3.5) and thus it can be rewritten as

$$K+2m_0H=n_0,\ 2m_0=\frac{\beta}{\alpha},\ n_0=\frac{\gamma}{\alpha}. \tag{3.6}$$

Hence the following result can be given.

Theorem 3.2. *Let M^2 be a linear Weingarten affine translation surface of first kind in $\mathbb{I}^3$ that satisfies* (3.6). *Then we have:*

(i) M^2 is the graph of

$$z\left(x,y\right)=c_1u^2+c_2v^2+c_3u+c_4v+c_5,\ c_1,...,c_5\in\mathbb{R}.$$

(ii) M^2 is the graph of either

$$z\left(x,y\right)=f\left(u\right)-\frac{m_0\left(a^2+b^2\right)}{2\left(ad-bc\right)^2}v^2+c_1v+c_2,\ f'''\neq 0,\ c_1,c_2\in\mathbb{R}$$

or

$$z\left(x,y\right)=g\left(v\right)-\frac{m_0\left(c^2+d^2\right)}{2\left(ad-bc\right)^2}u^2+c_1u+c_2,\ g'''\neq 0,\ c_1,c_2\in\mathbb{R},$$

where (u,v) is the affine parameter coordinates given by (3.1).

Proof. Substituting (3.2) in (3.6) gives

$$\left(ad-bc\right)^2f''g''+m_0\left(a^2+b^2\right)f''+m_0\left(c^2+d^2\right)g''=n_0. \tag{3.7}$$

After taking derivative of (3.7) with respect to u and v, we deduce $f'''g'''=0$. If both f''' and g''' are zero then we easily obtain the first statement of the theorem. Otherwise, we have the second statement of the theorem. This proves the theorem. □

Example 3.1. Consider the affine translation surface of first kind in $\mathbb{I}^3$ with

$$z\left(x,y\right)=\cos\left(x-y\right)+\left(x+y\right)^2,\ -\frac{\pi}{6}\leq x,y\leq\frac{\pi}{6}.$$

This surface plotted as in Fig. 1 satisfies the conditions to be Weingarten and linear Weingarten.

4. Affine translation surfaces satisfying $\triangle^{I,II} r_i = \lambda_i r_i$

This section is devoted to classify the affine translation surfaces of first kind in $\mathbb{I}^3$ that satisfy the conditions $\triangle^{I,II} r_i = \lambda_i r_i$, $\lambda_i \in \mathbb{R}$. For this, we get a local parameterization on such a surface as follows:

$$\begin{aligned} r(x,y) &= (r_1(x,y), r_2(x,y), r_3(x,y)) \\ &= (x, y, f(ax+by) + g(cx+dy)). \end{aligned} \tag{4.1}$$

Thus we first give the following result.

Theorem 4.1. *Let M^2 be an affine translation surface of first kind in $\mathbb{I}^3$ that satisfies $\triangle^I r_i = \lambda_i r_i$. Then it is the graph of one of the following functions:*

(i) $(\lambda_1, \lambda_2, \lambda_3) = (0,0,0)$,

$$z(x,y) = c_1 u^2 - \frac{c_1 (a^2+b^2)}{(c^2+d^2)} v^2 + c_3 u + c_4 v + c_5;$$

(ii) $(\lambda_1, \lambda_2, \lambda_3) = (0,0,\lambda > 0)$,

$$z(x,y) = c_1 e^{\sqrt{\frac{\lambda}{a^2+b^2}} u} + c_2 e^{-\sqrt{\frac{\lambda}{a^2+b^2}} u} + c_3 e^{\sqrt{\frac{\lambda}{c^2+d^2}} v} + c_4 e^{-\sqrt{\frac{\lambda}{c^2+d^2}} v};$$

(iii) $(\lambda_1, \lambda_2, \lambda_3) = (0,0,\lambda < 0)$,

$$\begin{aligned} z(x,y) = {} & c_1 \cos\left(\sqrt{\tfrac{-\lambda}{a^2+b^2}} u\right) + c_2 \sin\left(\sqrt{\tfrac{-\lambda}{a^2+b^2}} u\right) + c_3 \cos\left(\sqrt{\tfrac{-\lambda}{c^2+d^2}} v\right) \\ & + c_4 \sin\left(\sqrt{\tfrac{-\lambda}{c^2+d^2}} v\right), \end{aligned}$$

where (u,v) is the affine parameter coordinates given by (3.1) and $c_1, ..., c_5 \in \mathbb{R}$.

Proof. It is easy to compute from (2.5) and (4.1) that

$$\triangle^I r_1 = \triangle^I r_2 = 0 \tag{4.2}$$

and

$$\triangle^I r_3 = (a^2+b^2) f'' + (c^2+d^2) g''. \tag{4.3}$$

Assuming $\triangle^I r_i = \lambda_i r_i$, $i = 1,2,3$, in (4.2) and (4.3) yields $\lambda_1 = \lambda_2 = 0$ and

$$(a^2+b^2) f'' + (c^2+d^2) g'' = \lambda (f+g), \ \lambda_3 = \lambda. \tag{4.4}$$

If $\lambda = 0$ in (4.4), then we derive

$$f(u) = c_1 u^2 + c_2 u + c_3$$

and

$$g(v) = -\frac{c_1 (a^2+b^2)}{(c^2+d^2)} v^2 + c_4 v + c_5, \ c_1, ..., c_5 \in \mathbb{R},$$

which proves the statement (i) of the theorem. If $\lambda \neq 0$ then (4.4) can be rewritten as

$$(a^2+b^2) f'' - \lambda f = \mu = -(c^2+d^2) g'' + \lambda g, \ \mu \in \mathbb{R}. \tag{4.5}$$

In the case $\lambda > 0$, by solving (4.5) we obtain

$$\begin{cases} f(u) = c_1 \exp\left(\sqrt{\tfrac{\lambda}{a^2+b^2}} u\right) + c_2 \exp\left(-\sqrt{\tfrac{\lambda}{a^2+b^2}} u\right) - \tfrac{\mu}{\lambda}, \\ g(v) = c_3 \exp\left(\sqrt{\tfrac{\lambda}{c^2+d^2}} v\right) + c_4 \exp\left(-\sqrt{\tfrac{\lambda}{c^2+d^2}} v\right) + \tfrac{\mu}{\lambda}, \end{cases}$$

where $c_1, ..., c_4 \in \mathbb{R}$. This gives the statement (ii) of the theorem. Otherwise, i.e. $\lambda < 0$, then we derive

$$\begin{cases} f(u) = c_1 \cos\left(\sqrt{\tfrac{-\lambda}{a^2+b^2}} u\right) + c_2 \sin\left(\sqrt{\tfrac{-\lambda}{a^2+b^2}} u\right) - \tfrac{\mu}{\lambda}, \\ g(v) = c_3 \cos\left(\sqrt{\tfrac{-\lambda}{c^2+d^2}} v\right) + c_4 \sin\left(\sqrt{\tfrac{-\lambda}{c^2+d^2}} v\right) + \tfrac{\mu}{\lambda} \end{cases}$$

for $c_1, ..., c_4 \in \mathbb{R}$. This completes the proof. □

Example 4.1. Take the affine translation surface of first kind in $\mathbb{I}^3$ with

$$z(x,y) = \cos(x+y) + \sin(x-y), \quad -\pi \leq x, y \leq \pi.$$

Then it holds $\triangle^I r_i = \lambda_i r_i$ for $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = -2$ and can be plotted as in Fig. 2.

Next, we consider the affine translation surface of first kind in $\mathbb{I}^3$ that satisfies $\triangle^{II} r_i = \lambda_i r_i$, $\lambda_i \in \mathbb{R}$. Then its Laplace operator with respect to the second fundamental form II has the form

$$\begin{aligned} \triangle^{II}\phi = \tfrac{(f''g'')^{-2}}{2(ad-bc)} \left[(-b\phi_x + a\phi_y)(f'')^2 g''' + (d\phi_x - c\phi_y) f''' (g'')^2 \right] \\ + \tfrac{(f''g'')^{-1}}{(ad-bc)^2} \left[\left(2ab\phi_{xy} - b^2\phi_{xx} - a^2\phi_{yy}\right) f'' + \left(2cd\phi_{xy} - d^2\phi_{xx} - c^2\phi_{yy}\right) g'' \right] \end{aligned} \tag{4.6}$$

for a smooth function ϕ and $f''g'' \neq 0$. Hence we have the following result.

Theorem 4.2. *Let M^2 be an affine translation surface of first kind in $\mathbb{I}^3$ that satisfies $\triangle^{II} r_i = \lambda_i r_i$. Then it is the graph of one of the following functions:*

(i) $(\lambda_1 \neq 0, \lambda_2 \neq 0, 0)$,

$$z(x,y) = \ln \left| x^{\frac{1}{\lambda_1}} y^{\frac{1}{\lambda_2}} \right| + c_1, \ c_1 \in \mathbb{R};$$

(ii) $(\lambda \neq 0, \lambda, 0)$,

$$z(x,y) = \ln \left| (uv)^{\frac{1}{\lambda}} \right| + c_1, \ c_1 \in \mathbb{R},$$

where (u,v) is the affine parameter coordinates given by (3.1).

Proof. Let us assume that $\triangle^{II} r_i = \lambda_i r_i$, $\lambda_i \in \mathbb{R}$. Then, from (4.1) and (4.6), we state the following system:

$$d\frac{f'''}{(f'')^2} - b\frac{g'''}{(g'')^2} = 2(ad-bc)\lambda_1 x, \tag{4.7}$$

$$-c\frac{f'''}{(f'')^2} + a\frac{g'''}{(g'')^2} = 2(ad-bc)\lambda_2 y, \tag{4.8}$$

$$\frac{f'''f'}{(f'')^2} + \frac{g'''g'}{(g'')^2} - 4 = 2\lambda_3(f+g). \tag{4.9}$$

To solve above system, by considering $ad - bc \neq 0$, we distinguish two cases based on the constants a, b, c, d:

Case (a) Two of a, b, c, d are zero. Without loss of generality we may assume that $b = c = 0$ and $a = d = 1$. Then the equations (4.7) and (4.8) reduce to

$$\frac{f'''}{(f'')^2} = 2\lambda_1 x \tag{4.10}$$

and

$$\frac{g'''}{(g'')^2} = 2\lambda_2 y. \tag{4.11}$$

If $\lambda_1 = \lambda_2 = 0$, then we obtain a contradiction from (4.9) due to the fact that f, g are non-constant functions. Thereby we need to consider the remaining cases:

Case (a.1) $\lambda_1 = 0$, i.e. $f''' = 0$. Then substituting (4.10) and (4.11) into (4.9) implies $\lambda_3 = 0$ and

$$g(y) = \frac{2}{\lambda_2} \ln y + c_1, \ c_1 \in \mathbb{R}.$$

However, this is not a solution of (4.11) and gives a contradiction.

Case (a.2) $\lambda_2 = 0$, i.e. $g''' = 0$. Hence we can similarly obtain that $\lambda_3 = 0$ and

$$f(x) = \frac{2}{\lambda_1} \ln x + c_1, \ c_1 \in \mathbb{R},$$

which gives a contradiction by considering it into (4.10).

Case (a.3) $\lambda_1\lambda_2 \neq 0$. By substituting (4.10) and (4.11) into (4.9) we deduce

$$\lambda_1 x f' + \lambda_2 y g' - 2 = \lambda_3 (f + g) . \tag{4.12}$$

Case (a.3.1) If $\lambda_3 = 0$, then (4.12) reduces to

$$\lambda_1 x f' + \lambda_2 y g' = 2. \tag{4.13}$$

By solving (4.13) we find

$$f(x) = \frac{\xi}{\lambda_1} \ln x + c_1 \text{ and } g(v) = \frac{2-\xi}{\lambda_2} \ln y + c_2,\ c_1, c_2 \in \mathbb{R},\ \xi \in \mathbb{R}^*. \tag{4.14}$$

Substituting (4.14) into (4.10) and (4.11) yields $\xi = 1$. This proves the first statement of the theorem.

Case (a.3.2) If $\lambda_3 \neq 0$ in (4.12) then we can rewrite it as

$$\lambda_1 x f' - \lambda_3 f - 2 = \mu = -\lambda_2 y g' + \lambda_3 g,\ \mu \in \mathbb{R}. \tag{4.15}$$

After solving (4.15) , we conclude

$$f(x) = -\frac{2+\mu}{\lambda_3} + c_1 x^{\frac{\lambda_3}{\lambda_1}} \tag{4.16}$$

and

$$g(y) = \frac{\mu}{\lambda_3} + c_2 y^{\frac{\lambda_3}{\lambda_2}},\ c_1, c_2 \in \mathbb{R}. \tag{4.17}$$

However, these are not solutions of (4.10) and (4.11), respectively. Indeed, by considering (4.16) and (4.17) into (4.10) and (4.11), we conclude $\lambda_3 = 0$ which implies that this case is not possible.

Case (b) At most one of a, b, c, d is zero. Suppose that $\lambda_1 = 0$ in (4.7). It follows from (4.7) that

$$\frac{f'''}{(f'')^2} = \frac{c_1}{d} \text{ and } \frac{g'''}{(g'')^2} = \frac{c_1}{b},\ c_1 \in \mathbb{R}, \tag{4.18}$$

where we may assume that $b \neq 0 \neq d$ since at most one of a, b, c, d can vanish. If $c_1 = 0$, then we derive a contradiction from (4.9) due to $f''g'' \neq 0$. Otherwise, considering (4.18) into (4.8) yields $\frac{c_1}{bd} = 2\lambda_2 y$, which is no possible since y is an independent variable. This implies that λ_1 must be non-zero and it can be similarly shown that λ_2 must be non-zero. Hence from (4.7) and (4.8) we can write

$$\frac{f'''}{(f'')^2} = 2(\lambda_1 a x + \lambda_2 b y) \tag{4.19}$$

and

$$\frac{g'''}{(g'')^2} = 2(\lambda_1 c x + \lambda_2 d y) . \tag{4.20}$$

Compatibility condition in (4.19) or (4.20) gives $\lambda_1 = \lambda_2$. Put $\lambda_1 = \lambda_2 = \lambda$. By substituting (4.19) and (4.20) into (4.9) we deduce

$$\lambda u f' + \lambda v g' - 2 = \lambda_3 (f + g) , \tag{4.21}$$

where (u, v) is the affine parameter coordinates given by (3.1).

Case (b.1) If $\lambda_3 = 0$, then (4.21) reduces to

$$\lambda u f' + \lambda v g' = 2. \tag{4.22}$$

By solving (4.22) we find

$$f(u) = \frac{\xi}{\lambda} \ln u + c_1 \text{ and } g(v) = \frac{2-\xi}{\lambda} \ln v + c_2,\ c_1, c_2 \in \mathbb{R},\ \xi \in \mathbb{R}^*. \tag{4.23}$$

Substituting (4.23) into (4.19) and (4.20) yields $\xi = 1$. This proves the second statement of the theorem.

Case (b.2) If $\lambda_3 \neq 0$ in (4.11), then we can rewrite it as

$$\lambda u f' - \lambda_3 f - 2 = \mu = -\lambda v g' + \lambda_3 g,\ \mu \in \mathbb{R}. \tag{4.24}$$

After solving (4.24) , we deduce

$$f(u) = -\frac{2+\mu}{\lambda_3} + c_1 u^{\frac{\lambda_3}{\lambda}} \tag{4.25}$$

and

$$g(v) = \frac{\mu}{\lambda_3} + c_2 v^{\frac{\lambda_3}{\lambda}},\ c_1, c_2 \in \mathbb{R}. \tag{4.26}$$

Considering (4.25) and (4.26) into (4.19) and (4.20), respectively, we find $\lambda_3 = 0$, however this is a contradiction.

□

Example 4.2. Given the affine translation surface of first kind in $\mathbb{I}^3$ as follows

$$z(x, y) = \ln(2x + y) + \ln(x - y),\ (x, y) \in [3, 5] \times [1, 2].$$

Then it holds $\triangle^{II} r_i = \lambda_i r_i$ for $(\lambda_1, \lambda_2, \lambda_3) = (1, 1, 0)$ and we plot it as in Fig. 3.

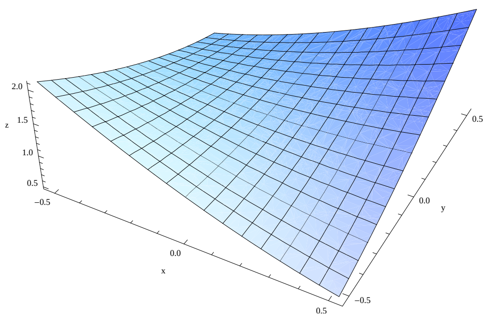

Figure 1. A (linear) Weingarten affine translation surface of first kind.

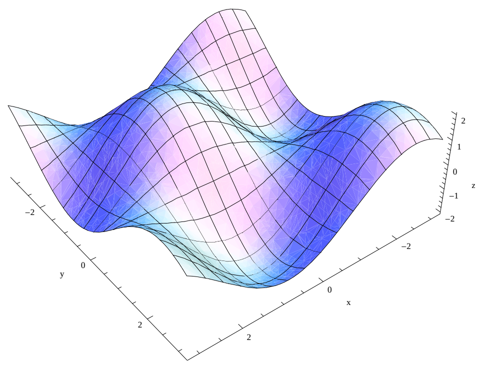

Figure 2. An affine translation surface of first kind with $\triangle^I r_i = \lambda_i r_i$, $(\lambda_1, \lambda_2, \lambda_3) = (0, 0, 2)$.

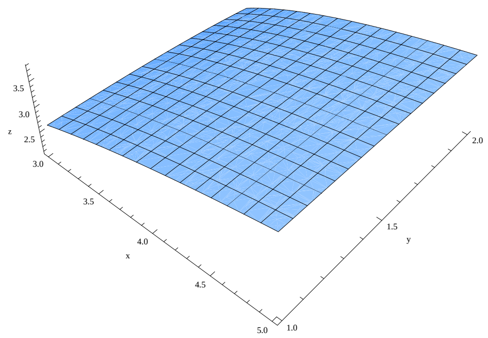

Figure 3. An affine translation surface of first kind with $\triangle^{II} r_i = \lambda_i r_i$, $(\lambda_1, \lambda_2, \lambda_3) = (1, 1, 0)$.

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Affiliations

MUHITTIN EVREN AYDIN
ADDRESS: Firat University, Dept. of Mathematics, 23200, Elazig-Turkey.
E-MAIL: meaydin@firat.edu.tr

MAHMUT ERGUT
ADDRESS: Namik Kemal University, Dept. of Mathematics, 59000, Tekirdag-Turkey.
E-MAIL: mergut@nku.edu.tr