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A Class of Nearly Sub-Weyl and Sub-Lyra Manifolds

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ABSTRACT

The authors introduce a nearly sub-Weyl and a nearly sub-Lyra manifold by virtue of a SNS-non-metric connection(defined below) in nearly sub-Riemannian manifolds. In particular, we show a SR-parallel curve associated with the nearly sub-Lyra connection is actually a minimizer of the horizontal length functional. A geometric characteristic of a SNS-non-metric connection is given in the last section.

Keywords: *Nearly sub-Riemmannian manifolds; Nearly sub-Weyl manifold; Nearly sub-Lyra manifold.*

AMS Subject Classification (2010): *Primary: 53C20; 53D11.*

1. Introduction

H.Weyl [12] introduced a generalization of Riemannian geometry in order to formulate a unified field theory. Weyl's theory provides an instructive example of non-Riemannian connection. It is well known that a Weyl connection is actually a torsion-free but non-metric connection, namely the so-called symmetric non-metric connection in transformation's theory. In 1951 G.Lyra [7] suggested a semi-symmetric metric connection and introduced the notion of "gauge", which made it possible to construct a geometrized theory of gravitation and electromagnetism along the lines of Weyl theory.

The study of various connections on Riemannian or non-Riemannian manifolds has been an active field over the past seven decades. In particular, since the formidable papers [1, 3, 4, 11, 13] were published in succession, these works stimulate such research fields to present a scene of prosperity, and demonstrate the abnormal importance of this topic. We have made some attempts in this field and obtained some results, see [6, 14].

In order to study sub-Riemannian geometry from the point of theory of transformation, we need to define an analogue of the Levi-Civita connection. Because of no metric defined on the entire tangent bundle, A.Bejauce defined a new sub-Riemannian connection in [2] and discussed the corresponding geometry that is very similar to Riemannian geometry, which motivate us to replace the non-holonomic connection in [6] by this new sub-Riemannian connection.

In this paper, we, based on the work by A.Bejauce [2], following the ideas of D.K.Sen and J.R.Vanstone in [10], consider the nearly sub-Lyra manifold and the nearly sub-Weyl manifold by virtue of a SNS-non-metric connection(defined below) in nearly sub-Riemannian manifolds, in contrast to our former papers, a non-holonomic connection(i.e. the projection of Riemannian connection on horizontal distribution) will be replaced by a horizontal sub-Riemannian connection.

The organization of this paper is as follows. In section 2, we will recall and give some necessary notations and terminologies about horizontal sub-Riemannian connection and nearly sub-Riemannian manifolds. Section 3 and Section 4 are concentrated on the sub-Weyl and sub-Lyra's geometries. Section 5 is devoted to the SNS-non-metric connection and its geometric characteristic. An example is given in the last section.

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2. Preliminaries

Let (M, V_0, g) be a n-dimensional sub-Riemannian manifold, in particular, when $V_0 = TM$, then M is degenerated into a Riemanian manifold and hence there are no abnormal geodesics(see [8] for details). Without loss of generality, we assume $V_0 \neq TM$, if V_0 is strongly bracket generating, then there does not exist non-trivial abnormal geodesics (see [9]). In order to study the sub-Riemannian geometry, we suppose that there exists a complementary distribution V_1 to V_0 in the tangent bundle TM of M. We note that V_1 exists on any paracompact sub- Riemannian manifold. Indeed, in this case, there exists a Riemannian metric g^* on M and V_1 is taken as the complementary orthogonal distribution to V_0 in TM, with respect to g^*. If not stated otherwise, we suppose that V_1 is an integrable distribution on M.

Throughout the paper, we denote by $F(M)$ the set of smooth functions on M, $\Gamma(V_0)$ the $C^\infty(M)$ -module of smooth sections on V_0 ,and $X, Y, Z, \cdots$ the vector fields in $\Gamma(TM)$, X_0 the projection of X on V_0, X_1 the projection of X on V_1. The repeated indices with one upper index and one lower index indicates summation over their range. We use the following ranges for indices: $i, j, k, h, \cdots \in \{1, \cdots, \ell\}$, $\alpha, \beta, \cdots \in \{\ell + 1, \cdots, n\}$.

We define a 3-multilinear mapping by

$$\begin{aligned} &\Omega : \Gamma(V_0) \times \Gamma(V_0) \times \Gamma(V_1) \to F(M) \\ &\Omega(X_0, Y_0, Z_1) = Z_1 g(X_0, Y_0) - g([Z_1, X_0]_0, Y_0) - g([Z_1, Y_0]_0, X_0), \end{aligned}$$

It is easy to check Ω is a tensor field by a direct computation.

Definition 2.1. We say that a sub-Riemannian manifold (M, V_0, g) is a nearly sub-Riemannian manifold if the tensor field Ω vanishes identically on M.

Theorem 2.1. *[2] Given a nearly sub-Riemannian manifold (M, V_0, g), then there exists a unique linear connection $\nabla : \Gamma(TM) \times \Gamma(V_0) \to \Gamma(V_0)$ satisfying*

$$(\nabla_Z g)(X_0, Y_0) = Z(g(X_0, Y_0)) - g(\nabla_Z X_0, Y_0) - g(X_0, \nabla_Z Y_0) = 0, \tag{2.1}$$

$$T(X, Y_0) = \nabla_X Y_0 - \nabla_{Y_0} X_0 - [X, Y_0]_0 = 0. \tag{2.2}$$

It is not hard to derive that the connection determined by Equation (2.1) and (2.2) is of the form

$$\begin{aligned} 2g(\nabla_{X_0} Y_0, Z_0) &= X_0 g(Y_0, Z_0) + Y_0 g(Z_0, X_0) - Z_0 g(X_0, Y_0) \\ &+ g([X_0, Y_0]_0, Z_0) - g([Y_0, Z_0]_0, X_0) + g([Z_0, X_0]_0, Y_0), \\ \nabla_{Z_1} X_0 &= [Z_1, X_0]_0, \end{aligned}$$

Its torsion is defined by

$$\begin{aligned} &T : \Gamma(TM) \times \Gamma(V_0) \to \Gamma(V_0) \\ &T(X, Y_0) = \nabla_X Y_0 - \nabla_{Y_0} X_0 - [X, Y_0]_0 \end{aligned}$$

By Theorem 2.1, we know

$$T(X_1, Y_0) = \nabla_{X_1} Y_0 - [X_1, Y_0]_0 = 0,$$

so (2.2) is equivalent to

$$T(X_0, Y_0) = \nabla_{X_0} Y_0 - \nabla_{Y_0} X_0 - [X_0, Y_0]_0 = 0. \tag{2.3}$$

Definition 2.2. A linear connection satisfying (2.1) and (2.2)(or (2.3)) is called a horizontal sub-Riemannian connection, in briefly, a HSR-connection.

Next we consider the curvature tensor R of ∇:

$$\begin{aligned} &R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(V_0) \to \Gamma(V_0) \\ &R(X, Y, Z_0) = \nabla_X \nabla_Y Z_0 - \nabla_X \nabla_Y Z_0 - \nabla_{[X,Y]} Z_0, \end{aligned}$$

Hereafter we call R the horizontal curvature tensor because of $R(X, Y, Z_0) \in \Gamma(V_0)$.

Now we consider the local coordinate (x^i, x^α), such that $(\partial/\partial x^\ell, \cdots, \partial/\partial x^n)$ is a local basis for V_1 and V_0 is locally given by the Pfaff system

$$\delta x^\alpha = dx^\alpha + A_i^\alpha dx^i,$$

where A_i^α are smooth functions locally defined on M. Thus

$$e_i = \frac{\partial}{\partial x^i} - A_i^\alpha \frac{\partial}{\partial x^\alpha},$$

form a local basis of V_0. We call (x^i, x^α) and $(e_i, \partial/\partial x^\alpha)$ an adapted coordinate system and an adapted frame field on M induced by the foliation determined by V_1. Then by a direct calculation, we obtain

$$\Omega(e_i, e_j, \frac{\partial}{\partial x^\alpha}) = \frac{\partial g_{ij}}{\partial x^\alpha},$$

where $g_{ij} = g(e_i, e_j)$ is the local component of the Riemannian metric g on V_0, so the sub-Riemannian manifold (M, V_0, g) is a nearly sub-Riemannian manifold if and only if $\frac{\partial g_{ij}}{\partial x^\alpha} = 0$, namely, g_{ij} is independent of x^α. We denote by

$$\nabla_{e_i} e_j = \nabla_i e_j = \{^k_{ij}\} e_k, \nabla_{\frac{\partial}{\partial x^\alpha}} e_j = \nabla_\alpha e_j = \{^k_{\alpha j}\} e_k, R(e_i, e_j, e_k) = R^h_{ijk} e_h,$$
$$R(\frac{\partial}{\partial x^\alpha}, e_j, e_k) = R^h_{\alpha jk} e_h, R(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}, e_k) = R^h_{\alpha\beta k} e_h,$$

According to Theorem 2.1, we obtain

$$\{^k_{ij}\} = \frac{1}{2} g^{kh} (\frac{\partial g_{ih}}{\partial x^j} + \frac{\partial g_{jh}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^h}), \{^k_{\alpha j}\} = 0,$$

and

$$R^h_{ijk} = e_i(\{^h_{jk}\}) - e_j(\{^h_{ik}\}) + \{^e_{jk}\}\{^h_{ie}\} - \{^e_{ik}\}\{^h_{je}\}, R^h_{\alpha jk} = 0, R^h_{\alpha\beta k} = 0. \tag{2.4}$$

Denote by $R_{ijkh} = R^l_{ijk} g_{lh}, R_{ik} = R_{ijkh} g^{jh}$, we further have

$$\begin{cases} R_{ijkh} = -R_{ijhk} = -R_{jikh}, \\ R_{ijkh} = R_{khij}, \\ R_{ijkh} + R_{jkih} + R_{kijh} = 0, \\ R_{ijkl,h} + R_{ijlh,k} + R_{ijhk,l} = 0, \end{cases} \tag{2.5}$$

where $R_{ijkl,h} = (\nabla_h R)(e_i, e_j, e_k, e_l)$ is the covariant derivative of R with respect to e_h (see [2]).

3. A nearly sub-Weyl manifold

Weyl's idea provides an instructive example of non-Riemannian connections, further G.B.Folland [5] had given a global formulation of Weyl manifolds thereby clarifying considerably many of Weyl's basic ideas. Based on these facts, we, in this subsection, aim at the geometries of the nearly sub-Riemannian manifold equipped with a Weyl structure.

Theorem 3.1. *(Existence) Given a nearly sub-Riemannian manifold (M, V_0, g) and a 1-form φ, then there exists a unique connection $\hat{\nabla} : \Gamma(TM) \times \Gamma(V_0) \to \Gamma(V_0)$ on M such that*

$$\begin{cases} (\hat{\nabla}_Z g)(X_0, Y_0) = -\varphi(Z) g(X_0, Y_0), \\ \hat{T}(X, Y_0) = \hat{\nabla}_X Y_0 - \hat{\nabla}_{Y_0} X_0 - [X, Y_0]_0 = 0. \end{cases} \tag{3.1}$$

Proof. From the second equation of (3.1), one can derive the connection $\hat{\nabla}$ is necessarily of the form,

$$\hat{\nabla}_{X_1} Y_0 = [X_1, Y_0]_0.$$

And (3.1) implies

$$g(\hat{\nabla}_{Z_0}X_0, Y_0) = Z_0(g(X_0, Y_0)) - g(X_0, \hat{\nabla}_{Y_0}Z_0) - g(X_0, [Z_0, Y_0]_0) - \varphi(X_0)g(Z_0, Y_0), \tag{3.2}$$

By cyclic permutation of X, Y, Z, one obtains another two equations (3.2′) and (3.2″), if we add (3.2′) and (3.2″), and subtract (3.2), one gets

$$\begin{aligned} 2g(\hat{\nabla}_{X_0}Y_0, Z_0) &= X_0(g(Y_0, Z_0)) + Y_0(g(Z_0, X_0)) - Z_0(g(X_0, Y_0)) \\ &\quad + g([X_0, Y_0]_0, Z_0) - g([Y_0, Z_0]_0, X_0) + g([Z_0, X_0]_0, Y_0) \\ &\quad + \varphi(X_0)g(Z_0, Y_0) + \varphi(Y_0)g(Z_0, X_0) - \varphi(Z_0)g(X_0, Y_0), \end{aligned}$$

Since g is non-degenerate on horizontal distribution V_0, it defines $\hat{\nabla}_{X_0}Y_0$ uniquely as follows,

$$\begin{cases} \hat{\nabla}_{X_0}Y_0 = \nabla_{X_0}Y_0 + \frac{1}{2}(\varphi(X_0)Y_0 + \varphi(Y_0)X_0 - g(X_0, Y_0)P) \\ \hat{\nabla}_{X_1}Y_0 = [X_1, Y_0]_0, \end{cases} \tag{3.3}$$

where P is a horizontal vector field defined by $g(X_0, P) = \varphi(X_0)$.

Conversely, the connection defined by (3.3) satisfies necessarily (3.1). Therefore, (3.3) define uniquely the connection.

This completes the proof. □

Definition 3.1. A connection is said to be a nearly sub-Weyl connection, or a nearly sub-Weyl transformation from the point of transformation's theory, if it satisfies (3.3). A nearly sub-Riemannian manifold (M, V_0, g) is said to be a nearly sub-Weyl manifold denoted by (M, V_0, g, φ), if M admits a nearly sub-Weyl connection.

According to Theorem 3.1, a nearly sub-Weyl connection is uniquely determined by the sub-Riemannian metric g and a 1-form φ. In our adapted coordinates, the coefficients of nearly sub-Weyl connection are given by

$$\hat{\Gamma}^k_{ij} = \{^{\,k}_{ij}\} + \frac{1}{2}(\varphi_i\delta^k_j + \varphi_j\delta^k_i - \varphi^k g_{ij}), \hat{\Gamma}^k_{\alpha j} = 0, \tag{3.4}$$

where $\varphi_i = \varphi(e_i)$ and $\varphi^k = \varphi_i g^{ik}$. Such a connection is called a nearly Weyl transformation from the point of transformation theory.

Definition 3.2. An absolutely continuous curve $\gamma : x^a = x^a(t)$ is said to be a horizontal invariant curve if it satisfies $\nabla_{\dot{\gamma}}\dot{\gamma}_0 = \alpha(t)\dot{\gamma}_0$, where t is any parameter and $\alpha(t)$ is a function related to γ. In particular, γ is called a sub-Riemannian parallel curve(in short, a SR-parallel curve) if $\nabla_{\dot{\gamma}}\dot{\gamma}_0 = 0$.

Remark 3.1. It is obvious that a SR-parallel curve is necessarily a horizontal invariant curve. In contrast to the path that Jiao and Zhao defined in [14], the horizontal invariant curve is not necessarily a horizontal curve(i.e. $\dot{\gamma} \in V_0$). On the other hand, this kind of SR-parallel curve is exactly a Riemannian geodesic when the horizontal bundle is the whole tangent bundle.

Remark 3.2. The image curve of a a horizontal invariant curve with respect to the HSR-connection is not a horizontal invariant curve with respect to the nearly sub-Weyl connection any more under a nearly sub-Weyl transformation.

In fact, for an absolutely continuous curve $\gamma : x^a = x^a(t)$ with tangent vector

$$\dot{\gamma} = \frac{dx^i}{dt}\frac{\partial}{\partial x^i} + \frac{dx^\alpha}{dt}\frac{\partial}{\partial x^\alpha} = \frac{dx^i}{dt}e_i + (A^\alpha_i\frac{dx^i}{dt} + \frac{dx^\alpha}{dt})\frac{\partial}{\partial x^\alpha},$$

If $\gamma : x^a = x^a(t)$ is a horizontal invariant curve with respect to the HSR-connection, then it satisfies the equation $\nabla_{\dot{\gamma}}\dot{\gamma}_0 = \alpha(t)\dot{\gamma}_0$, namely

$$\nabla_{\dot{\gamma}}\dot{\gamma}_0 = \nabla_{\dot{\gamma}_0}\dot{\gamma}_0 + \nabla_{\dot{\gamma}_1}\dot{\gamma}_0 = \alpha(t)\dot{\gamma}_0,$$

Next we compute the second term $\nabla_{\dot{\gamma}_1}\dot{\gamma}_0$. According to the definition of HSR-connection, one has

$$\begin{aligned} \nabla_{\dot{\gamma}_1}\dot{\gamma}_0 &= [\dot{\gamma}_1, \dot{\gamma}_0]_0 \\ &= [(A^\alpha_i\frac{dx^i}{dt} + \frac{dx^\alpha}{dt})\frac{\partial}{\partial x^\alpha}, \frac{dx^i}{dt}e_i]_0 \\ &= \{(A^\alpha_i\frac{dx^i}{dt} + \frac{dx^\alpha}{dt})\frac{\partial}{\partial x^\alpha}(\frac{dx^i}{dt})e_i - \frac{dx^i}{dt}e_i(A^\alpha_i\frac{dx^i}{dt} + \frac{dx^\alpha}{dt})\frac{\partial}{\partial x^\alpha} \\ &\quad + (A^\alpha_i\frac{dx^i}{dt} + \frac{dx^\alpha}{dt})\frac{dx^i}{dt}\frac{\partial}{\partial x^\alpha}(A^\beta_i)\frac{\partial}{\partial x^\beta}\}_0 \\ &= 0. \end{aligned}$$

in the adapted frame system, we derive the equations of horizontal invariant curve with respect to the HSR-connection

$$\frac{d^2x^k}{dt^2} + \{^k_{ij}\}\frac{dx^i}{dt}\frac{dx^j}{dt} = \alpha(t)\frac{dx^k}{dt}. \tag{3.5}$$

Now we substitute (3.4) into Equation (3.5), then one gets

$$\frac{d^2x^k}{dt^2} + \hat{\Gamma}^k_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} + \frac{1}{2}\varphi^k g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = \beta(t)\frac{dx^k}{dt}.$$

Then, γ is not a horizontal invariant curve with respect to the nearly sub-Weyl connection. Hence the proof is finished.

4. A nearly sub-Lyra manifold

G.Lyra [7] suggested a modification of Riemannian geometry which bears a remarkable resemblance to Weyl's geometry. According to Lyra's ideas, the vector $\overrightarrow{PP'}$ between two neighbouring points $P(x^a)$ and $P'(x^a + dx^a)$ has the components $\xi^a = x^0 dx^a$, where $x^0(x^i)$ is a gauge function. The adapted coordinate system (x^a) together with the gauge x^0 form a local adapted reference system $(x^0; x^a)$ which induces a local natural basis $\{\tilde{e}_i = \frac{1}{x^0}\frac{\partial}{\partial x^i}\}$. We apply this idea into our sub-Riemannian geometry. We assume $\tilde{g}$ be a natural extension of sub-Riemannian metric g such that $V_0 = span\{\frac{\partial}{\partial x^i} : i = 1, 2, \cdots, \ell\}$, $V_1 = span\{\frac{\partial}{\partial x^\alpha} : \alpha = \ell + 1, \cdots, n\}$, where x_0 is independent of x^α, then $\{\tilde{e}_i = \frac{1}{x^0}\frac{\partial}{\partial x^i}\}$ form the local reference vector fields of V_0, and the Lie bracket of any two vector fields in local reference vector fields,

$$[\tilde{e}_i, \tilde{e}_j] = \frac{1}{2}(\phi^0_i \delta^k_j - \phi^0_j \delta^k_i)\tilde{e}_k,$$

where $\phi^0_i = 2\frac{\partial}{\partial x^i}(\frac{1}{x^0})$. If we denote the components of the sub-Riemannian metric tensor g, in a local reference system, by

$$g_{ij} = g(\tilde{e}_i, \tilde{e}_j),$$

then the metric form is

$$ds^2 = (x^0)^2 g_{ij} dx^i dx^j,$$

Theorem 4.1. *(Existence) Given a nearly sub-Riemannian manifold (M, V_0, g) and a 1-form ϕ with $\phi(X_1) = 0$, then Then there exists a unique connection $\tilde{\nabla} : \Gamma(TM) \times \Gamma(V_0) \to \Gamma(V_0)$ on M such that*

$$\begin{cases} (\tilde{\nabla}_Z g)(X_0, Y_0) = 0, \\ \tilde{T}(X, Y_0) = \tilde{\nabla}_X Y_0 - \tilde{\nabla}_{Y_0} X_0 - [X, Y_0]_0 = \frac{1}{2}(\phi(Y_0)X_0 - \phi(X)Y_0). \end{cases} \tag{4.1}$$

Proof. By the same method as Theorem 3.1, one can obtain an unique connection determined by Equations (4.1),

$$\begin{cases} \tilde{\nabla}_{X_0} Y_0 = \nabla_{X_0} Y_0 + \frac{1}{2}(\phi(Y_0)X_0 - g(X_0, Y_0)Q), \\ \tilde{\nabla}_{X_1} Y_0 = [X_1, Y_0]_0. \end{cases} \tag{4.2}$$

where Q is a horizontal vector field defined by $g(X_0, Q) = \phi(X_0)$. □

Definition 4.1. A connection is called a nearly sub-Lyra connection if it satisfies (4.2), or a nearly sub-Lyra transformation from the point of transformation's theory. A nearly sub-Riemannian manifold (M, V_0, g) is said to be a nearly sub-Lyra manifold denoted by (M, V_0, g, ϕ), if M admits a nearly sub-Lyra connection.

Theorem 4.1 implies, a nearly sub-Riemannian manifold is turned out to be a nearly sub-Lyra manifold under a nearly sub-Lyra transformation.

For an absolutely continuous curve $\gamma : x^a = x^a(t), t \in [0, 1]$, we define the horizontal length of γ by

$$L_0(\gamma) = \int_0^1 \sqrt{g(\dot{\gamma}_0, \dot{\gamma}_0)}dt,$$

and

$$d_0(p, q) = inf\{L_0(\gamma) : \gamma \text{ is an absolutely curve with } \gamma(0) = p, \gamma(1) = q\},$$

A horizontal length minimizer is a curve that realizes the distance $d_0(p, q)$.

Theorem 4.2. *Let (M, V_0, g, ϕ) be a nearly sub-Lyra manifold, then a SR-parallel curve is actually a minimizer of the horizontal length functional.*

Proof. Now we take $X_0 = \tilde{e}_i, Y_0 = \tilde{e}_j, Z_0 = \tilde{e}_k$ in Equation (4.2) and denote the coefficients by $\tilde{\nabla}_{\tilde{e}_i}\tilde{e}_j = \tilde{\Gamma}^k_{ij}\tilde{e}_k$, we then arrive at

$$\tilde{\Gamma}^k_{ij} = \frac{1}{x^0}\{^{\,k}_{ij}\} + \frac{1}{2}(\phi_i\delta^k_j - \phi^k g_{ij}), \tilde{\Gamma}^k_{\alpha j} = 0, \tag{4.3}$$

where $\phi_i = \phi(\tilde{e}_i) + \phi^0_i$, $\phi^k = \phi_i g^{ik}$.

Since a horizontal length minimizer is defined by the extremal curves of the problem in the calculus of variations:

$$\delta(\int_0^1 ds) = \delta(\int_0^1 \sqrt{(x^0)^2 g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}dt) = 0,$$

where s is the arc-length and t is an arbitrary parameter, the Lagrangian

$$L(x^k, \dot{x}^k, t) = \sqrt{(x^0)^2 g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}} = \frac{ds}{dt},$$

then

$$\begin{aligned}
\frac{\partial L}{\partial x^k} &= \frac{1}{2}\frac{\partial}{\partial x^k}\{(x^0)^2 g_{ij}\}\frac{dx^i}{dt}\frac{dx^j}{ds},\\
\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}^k}) &= \{\frac{\partial}{\partial x^i}[(x^0)^2 g_{kj}]\frac{dx^i}{ds}\frac{dx^j}{ds} + (x^0)^2 g_{kj}\frac{d^2x^j}{ds^2}\}\frac{ds}{dt}.
\end{aligned}$$

and hence the Euler-Lagrange equations are equivalent to

$$\begin{aligned}
\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}^k}) - \frac{\partial L}{\partial x^k} = 0 \quad &\Leftrightarrow \quad \frac{\partial}{\partial x^i}[(x^0)^2 g_{kj}]\frac{dx^i}{ds}\frac{dx^j}{ds} + (x^0)^2 g_{kj}\frac{d^2x^j}{ds^2} = \frac{1}{2}\frac{\partial}{\partial x^k}\{(x^0)^2 g_{ij}\}\frac{dx^i}{ds}\frac{dx^j}{ds}\\
&\Leftrightarrow \quad (x^0)^2 g_{kj}\frac{d^2x^j}{ds^2} + (2x^0\frac{\partial x^0}{\partial x^i}g_{kj} + (x^0)^2\frac{\partial g_{kj}}{\partial x^i})\frac{dx^i}{ds}\frac{dx^j}{ds}\\
&\qquad = x^0\frac{\partial x^0}{\partial x^k}g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} + \frac{1}{2}(x^0)^2\frac{\partial g_{ij}}{\partial x^k}\frac{dx^i}{ds}\frac{dx^j}{ds}\\
&\Leftrightarrow \quad (x^0)^2 g_{kj}\frac{d^2x^j}{ds^2} + 2x^0\frac{\partial x^0}{\partial x^i}g_{kj}\frac{dx^i}{ds}\frac{dx^j}{ds} + \frac{1}{2}(x^0)^2(\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j})\frac{dx^i}{ds}\frac{dx^j}{ds}\\
&\qquad = x^0\frac{\partial x^0}{\partial x^k}g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} + \frac{1}{2}(x^0)^2\frac{\partial g_{ij}}{\partial x^k}\frac{dx^i}{ds}\frac{dx^j}{ds}\\
&\Leftrightarrow \quad g_{kj}\frac{d^2x^j}{ds^2} + \frac{1}{x^0}\{2\frac{\partial x^0}{\partial x^i}g_{kj} - \frac{\partial x^0}{\partial x^k}g_{ij}\}\frac{dx^i}{ds}\frac{dx^j}{ds}\\
&\qquad + \frac{1}{2}(\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k})\frac{dx^i}{ds}\frac{dx^j}{ds} = 0\\
&\Leftrightarrow \quad \frac{d^2x^l}{ds^2} + \frac{1}{x^0}\{2\frac{\partial x^0}{\partial x^i}g_{kj} - \frac{\partial x^0}{\partial x^k}g_{ij}\}\frac{dx^i}{ds}\frac{dx^j}{ds}g^{kl} + \{^{\,l}_{ij}\}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0\\
&\Leftrightarrow \quad \frac{d^2x^l}{ds^2} + x^0\{-\frac{\partial}{\partial x^i}(\frac{1}{x^0})\delta^l_j - \frac{\partial}{\partial x^j}(\frac{1}{x^0})\delta^l_i + \frac{\partial}{\partial x^k}(\frac{1}{x^0})g_{ij}g^{kl}\}\frac{dx^i}{ds}\frac{dx^j}{ds}\\
&\qquad + \{^{\,l}_{ij}\}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0\\
&\Leftrightarrow \quad \frac{d^2x^k}{ds^2} + \{^{\,k}_{ij}\}\frac{dx^i}{ds}\frac{dx^j}{ds} + \frac{x^0}{2}(\phi^0_i\delta^k_j + \phi^0_j\delta^k_i - (\phi^0)^k g_{ij})\frac{dx^i}{ds}\frac{dx^j}{ds} = 0.
\end{aligned}$$

In particular, if we choose the normal gauge $x^0 = 1$, then we have

$$\frac{d^2x^k}{ds^2} + \{^{\,k}_{ij}\}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0. \tag{4.4}$$

This completes the proof. □

Remark 4.1. Let (M, V_0, g, ϕ) be a nearly sub-Lyra manifold, if $\gamma(s)$ whose horizontal tangent vector field is $\dot{\gamma}_0 = x^0 \frac{dx^i}{ds}\tilde{e}_i$ is a SR-parallel curve with respect to the nearly sub-Lyra connection, then it satisfies $\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma}_0 = 0$, i.e.

$$\frac{d^2x^k}{ds} + \{^k_{ij}\}\frac{dx^i}{ds}\frac{dx^j}{ds} + \frac{x^0}{2}(\phi_i\delta^k_j + \phi_j\delta^k_i - \phi^k g_{ij})\frac{dx^i}{ds}\frac{dx^j}{ds} = \frac{x^0}{2}(\phi_i - \phi^0_i)\frac{dx^i}{ds}\frac{dx^k}{ds}. \tag{4.5}$$

where s is arc-length parameter. By comparing Equation(4.4) with Equation(4.5), one obtains that a SR-parallel curve with respect to the nearly sub-Lyra connection does not coincide with the horizontal length minimizer associated with the metric.

5. A geometric characterization of a SNS-non-metric connection

In this subsection, we will considered a class of non-symmetric connection, in briefly, a SNS-non-metric connection, and give a geometric property of a SNS-non-metric connection.

Definition 5.1. Let (M, V_0, g) be a nearly sub-Riemannian manifold, a linear connection $D : \Gamma(TM) \times \Gamma(V_0) \to \Gamma(V_0)$ is said to be a semi-symmetric non-metric connection, in briefly, a SNS-non-metric connection, if it satisfies

$$\begin{cases}(D_Z g)(X_0, Y_0) = -\pi(X_0)g(Y_0, Z_0) - \pi(Y_0)g(X_0, Z_0),\\ T_D(X, Y_0) = \pi(Y_0)X_0 - \pi(X)Y_0.\end{cases} \tag{5.1}$$

where π is a 1-form.

One can derive that Equations (5.1) determined uniquely a SNS-non-metric connection,

$$\begin{aligned} 2g(D_{X_0}Y_0, Z_0) &= X_0(g(Y_0, Z_0)) + Y_0(g(Z_0, X_0)) - Z_0(g(X_0, Y_0)) + g([X_0, Y_0]_0, Z_0)\\ &= -g([Y_0, Z_0]_0, X_0) + g([Z_0, X_0]_0, Y_0) + 2\pi(Y_0)g(X_0, Z_0),\\ D_{X_1}Y_0 &= [X_1, Y_0]_0, \end{aligned}$$

namely,

$$D_{X_0}Y_0 = \nabla_{X_0}Y_0 + \pi(Y_0)X_0, D_{X_1}Y_0 = [X_1, Y_0]_0. \tag{5.2}$$

in our adapted frame system, it can be rewritten by

$$D^k_{ij} = \{^k_{ij}\} + \pi_j\delta^k_i, D^k_{\alpha j} = 0. \tag{5.3}$$

Definition 5.2. For two classes of SNS-non-metric connection D_1 and D_2, let $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ be their symmetrization of connection coefficients, if the horizontal invariant curve associated with $\bar{D}_1$ corresponds always to that associated with $\bar{D}_2$, then we say D_1 is a SR-projective transformation of D_2(or, D_2 is a SR-projective transformation of D_1).

Theorem 5.1. *The SNS-non-metric connection (5.2) is essentially a SR-projective transformation.*

Proof. We denote the symmetrization of connection coefficients of (5.2) by $\bar{D}^k_{ij}$, then

$$\bar{D}^k_{ij} = \frac{D^k_{ij} + D^k_{ji}}{2} = \{^k_{ij}\} + \frac{1}{2}(\pi_i\delta^k_j + \pi_j\delta^k_i), \tag{5.4}$$

If $\gamma : x^a = x^a(t)$ is a horizontal invariant curve associated with the HSR-connection, then it satisfies Equations (5.2), one can obtain by substituting Equations (5.4) into Equations (5.2),

$$\frac{d^2x^k}{dt^2} + \bar{D}^k_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = \beta(t)\frac{dx^k}{dt}. \tag{5.5}$$

where $\beta(t) = \alpha(t) + \pi_i\frac{dx^i}{dt}$.

The converse statement is also true by simple computation, which means a horizontal invariant curve associated with the SR-connection corresponds that with respect to the SNS-non-metric connection, and hence the proof is finished. □

Theorem 5.2. *A connection transformation between a nearly sub-Weyl manifold and a nearly sub-Lyra manifold keeps the horizontal invariant curves unchanged.*

Proof. In a local reference system with the normal gauge $x^0 = 1$, by comparing the nearly sub-Weyl connection (3.4) with the nearly sub-Lyra connection (4.3), one has

$$\tilde{\Gamma}^k_{ij} = \hat{\Gamma}^k_{ij} - \frac{1}{2}\delta^k_i \varphi_j.$$

if we choose the 1-form ϕ in the nearly sub-Lyra connection (4.3) is exactly the 1-form φ in the nearly sub-Weyl connection (3.4). Therefore the proof follows from Theorem 5.1. □

At the end of this paper, we give a decomposition of a SNS-non-metric connection.

Theorem 5.3. *A SNS-non-metric connection always decompose into a nearly sub-Weyl connection and a nearly sub-Lyra connection.*

Proof. In a local reference system with the normal gauge $x^0 = 1$, by comparing the nearly sub-Weyl connection (3.4) and the nearly sub-Lyra connection (4.3) with the SNS-non-metric connection (5.3), one has

$$\begin{aligned} D^k_{ij} &= \{^k_{ij}\} + \pi_j \delta^k_i \\ &= [\frac{1}{2}\{^k_{ij}\} + (\pi_j\delta^k_i + \pi_i\delta^k_j - \pi^k g_{ij})] + [\frac{1}{2}\{^k_{ij}\} + (-\pi_i\delta^k_j + \pi^k g_{ij})] \\ &= \frac{1}{2}[\{^k_{ij}\} + \frac{1}{2}(4\pi_j\delta^k_i + 4\pi_i\delta^k_j - 4\pi^k g_{ij})] + \frac{1}{2}[\{^k_{ij}\} + \frac{1}{2}((-2\pi_i)\delta^k_j - (-2\pi^k)g_{ij})] \\ &= \frac{1}{2}\hat{\Gamma}^k_{ij} + \frac{1}{2}\tilde{\Gamma}^k_{ij}. \end{aligned}$$

This finishes the proof. □

6. Examples

Example 6.1. (Almost contact metric manifold)

Let M be a $(2n+1)$-dimensional almost contact manifold endowed with an almost contact strucuture (φ, ξ, η), where φ is a $(1,1)$-tensor field, ξ is a vector field and η is a 1-form such that

$$\varphi^2 = -I + \eta \otimes \xi, \eta(\xi) = 1.$$

If the Riemannian metric g satisfies

$$g(X, \varphi Y) = -g(\varphi X, Y), g(X, \xi) = \eta(\xi).$$

then (φ, ξ, η, g) is called an almost contact metric structure and M is called an almost contact metric manifold. Now we define a linear connection on such manifold

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X,$$

where ∇ is the levi-Civita connection associated with Riemannian metric g, then we obtain

$$T(X,Y) = \eta(Y)X - \eta(X)Y, \alpha = -\eta \otimes \eta.$$

which shows that $\tilde{\nabla}$ is a semi-symmetric non-metric connection.

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