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Classification of Lagrangian H-umbilical Surfaces of Constant Curvature in Complex Lorentzian Plane

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ABSTRACT

Chen and Fastenakels classified all flat Lagrangian Surfaces in Complex Lorentzian Plane $\mathbf{C}_1^2$ in [7] . In this article, we completely classify non-flat Lagrangian H-umbilical Surfaces of constant curvature in Complex Lorentzian Plane $\mathbf{C}_1^2$.

Keywords: *Lagrangian submanifold, H-umbilical submanifold, Complex Lorentzian plane*
AMS Subject Classification (2010): *Primary 53C40, 53C15, 53C25.*

1. Introduction

Let $L : M \to \mathbf{C}^n$ be a Lagrangian isometric immersion. For $n \geq 3$, if L is a Lagrangian H-umbilical immersion of constant sectional curvature, then L is flat or $\lambda = 2\mu \neq 0$. Therefore L is flat or locally a Lagrangian pseudo-sphere [Theorem 3.1 in [2]] .

The situation in $n = 2$ is much more complicated. Lagrangian H-umbilical Surfaces with $\lambda = 2\mu$ in complex Euclidean plane consist of a much bigger famliy of surfaces including the Lagrangian pseudo-sphere. Lagrangian H-umbilical Surfaces of constant curvature in complex Euclidean plane are completely classified in [3] .

In [4] B.-Y. Chen proved that for $n \geq 3$, if L is a Lagrangian H-umbilical submanifold of constant sectional curvature in the indefinite complex Euclidean space $\mathbf{C}_\mathbf{k}^\mathbf{n}$, then L is flat or $\lambda = 2\mu \neq 0$. Hence L is flat, or locally either a Lagrangian pseudo-Riemannian sphere or a Lagrangian pseudo-hyperbolic space [Theorem 4.1 in [4]].

For $n = 2$, Chen and Fastenakels classified all flat Lagrangian Surfaces in Complex Lorentzian Plane $\mathbf{C}_1^2$ in [7]. In this article, we completely classify non-flat Lagrangian H-umbilical surfaces of constant curvature in complex Lorentzian plane $\mathbf{C}_1^2$. Similar to Riemanian case, Lagrangian H-umbilical surfaces with $\lambda = 2\mu \neq 0$ in complex Lorentzian plane come from two large families of surfaces containing Lagrangian pseudo-Riemannian 2 sphere and Lagrangian pseudo-hyperbolic 2 space. We also determine all the cases without the condition $\lambda = 2\mu \neq 0$. Our results complete the classification of Lagrangian H-umbilical submanifolds of constant sectional curvature in indefinite complex Euclidean spaces.

2. Preliminaries

Let $L : M \to \mathbf{C}_1^2$ be an isometric immersion of a 2-dimensional pseudo-Riemannian manifold M into the complex Lorentzian plane $\mathbf{C}_1^2$. Then M is called a Lagrangian (or totally real) submanifold if the almost complex structure J of $\mathbf{C}_1^2$ carries each tangent space of M into its corresponding normal space. The formulas of Gauss

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and Weingarten are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \qquad (2.1)$$
$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi,$$

for tangent vector fields X and Y and normal vector fields ξ, where D is the normal connection. The second fundamental form h is related to A_ξ by

$$\langle h(X,Y), \xi\rangle = \langle A_\xi X, Y\rangle .$$

The mean curvature vector of M in $\mathbf{C}^2_1$ is defined by

$$H = \frac{1}{2}\,\text{trace}\, h$$

The Gauss and Codazzi equations are given by

$$\langle R(X,Y)Z,W\rangle = \langle h(X,W), h(Y,Z)\rangle - \langle h(X,Z), h(Y,W)\rangle ,$$
$$(\nabla h)(X,Y,Z) = (\nabla h)(Y,X,Z),$$

where (∇h) is defined by

$$(\nabla h)(X,Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

When M is a Lagrangian surface in $\mathbf{C}^2_1$, we have

$$D_X JY = J\nabla_X Y,$$
$$\langle h(X,Y), JZ\rangle = \langle h(Y,Z), JX\rangle = \langle h(Z,X), JY\rangle .$$

It is well known that there exist no totally umbilical Lagrangian submanifolds in a complex or psuedo complex space-form with $n \geq 2$ except the totally geodesic ones (see [8]). To investigate the "simplest" Lagrangian submanifolds next to the totally geodesic ones in complex or psuedo complex space-forms, B.-Y. Chen introduced the concept of *Lagrangian H-umbilical submanifolds* in [2, 4].

If $L : M \to \mathbf{C}^2_1$ is a Lagrangian H-umbilical surface, the second fundamental form takes the following form:

$$h(e_1, e_1) = \lambda J e_1, \quad h(e_1, e_2) = \mu J e_2, \quad h(e_2, e_2) = -\mu J e_1 \qquad (2.2)$$

for some suitable functions λ and μ with respect to some suitable orthonormal local frame field.

For vectors in $\mathbf{C}^2_1$, we have the following lemma (Lemma 2.3 in [6])

Lemma 2.1. *Let u, v be any two vectors in $\mathbf{C}^2_1$ and let a, b be any two complex numbers. Then we have*

$$\langle au, bv\rangle = \langle a, b\rangle \langle u, v\rangle + \langle ia, b\rangle \langle u, iv\rangle ,$$
$$\langle au, ibv\rangle = \langle a, b\rangle \langle u, iv\rangle + \langle a, ib\rangle \langle u, v\rangle ,$$

where $\langle a, b\rangle$ and $\langle u, v\rangle$ are cononical product for complex numbers and cononical inner product for vectors in $\mathbf{C}^2_1$.

3. Lagrangian H-umbilical Surfaces of constant curvature in $\mathbf{C}^2_1$

Let $L : M \to \mathbf{C}^2_1$ be a Lagrangian H-umbilical surface of constant curvature whose second fundamental form satisfies (2.2) with respect to some orthonormal local frame field $\{e_1, e_2\}$. Since the complex structure J interchanges the tangent and normal spaces of M in $\mathbf{C}^2_1$, M has real index 1 (i.e. M is Lorentzian [1] or [6]). We divide the classification into two cases: e_1 is time-like or e_1 is space-like.

Assuming e_1 is time-like, we have

Theorem 3.1. *Let $L : M \to \mathbf{C}_1^2$ be a Lagrangian H-umbilical surface of non-zero constant curvature K whose second fundamental form satisfies (2.2) with respect to some orthonormal local frame field $\{e_1, e_2\}$. If e_1 is time-like, then one of the following four statements holds:*

(1) $K = -b^2$ and L is congruent to the Lagrangian immersion

$$L(s,t) = e^{2ibs} z(t) + \int_0^t z'(t) e^{-2i\theta(t)} dt \tag{3.1}$$

where b is a positive number, $\theta(t)$ a function on (α, β) containing 0, and $z(t)$ a $\mathbf{C}_1^2$ valued solution to the ordinary differential equation:

$$z''(t) - i\theta'(t) z'(t) - b^2 z(t) = 0$$

(2) $K = -b^2 < -1$ and L is congruent to

$$\begin{aligned} L(s,t) = {} & \frac{\cos(bs)}{\sqrt{b^2-1}} \exp\left(i \sin^{-1}\left(\frac{b\sin(bs)}{\sqrt{b^2-1}}\right) - \frac{i}{b}\tan^{-1}\left(\frac{\sin(bs)}{\sqrt{b^2\cos^2(bs)-1}}\right)\right) \\ & \times \left(-ic_1 + \cosh(\sqrt{b^2-1}t),\ -ic_2 + \sinh(\sqrt{b^2-1}t)\right) \\ & + \left(\int_0^s \exp\left(2i\sin^{-1}\left(\frac{b\sin(bs)}{\sqrt{b^2-1}}\right) - \frac{i}{b}\tan^{-1}\left(\frac{\sin(bs)}{\sqrt{b^2\cos^2(bs)-1}}\right)\right) ds\right)(c_1, c_2) \end{aligned} \tag{3.2}$$

for some constants c_1, c_2.

(3) $K = -b^2$ and L is congruent to

$$\begin{aligned} L(s,t) = {} & \frac{\cos(bs)}{\sqrt{b^2+1}} \exp\left(i \sin^{-1}\left(\frac{b\sin(bs)}{\sqrt{b^2+1}}\right) + \frac{i}{b}tanh^{-1}\left(\frac{\sin(bs)}{\sqrt{b^2\cos^2(bs)+1}}\right)\right) \\ & \times \left(-ic_1 + \cosh(\sqrt{b^2+1}t),\ -ic_2 + \sinh(\sqrt{b^2+1}t)\right) \\ & + \left(\int_0^s \exp\left(2i\sin^{-1}\left(\frac{b\sin(bs)}{\sqrt{b^2+1}}\right) + \frac{i}{b}tanh^{-1}\left(\frac{\sin(bs)}{\sqrt{b^2\cos^2(bs)+1}}\right)\right) ds\right)(c_1, c_2) \end{aligned} \tag{3.3}$$

for some constants c_1, c_2.

(4) $K = b^2$ and L is congruent to

$$\begin{aligned} L(s,t) = {} & \frac{\cosh(bs)}{\sqrt{1-b^2}} \exp\left(i \sin^{-1}\left(\frac{b\sinh(bs)}{\sqrt{1-b^2}}\right) + \frac{i}{b}\tan^{-1}\left(\frac{\sinh(bs)}{\sqrt{1-b^2\cosh^2(bs)}}\right)\right) \\ & \times \left(-ic_1 + \cosh(\sqrt{1-b^2}t),\ -ic_2 + \sinh(\sqrt{1-b^2}t)\right) \\ & + \left(\int_0^s \exp\left(2i\sin^{-1}\left(\frac{b\sinh(bs)}{\sqrt{1-b^2}}\right) + \frac{i}{b}\tan^{-1}\left(\frac{\sinh(bs)}{\sqrt{1-b^2\cosh^2(bs)}}\right)\right) ds\right)(c_1, c_2) \end{aligned} \tag{3.4}$$

for some constants c_1, c_2.

Proof. Let $L : M \to \mathbf{C}_1^2$ be a Lagrangian H-umbilical surface of non-zero constant curvature K whose second fundamental form satisfies (2.2) with respect to some orthonormal local frame field $\{e_1, e_2\}$. Since e_1 is time-like, from (2,2), Gauss and Codazzi equations we find ([9] and [4]):

$$\begin{aligned} & e_1\mu = (\lambda - 2\mu)\omega_1^2(e_2), \\ & e_2\lambda = (2\mu - \lambda)\omega_1^2(e_1), \\ & e_2\mu = -3\mu\omega_1^2(e_1), \\ & K = \mu(\mu - \lambda) = constant. \end{aligned} \tag{3.5}$$

Differentiating with respect to e_2 the last equation of (3.5), we have

$$0 = e_2K = (2\mu - \lambda)\, e_2\mu - \mu e_2\lambda = 4\mu(2\mu - \lambda)\omega_2^1(e_1) \tag{3.6}$$

If $\mu(\lambda - 2\mu) = 0$, then $K = 0$ or $\lambda = 2\mu$. Since $K \neq 0$ (flat Lagrangian surfaces in Complex Lorentzian Plane $\mathbf{C}_1^2$ are completely classified in [7]), we have $\lambda = 2\mu$. Thus, by Theorem 3.2 of [9], we have statement (1). Lagrangian H-umbilical surfaces with $\lambda = 2\mu$ in Complex Lorentzian Plane $\mathbf{C}_1^2$ are completely classified in [9] and statement (1) is one of the main results.

If $\mu(\lambda - 2\mu) \neq 0$, from (3.6) we get $\omega_2^1(e_1) = 0$. Hence from (3.5) we have $e_2\lambda = e_2\mu = 0$.

Since $\nabla_{e_1}e_1 = \omega_1^2(e_1)e_2 = 0$, the integral curves of e_1 are geodesic in M. Thus, there exists a local coordinate system $\{s, u\}$ on M such that the metric tensor is given by

$$g = -ds^2 + G^2(s,u)du^2 \tag{3.7}$$

for some function G with $\partial/\partial s = e_1, \partial/\partial u = Ge_2$.
From (3.7) we have

$$\nabla_{\partial/\partial u}\frac{\partial}{\partial s} = (lnG)_s\frac{\partial}{\partial u}, \quad \omega_1^2(e_2) = \frac{G_s}{G} \tag{3.8}$$

Since $e_2\lambda = e_2\mu = 0$, we get $\lambda = \lambda(s)$ and $\mu = \mu(s)$.

From the first and the last equations of (3.5), we have

$$(lnG)_s = \omega_1^2(e_2) = \frac{\mu'}{\lambda - 2\mu} = \frac{-\mu\mu'}{K + \mu^2}$$

Solving this equation gives $G = F(u)/\sqrt{|K + \mu^2|}$ for some function F.

Therefore, (3.7) becomes

$$g = -ds^2 + \frac{F^2(u)}{|K + \mu^2(s)|}du^2 \tag{3.9}$$

If t is an anti-derivative of $F(u)$, we have from (3.9)

$$g = -ds^2 + \frac{dt^2}{|K + \mu^2(s)|}, \qquad G^2(s) = \frac{1}{|K + \mu^2(s)|} \tag{3.10}$$

Case (a): $K = -b^2 < 0$

From (3.7), the Gauss curvature K of M is given by (p.81 in [10]):

$$K = G_{ss}/G$$

Therefore we have $G_{ss} + b^2G = 0$. Solving this equation yields

$$G = A\cos(bs) + B\sin(bs)$$

for some constants A and B, not both zero. Thus, we have

$$g = -ds^2 + r^2\ \cos^2(bs + c)du^2 \tag{3.11}$$

fo some constants $r \neq 0$ and c. After a suitable translation in s and a suitable dilation in t, (3.11) becomes

$$g = -ds^2 + \cos^2(bs)dt^2 \tag{3.12}$$

From (3.12) we have

$$\begin{aligned} &\nabla_{\partial/\partial s}\frac{\partial}{\partial s} = 0, \qquad \nabla_{\partial/\partial s}\frac{\partial}{\partial t} = -b\,\tan(bs)\frac{\partial}{\partial t}, \\ &\nabla_{\partial/\partial t}\frac{\partial}{\partial t} = -\frac{b}{2}\sin(2bs)\frac{\partial}{\partial s}. \end{aligned} \tag{3.13}$$

Case (a-1): $K = -b^2 < -\mu^2$ and $b > 0$.

From (3.10) and (3.12), we have $|K + \mu^2(s)| = b^2 - \mu^2 = \sec^2(bs) \geq 1$.

Without loss of generality, we assume that

$$\mu = \sqrt{b^2 - \sec^2(bs)}, \qquad \lambda = \frac{2b^2 - \sec^2(bs)}{\sqrt{b^2 - \sec^2(bs)}} \tag{3.14}$$

From (2.2), $(3.12) - (3.14)$, and Gauss formula we see that the immersion satisfies the following system of PDEs:

$$\begin{aligned} L_{ss} &= i\,\frac{2b^2 - \sec^2(bs)}{\sqrt{b^2 - \sec^2(bs)}}\;L_s, \\ L_{st} &= \left(i\,\sqrt{b^2 - \sec^2(bs)} - b\,\tan(bs)\right)L_t \\ L_{tt} &= -\left(i\sqrt{b^2\cos^2(bs) - 1} + b\sin(bs)\right)\cos(bs)L_s \end{aligned} \tag{3.15}$$

After solving the second equation of (3.15), we have

$$L_t = F(t)\cos(bs)\exp\left(i\sin^{-1}\left(\frac{b\sin(bs)}{\sqrt{b^2-1}}\right) - \frac{i}{b}\tan^{-1}\left(\frac{\sin(bs)}{\sqrt{b^2\cos^2(bs)-1}}\right)\right) \tag{3.16}$$

for some $\mathbf{C}^2_1$-valued function $F(t)$. Thus, we have

$$L \;=\; A(s) + B(t)\cos(bs)\exp\left(i\sin^{-1}\left(\frac{b\sin(bs)}{\sqrt{b^2-1}}\right) - \frac{i}{b}\tan^{-1}\left(\frac{\sin(bs)}{\sqrt{b^2\cos^2(bs)-1}}\right)\right) \tag{3.17}$$

where $B(t)$ is an anti-derivative of $F(t)$ and $A(s)$ is a $\mathbf{C}^2_1$-valued function. From (3.17) we have

$$\begin{aligned} L_s \;=\;& A'(s) - B(t)\left(b\sin(bs) - i\sqrt{b^2\cos^2(bs)-1}\right) \\ &\times \exp\left(i\sin^{-1}\left(\frac{b\sin(bs)}{\sqrt{b^2-1}}\right) - \frac{i}{b}\tan^{-1}\left(\frac{\sin(bs)}{\sqrt{b^2\cos^2(bs)-1}}\right)\right), \end{aligned} \tag{3.18}$$

$$\begin{aligned} L_{ss} \;=\;& A''(s) - B(t)\frac{(\sqrt{b^2 - \sec^2(bs)} + ib\tan(bs))(2b^2\cos^2(bs) - 1)}{\sqrt{b^2\cos^2(bs)-1}} \\ &\times \exp\left(i\sin^{-1}\left(\frac{b\sin(bs)}{\sqrt{b^2-1}}\right) - \frac{i}{b}\tan^{-1}\left(\frac{\sin(bs)}{\sqrt{b^2\cos^2(bs)-1}}\right)\right) \end{aligned} \tag{3.19}$$

Placing (3.18) and (3.19) into the first equation of (3.15), we get

$$A''(s) = i\,\frac{2b^2 - \sec^2(bs)}{\sqrt{b^2 - \sec^2(bs)}}\;A'(s) \tag{3.20}$$

Solving this equation, we have

$$A'(s) = C\exp\left(2i\sin^{-1}\left(\frac{b\sin(bs)}{\sqrt{b^2-1}}\right) - \frac{i}{b}\tan^{-1}\left(\frac{\sin(bs)}{\sqrt{b^2\cos^2(bs)-1}}\right)\right) \tag{3.21}$$

for some vector C in $\mathbf{C}^{\mathbf{2}}_{\mathbf{1}}$. Hence, we have

$$A(s) = C\int_0^s \exp\left(2i\sin^{-1}\left(\frac{b\sin(bs)}{\sqrt{b^2-1}}\right) - \frac{i}{b}\tan^{-1}\left(\frac{\sin(bs)}{\sqrt{b^2\cos^2(bs)-1}}\right)\right)ds + E \tag{3.22}$$

for some vector E in $\mathbf{C}^{\mathbf{2}}_{\mathbf{1}}$.

By a suitable translation, we may assume $E = 0$. Hence, we have from (3.17)

$$\begin{aligned} L = {}& C\int_0^s \exp\left(2i\sin^{-1}\left(\frac{b\sin(bs)}{\sqrt{b^2-1}}\right) - \frac{i}{b}\tan^{-1}\left(\frac{\sin(bs)}{\sqrt{b^2\cos^2(bs)-1}}\right)\right)ds \\ &+ B(t)\cos(bs)\exp\left(i\sin^{-1}\left(\frac{b\sin(bs)}{\sqrt{b^2-1}}\right) - \frac{i}{b}\tan^{-1}\left(\frac{\sin(bs)}{\sqrt{b^2\cos^2(bs)-1}}\right)\right), \end{aligned} \tag{3.23}$$

Thus, we have

$$L_s = \left(C + i\sqrt{b^2-1}B(t)\right)\exp\left(2i\sin^{-1}\left(\frac{b\sin(bs)}{\sqrt{b^2-1}}\right) - \frac{i}{b}\tan^{-1}\left(\frac{\sin(bs)}{\sqrt{b^2\cos^2(bs)-1}}\right)\right) \tag{3.24}$$

$$L_t = B'(t)\cos(bs)\exp\left(i\sin^{-1}\left(\frac{b\sin(bs)}{\sqrt{b^2-1}}\right) - \frac{i}{b}\tan^{-1}\left(\frac{\sin(bs)}{\sqrt{b^2\cos^2(bs)-1}}\right)\right) \tag{3.25}$$

Placing (3.24) and (3.25) into the last equation of (3.15), we get

$$B''(t) - (b^2-1)B(t) = -i\sqrt{b^2-1}\,C \tag{3.26}$$

From (3.26) we have

$$B(t) = C_1\cosh(\sqrt{b^2-1}\,t) + C_2\sinh(\sqrt{b^2-1}\,t) - \frac{i}{\sqrt{b^2-1}}\,C \tag{3.27}$$

Combining (3.23) and (3.27), we have

$$
\begin{aligned}
L = \; & C \int_0^s \exp\left(2i\sin^{-1}\left(\frac{b\sin(bs)}{\sqrt{b^2-1}}\right) - \frac{i}{b}\tan^{-1}\left(\frac{\sin(bs)}{\sqrt{b^2\cos^2(bs)-1}}\right)\right) ds \\
& + \left(C_1\cosh(\sqrt{b^2-1}\,t) + C_2\sinh(\sqrt{b^2-1}\,t) - \frac{i}{\sqrt{b^2-1}}\,C\right)\cos(bs) \\
& \times \exp\left(i\sin^{-1}\left(\frac{b\sin(bs)}{\sqrt{b^2-1}}\right) - \frac{i}{b}\tan^{-1}\left(\frac{\sin(bs)}{\sqrt{b^2\cos^2(bs)-1}}\right)\right),
\end{aligned}
\tag{3.28}
$$

Therefore, we obtain statement (2) by choosing suitable initial conditions, i.e. the immersion is congruent to (3.2).

Case (a-2): $K = -b^2 > -\mu^2$ and $b > 0$.

From (3.10) and (3.12), we have $|K + \mu^2(s)| = \mu^2 - b^2 = \sec^2(bs)$.

Hence, without loss of generality, we assume that

$$
\mu = \sqrt{b^2 + \sec^2(bs)}, \qquad \lambda = \frac{2b^2 + \sec^2(bs)}{\sqrt{b^2 + \sec^2(bs)}}
\tag{3.29}
$$

From (2.2), (3.12), (3.14), (3.29), and Gauss formula we see that the immersion satisfies the following system of PDEs:

$$
\begin{aligned}
L_{ss} &= i\,\frac{2b^2 + \sec^2(bs)}{\sqrt{b^2 + \sec^2(bs)}}\;L_s, \\
L_{st} &= \left(i\,\sqrt{b^2 + \sec^2(bs)} - b\,\tan(bs)\right) L_t \\
L_{tt} &= -\left(i\sqrt{b^2\cos^2(bs)+1} + b\sin(bs)\right)\cos(bs) L_s
\end{aligned}
\tag{3.30}
$$

After solving the PDE system (3.30) in the same way as in Case (a-1) and after choosing suitable initial conditions, we obtain statement (3), i.e. the immersion is congruent to (3.3).

Case (b): $K = b^2 > 0$ and $b > 0$.

From the Gauss curvature $K = G_{ss}/G$, we have $G_{ss} - b^2 G = 0$. Solving this equation yields

$$G = A\cosh(bs) + B\sinh(bs)$$

for some constants A and B, not both zero.

After applying a suitable translation in s and a suitable dilation in t, we have

$$
g = -ds^2 + \cosh^2(bs)dt^2
\tag{3.31}
$$

From which we have

$$
\begin{aligned}
&\nabla_{\partial/\partial s}\frac{\partial}{\partial s} = 0, \qquad \nabla_{\partial/\partial s}\frac{\partial}{\partial t} = b\,\tanh(bs)\frac{\partial}{\partial t}, \\
&\nabla_{\partial/\partial t}\frac{\partial}{\partial t} = \frac{b}{2}\sinh(2bs)\frac{\partial}{\partial s}.
\end{aligned}
\tag{3.32}
$$

From (3.10) and (3.31), we have $|K + \mu^2(s)| = b^2 + \mu^2 = sech^2(bs)$.

Therefore, we have $b^2 \leq sech^2(bs) \leq 1$. Without loss of generality, we assume that

$$\mu = \sqrt{sech^2(bs) - b^2}, \quad \lambda = \frac{sech^2(bs) - 2b^2}{\sqrt{sech^2(bs) - b^2}} \tag{3.33}$$

From (2.2), (3.31)-(3.33), and Gauss formula we see that the immersion satisfies the following system of PDEs:

$$\begin{aligned} L_{ss} &= i\ \frac{sech^2(bs) - 2b^2}{\sqrt{sech^2(bs) - b^2}}\ L_s, \\ L_{st} &= \left(i\ \sqrt{sech^2(bs) - b^2} + b\ \tanh(bs)\right) L_t \\ L_{tt} &= -\left(i\sqrt{1 - b^2\cosh^2(bs)} - b\sinh(bs)\right)\cosh(bs) L_s \end{aligned} \tag{3.34}$$

After solving the PDE system (3.34) in the same way as in Case (a-1) and after choosing suitable initial conditions, we obtain statement (4), i.e. the immersion is congruent to (3.4). □

If e_1 is space-like, we have

Theorem 3.2. *Let $L : M \to \mathbf{C}^2_1$ be a Lagrangian H-umbilical surface of non-zero constant curvature K whose second fundamental form satisfies (2.2) with respect to some orthonormal local frame field $\{e_1, e_2\}$. If e_1 is space-like, then one of the following four statements holds:*

(1) $K = b^2$ and L is congruent to the Lagrangian immersion

$$W(s,t) = e^{2ibs} z(t) + \int_0^t z'(t) e^{-2i\theta(t)} dt \tag{3.35}$$

where b is a positive number, $\theta(t)$ a function on (α, β) containing 0, and $z(t)$ a $\mathbf{C}^2_1$ valued solution to the ordinary differential equation:

$$z''(t) - i\theta'(t) z'(t) - b^2 z(t) = 0$$

(2) $K = b^2 > 1$ and L is congruent to

$$\begin{aligned} W(s,t) = {} & \frac{\cos(bs)}{\sqrt{b^2 - 1}}\ \exp\left(i\sin^{-1}\left(\frac{b\sin(bs)}{\sqrt{b^2-1}}\right) - \frac{i}{b}\tan^{-1}\left(\frac{\sin(bs)}{\sqrt{b^2\cos^2(bs) - 1}}\right)\right) \\ & \times \left(-ic_1 + \cosh(\sqrt{b^2-1}t),\ -ic_2 + \sinh(\sqrt{b^2-1}t)\right) \\ & + \left(\int_0^s \exp\left(2i\sin^{-1}\left(\frac{b\sin(bs)}{\sqrt{b^2-1}}\right) - \frac{i}{b}\tan^{-1}\left(\frac{\sin(bs)}{\sqrt{b^2\cos^2(bs) - 1}}\right)\right) ds\right)(c_1, c_2) \end{aligned} \tag{3.36}$$

for some constants c_1, c_2.

(3) $K = b^2$ and L is congruent to

$$\begin{aligned} W(s,t) = {} & \frac{\cos(bs)}{\sqrt{b^2 + 1}}\ \exp\left(i\sin^{-1}\left(\frac{b\sin(bs)}{\sqrt{b^2+1}}\right) + \frac{i}{b}tanh^{-1}\left(\frac{\sin(bs)}{\sqrt{b^2\cos^2(bs) + 1}}\right)\right) \\ & \times \left(-ic_1 + \cosh(\sqrt{b^2+1}t),\ -ic_2 + \sinh(\sqrt{b^2+1}t)\right) \\ & + \left(\int_0^s \exp\left(2i\sin^{-1}\left(\frac{b\sin(bs)}{\sqrt{b^2+1}}\right) + \frac{i}{b}tanh^{-1}\left(\frac{\sin(bs)}{\sqrt{b^2\cos^2(bs) + 1}}\right)\right) ds\right)(c_1, c_2) \end{aligned} \tag{3.37}$$

for some constants c_1, c_2.

(4) $K = -b^2$ *and* L *is congruent to*

$$W(s,t) = \frac{\cosh(bs)}{\sqrt{1-b^2}} \exp\left(i \sin^{-1}\left(\frac{b\sinh(bs)}{\sqrt{1-b^2}}\right) + \frac{i}{b}\tan^{-1}\left(\frac{\sinh(bs)}{\sqrt{1-b^2\cosh^2(bs)}}\right)\right)$$

$$\times\left(-ic_1 + \cosh(\sqrt{1-b^2}t),\ -ic_2 + \sinh(\sqrt{1-b^2}t)\right) \tag{3.38}$$

$$+\left(\int_0^s \exp\left(2i\sin^{-1}\left(\frac{b\sinh(bs)}{\sqrt{1-b^2}}\right) + \frac{i}{b}\tan^{-1}\left(\frac{\sinh(bs)}{\sqrt{1-b^2\cosh^2(bs)}}\right)\right)ds\right)(c_1,c_2)$$

for some constants c_1, c_2.

Proof. Let $L : M \to \mathbf{C}^2_1$ be a Lagrangian H-umbilical surface of non-zero constant curvature K whose second fundamental form satisfies (2.2) with respect to some orthonormal local frame field $\{e_1, e_2\}$. Since e_1 is space-like, from (2,2), Gauss and Codazzi equations we find ([9] and [4]):

$$\begin{aligned} e_1\mu &= -(\lambda - 2\mu)\omega_1^2(e_2), \\ e_2\lambda &= -(2\mu - \lambda)\omega_1^2(e_1), \\ e_2\mu &= 3\mu\omega_1^2(e_1), \\ K &= \mu(\lambda - \mu) = constant. \end{aligned} \tag{3.39}$$

Differentiating with respect to e_2 the last equation of (3.39), we have $0 = \mu(2\mu - \lambda)\omega_2^1(e_1)$.

If $\mu(\lambda - 2\mu) = 0$, then $K = 0$ or $\lambda = 2\mu$. Since $K \neq 0$, we have $\lambda = 2\mu$. Thus, by Theorem 3.2 of [9], we have statement (1).

If $\mu(\lambda - 2\mu) \neq 0$, we get $\omega_2^1(e_1) = 0$. Hence from (3.39) we have $e_2\lambda = e_2\mu = 0$.

Since $\nabla_{e_1} e_1 = -\omega_1^2(e_1)e_2 = 0$, the integral curves of e_1 are geodesic in M. Thus, there exists a local coordinate system $\{s, u\}$ on M such that the metric tensor is given by

$$g = ds^2 - G^2(s,u)du^2 \tag{3.40}$$

for some function G with $\partial/\partial s = e_1, \partial/\partial u = Ge_2$.

Since $< e_2, e_2 >= -1$, from (3.40) we have

$$\nabla_{\partial/\partial u}\frac{\partial}{\partial s} = (lnG)_s\frac{\partial}{\partial u}, \quad \omega_1^2(e_2) = -\frac{G_s}{G} \tag{3.41}$$

Since $e_2\lambda = e_2\mu = 0$, we get $\lambda = \lambda(s)$ and $\mu = \mu(s)$.

From the first and the last equations of (3.39), we have

$$(lnG)_s = -\omega_1^2(e_2) = \frac{\mu'}{\lambda - 2\mu} = \frac{\mu\mu'}{K - \mu^2}$$

Solving this equation gives $G = F(u)/\sqrt{|K-\mu^2|}$ for some function F.

Therefore, (3.40) becomes

$$g = ds^2 - \frac{F^2(u)}{|K - \mu^2(s)|} du^2 \tag{3.42}$$

If t is an anti-derivative of $F(u)$, we have from (3.42)

$$g = ds^2 - \frac{dt^2}{|K - \mu^2(s)|}, \qquad G^2(s) = \frac{1}{|K - \mu^2(s)|} \tag{3.43}$$

The rest of the proof is almost the same as in Case (a) and (b) in Theorem 3.1. Finally we obtain statement (2), (3) and (4) if e_1 is space-like. □

Remark 3.1. Flat Lagrangian Surfaces in Complex Lorentzian Plane $\mathbf{C}_1^2$ are completely classified in [7].

Remark 3.2. Theorem 3.1 and Theorem 3.2 completely classify Lagrangian H-umbilical Surfaces of non-zero constant curvature in Complex Lorentzian Plane $\mathbf{C}_1^2$

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