

Parallel Projected Sphere on a Plane: A New Plane–Geometric Investigation

Thomas L. Toulías* and George E. Lefkaditis

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ABSTRACT

Consider three concentric ellipses E_i , $i = 1, 2, 3$, each defined by a pair of conjugate semi–diameters taken from a given bundle of three coplanar line segments (where only two of them are permitted to coincide). In a proof by G. A. Peschka of the Karl Pohlke’s Fundamental Theorem of Axonometry, a parallel projection of a sphere onto a plane, say \mathbb{E} , is adopted to show that a new concentric (to E_i) ellipse E exists, “circumscribing” all E_i , i.e. E is simultaneously tangent to all $E_i \subset \mathbb{E}$, $i = 1, 2, 3$. Motivated by the above, this paper investigates the plane–geometric problem of determining all the existing circumscribing ellipses (like E) of E_i , $i = 1, 2, 3$, exclusively from the Analytic Plane Geometry’s point of view (unlike the sphere’s parallel projection that requires the adoption of a three–dimensional space). It is proved that, at most, two circumscribing ellipses (of E_i) exist. One of them is always existing while, under certain conditions, another circumscribing ellipse (of E_i), say $E^* (\neq E)$, can also exist. Moreover, in case this second circumscribing ellipse E^* does not exist, then a hyperbola (concentric to E_i) exists instead, and is (simultaneously) tangent to all E_i , $i = 1, 2, 3$. The above results and their calculations are demonstrated by various examples and figures.

Keywords: Pohlke’s Theorem; conjugate semi–diameters of an ellipse; affine transformations; orthogonal decomposition of a symmetric matrix.

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1. Introduction

This work is motivated by a work of Müller and Kruppa [4], where they suggested a proof (out of many in the literature) of Karl Pohlke’s Theorem, also known as the Axonometry’s Fundamental Theorem. Recalling Pohlke’s Theorem, we consider a bundle of three arbitrary chosen coplanar line segments, say OP , OQ and OR , where only one of them can be of zero length (non–degenerated segments). Under the assumption that the points $O, P, Q, R \in \mathbb{E}$ are not collinear (belonging to a plane \mathbb{E}), their corresponding line segments, as above, can always be considered as parallel projections of three other line segments of the three–dimensional space, say O^*P^* , O^*Q^* and O^*R^* respectively, having equal length and being orthogonal with each other; see Fig. 1. The orthogonal projection can also be considered as a special case of the above parallel projection. For further reading about Pohlke’s Theorem see also [1] and [6].

The specific method in [4] of the proof of Pohlke’s Theorem (which consists of finding the orthogonal segments O^*P^* , O^*Q^* and O^*R^*) is based on the adoption of a parallel projection, say \mathcal{P} , applied on an appropriate sphere S onto plane \mathbb{E} (on which OP , OQ and OR lie). The following property of three concentric ellipses (proven through the use of \mathcal{P}) is stated here as a Proposition, [4, pg. 244]:

Proposition 1.1. *Consider four non–collinear points O, P, Q, R on a plane, forming a bundle of three line segments OP , OQ and OR , where only two of them are permitted to coincide. If the pairs (OP, OR) , (OQ, OR) and (OP, OQ) are considered to be pairs of conjugate semi–diameters defining the ellipses, say E_1 , E_2 and E_3 respectively, then a new ellipse E (concentric to E_i) exists and is tangent to all E_i , $i = 1, 2, 3$.*

The use of sphere, for the proving of the above, appears for the first time in a work by J. W. v. Deschanden and subsequently by G. A. Peschka in his elementary proof of Pohlke’s Fundamental Theorem of Axonometry;

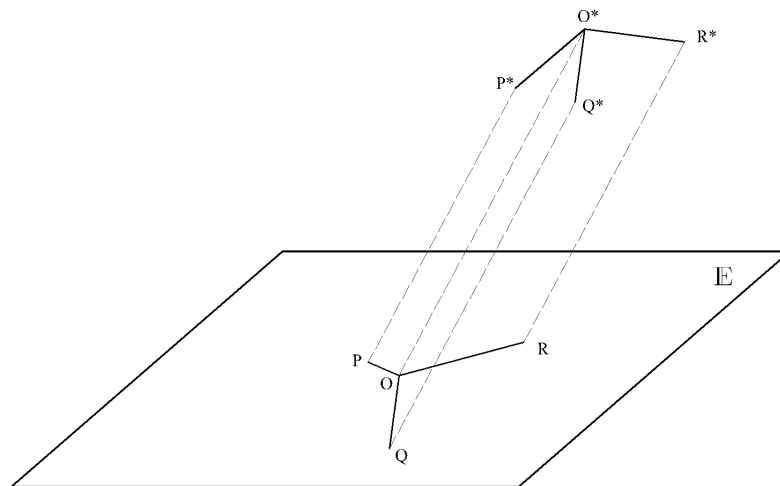


Figure 1. Parallel projection for the Pohlke's Theorem.

see [5]. Under this parallel projection \mathcal{P} , a cylindrical surface is created, tangent to sphere S and around its maximum circle, say $k \subset S$; see Fig. 2. Hence, k is the contour of sphere S through \mathcal{P} , and it is parallel-projected (via \mathcal{P}) onto an ellipse E of plane \mathbb{E} , while the orthogonal line segments O^*P^* , O^*Q^* and O^*R^* are parallel-projected onto radii OP , OQ and OR of the ellipses E_i , $i = 1, 2, 3$. Therefore E is circumscribing all E_i , $i = 1, 2, 3$. According to proof of Proposition 1.1, these ellipses are defined by three pairs of conjugate semi-diameters, or conjugate radii, (OP, OR) , (OQ, OR) and (OP, OQ) respectively, which are parallel projections, through \mathcal{P} , of the corresponding three maximum circles on sphere S . These maximum circles belong to planes (perpendicular to each other) spanned, respectively, by $\{O^*P^*, O^*R^*\}$, $\{O^*Q^*, O^*R^*\}$ and $\{O^*P^*, O^*Q^*\}$. Figure 2 demonstrates the above projections (used by G. A. Peschka) for the proof of Proposition 1.1.

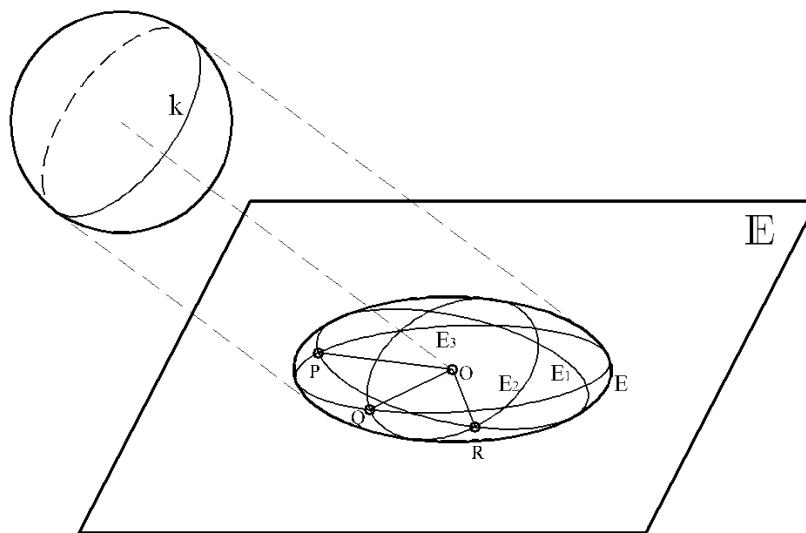


Figure 2. Pohlke's Theorem through a parallel-projected sphere onto plane \mathbb{E} .

One can notice that the problem in Proposition 1.1 is, by its nature, a plane-geometric (two-dimensional) problem. Moreover, the question of how many common tangent ellipses circumscribing all E_i (like E) exist, is open. The problem, of finding the E_i 's "circumscribing" ellipse E is the problem of finding a "common tangential ellipse" (c.t.e.) E of all the ellipses E_i , $i = 1, 2, 3$. A visualization of a c.t.e. is given in Fig. 3. For the topic of concentric circumscribing (tangent) ellipses see [2] among others.

The present paper, in particular, provides a thorough investigation of the the two-dimensional problem of finding a c.t.e. of three given concentric ellipses is addressed, where each of these three ellipses is defined by

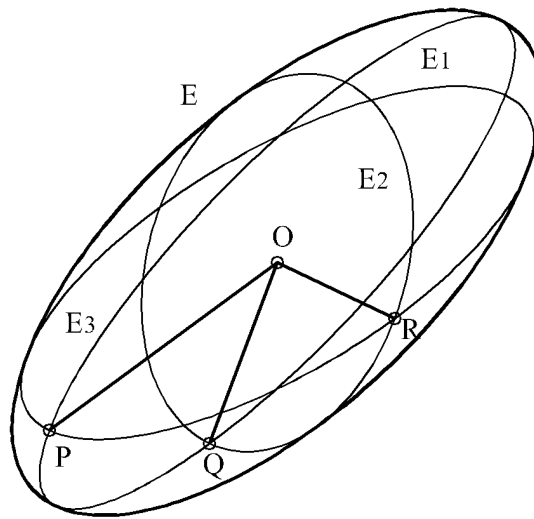


Figure 3. A common tangential ellipse E of all E_i , $i = 1, 2, 3$.

two conjugate radii which belong to a given bundle of three (coplanar) non-degenerated line segments (where only two of them can coincide). Useful formulas are also provided and various examples demonstrate the results. Note that Lefkaditis *et al.* in [3] proposed a construction method of an enveloping ellipse E utilizing Synthetic Projective Geometry of the Plane. Furthermore, we present also a method which provides (not some but) all the existing common tangential ellipses of E_i , $i = 1, 2, 3$. In fact, it is proved that at, most, two common tangential ellipses can exist. One of them always exists (as it is also expected from the sphere's parallel projection \mathcal{P}) while, under certain assumptions, one more c.t.e., say E^* ($\neq E$), can also exist. It is worth mentioning that, when there is only one c.t.e. of E_i (meaning that a second c.t.e. ellipse E^* does not exist), then a common tangential hyperbola exists instead, i.e. a hyperbola (concentric to E and E_i) which is (simultaneously) tangent to all E_i , $i = 1, 2, 3$.

Affine transformations play a key role in this investigation. A study of certain affine transformations is carried out in Section 2, that helps simplify the corresponding formulations. In Section 3, these transformations first apply for the special case where one of the ellipses E_i is assumed to be a circle, and then the investigation is generalized for the case where all E_i are ellipses in general.

2. Ellipse as an Affinic Circle

In this section, a certain affine transformation is studied transforming circles into ellipses, which is needed for the development of our study in Section 3. Firstly, we consider the following preliminary Lemma concerning the rotation of an ellipse around its center.

Lemma 2.1. *Let E be an ellipse with principal radii $0 < b < a$, centered at the origin O of an orthonormal coordinate system. The analytical expression of the rotated ellipse E_θ by an angle $\theta \in [-\pi, \pi]$ around its center O , is then given by*

$$E_\theta : (a^2 \sin^2 \theta + b^2 \cos^2 \theta)x^2 - (a^2 - b^2)(\sin 2\theta)xy + (a^2 \cos^2 \theta + b^2 \sin^2 \theta)y^2 = a^2 b^2. \quad (2.1)$$

Proof. Consider the canonical form of E : $(x/a)^2 + (y/b)^2 = 1$. Then, the result is obtained through the affine transformation (rotation) R_θ defined by $\mathbf{x}' = R_\theta(\mathbf{x}) := \mathbf{R}_\theta \mathbf{x}$, where $\mathbf{x} := (x, y)^T \in \mathbb{R}^{2 \times 1}$ and $\mathbf{x}' := (x', y')^T \in \mathbb{R}^{2 \times 1}$ denote the coordinates' vectors of points X and X' , while

$$\mathbf{R}_\theta := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \in \mathbb{R}_\perp^{2 \times 2}, \quad (2.2)$$

is the usual rotation (orthonormal) matrix which defines the rotation $X' := R_\theta(X)$ of the Euclidean plane \mathbb{R}^2 by an angle θ around its origin O . Thus, applying R_θ to the matrix form of ellipse E , i.e. $E : \mathbf{a}\mathbf{x}^T = 1$, $\mathbf{a} := (a, b) \in \mathbb{R}^2$, we obtain the rotated ellipse $E_\theta := R_\theta(E)$, expressed by (2.1). \square

A given circle can be transformed into an ellipse through an axis-invariant affine transformation which shall be called “axis shear”. Specifically, we shall call “ x -shear” the axis shear that preserves the horizontal axis $x'Ox$ of an given coordinate system (c.s.), while “ y -shear” shall be called the axis shear which preserves the corresponding c.s.’s vertical axis $y'Oy$. Hence, the y -shear transforms the orthonormal vector base $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2\}$ into $\mathcal{B}' = \{\mathbf{u}, \mathbf{e}_2\}$, while the x -shear transforms the vector base \mathcal{B} into $\mathcal{B}' = \{\mathbf{e}_1, \mathbf{u}\}$, $\mathbf{u} \in \mathbb{R}^{2 \times 1} \setminus \{\mathbf{0}\}$, where \mathbf{e}_1 and \mathbf{e}_2 denote the usual orthonormal vectors $\mathbf{e}_1 := (1, 0)^T$ and $\mathbf{e}_2 := (0, 1)^T$ of \mathbb{R}^2 . Note that the c.s. spanned by the vector base \mathcal{B} is an orthonormal c.s. (o.s.c.). The vector $\mathbf{u} \neq \mathbf{0}$ shall be called as the “shearing vector”, while the \mathbf{u} ’s angle (with respect to the adopted o.c.s.) $\omega := \angle(\mathbf{e}_1, \mathbf{u}) \in (-\pi, \pi)$ shall often be called as the “shearing angle”. In particular, the x -shear, say S_x , is an affine transformation defined through its matrix representation $\mathbf{x}' = S_x(\mathbf{x}) := \mathbf{N}_x \mathbf{x}$ where the transformation matrix \mathbf{N}_x is given by

$$\mathbf{N}_x = (\mathbf{e}_1, \mathbf{u}) := \begin{pmatrix} 1 & u \cos \omega \\ 0 & u \sin \omega \end{pmatrix}, \tag{2.3}$$

with $\omega \in (-\pi, \pi)$ being the shearing angle of the x -shear S_x . Respectively, the y -shear, say S_y , is defined by $\mathbf{x}' = S_y(\mathbf{x}) := \mathbf{N}_y \mathbf{x}$, where the transformation matrix \mathbf{N}_y is given by

$$\mathbf{N}_y = (\mathbf{u}, \mathbf{e}_2) := \begin{pmatrix} u \cos \omega & 0 \\ u \sin \omega & 1 \end{pmatrix}, \tag{2.4}$$

with $\omega \in (-\pi, \pi)$ being the corresponding S_y ’s shearing angle, which is the same as in (2.3) (as S_y is referring to the same shearing vector \mathbf{u} as S_x). The x -, or y -shears, are therefore completely defined through a given shearing vector $\mathbf{u} \in \mathbb{R}^{2 \times 1}$, i.e. by its length $u \in \mathbb{R}_+$, and its angle $\omega \in (-\pi, \pi)$ (with respect to some c.s.).

For the ellipse derived from a y -shear of a circle we consider the following. Note that with the term “directive angle” of an ellipse we shall refer hereafter to the angle formed by the ellipse’s major axis with respect to the horizontal $x'Ox$ axis of an adopted c.s. In general, the directive angle of an ellipse with respect to a given line shall refer to the angle between the ellipse’s major axis and this given line.

Lemma 2.2. *Let C be a circle of radius $\rho > 0$ centered at the origin O (of an o.c.s.). The y -sheared circle $S_y(C)$ corresponds to an ellipse, centered also at the origin O , with analytical expression*

$$S_y(C) : (1 + u^2 \sin^2 \omega)x^2 - u^2(\sin 2\omega)xy + u^2(\cos^2 \omega)y^2 = \rho^2 u^2 \cos^2 \omega. \tag{2.5}$$

where $u > 0$ and $\omega \in [-\pi, \pi]$ are the corresponding S_y ’s shearing vector length and angle. The principal radii $0 < b < a$ of $S_y(C)$ are given by

$$a, b = \frac{\sqrt{2}\rho u |\cos \omega|}{\sqrt{u^2 + 1 \pm \sqrt{u^4 + 1 - 2u^2 \cos 2\omega}}}, \tag{2.6}$$

where the minus sign corresponds to the major radius a while the plus sign to the minor radius b . Note that the surface area of $S_y(C)$ adopts the compact form $A = \pi u \rho^2 |\cos \omega|$. Moreover, it holds that $|\omega| < |\theta|$ where θ is the directive angle of the ellipse $S_y(C)$, which is given through the $S_y(C)$ ’s major axis slope $\tan \theta$, by

$$\tan \theta = \frac{u^2 \sin 2\omega}{u^2 \cos 2\omega - 1 + \sqrt{u^4 + 1 - 2u^2 \cos 2\omega}}. \tag{2.7}$$

Proof. For the y -shear S_y , defined by $\mathbf{x}' = S_y(\mathbf{x}) := \mathbf{N}_y \mathbf{x}$, with transformation matrix \mathbf{N}_y as in (2.4), we have that $\mathbf{x} = S_y^{-1}(\mathbf{x}') = \mathbf{N}_y^{-1} \mathbf{x}'$, i.e.

$$\begin{pmatrix} x \\ y \end{pmatrix} =: \mathbf{x} = \mathbf{N}_y^{-1} \mathbf{x}' = \begin{pmatrix} (u \cos \omega)^{-1} & 0 \\ -\tan \omega & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{x'}{u \cos \omega} \\ y' - x' \tan \omega \end{pmatrix}, \tag{2.8}$$

and thus the matrix representation form $C : \mathbf{x}^T \mathbf{x} = \rho^2$ of the circle $C : x^2 + y^2 = \rho^2$ implies that $S_y(C) : \mathbf{x}'^T (\mathbf{N}_y^{-1})^T \mathbf{N}_y^{-1} \mathbf{x}' = \rho^2$ or, equivalently, $S_y(C) : \mathbf{x}^T (\mathbf{N}_y^{-1})^T \mathbf{N}_y^{-1} \mathbf{x} = \rho^2$ (as $S_y(C)$ is referred again to the adopted o.c.s.). Therefore, $S_y(C) : x^2 + u^2(y \cos \omega - x \sin \omega)^2 = \rho^2 u^2 \cos^2 \omega$, and hence (2.5) is obtained. The ellipse $S_y(C)$ is centered also at the origin O . This is true because x - and y -shears preserve the center O as it belongs both to the preserved coordinates’ axes $x'Ox$ and $y'Oy$.

In order to obtain the principal radii $0 < b < a$, the ellipse $S_y(C)$ has to be rotated by its directive angle, say θ , around its center O until its canonical form is obtained (on the adopted o.c.s.), i.e. until $S_y(C)$ is transformed

into $E_y: \tilde{\mathbf{x}}^T \tilde{\mathbf{D}} \tilde{\mathbf{x}} = 1$, $\tilde{\mathbf{x}} \in \mathbb{R}^{2 \times 1}$, where $\tilde{\mathbf{D}} \in \mathbb{R}_{\text{diag}}^{2 \times 2}$ is a real diagonal 2×2 matrix. For this purpose, we consider the rotation transform

$$\tilde{\mathbf{x}} = R_\theta(\mathbf{x}) := \mathbf{R}_\theta \mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^{2 \times 1}, \quad (2.9)$$

with rotation matrix $\mathbf{R}_\theta \in \mathbb{R}_{\perp}^{2 \times 2}$ as in (2.2), that provides the ellipse $S_y(C)$, as in (2.5), from its requested canonical form $E_y: \tilde{\mathbf{x}}^T \tilde{\mathbf{D}} \tilde{\mathbf{x}} = 1$, i.e. $R_\theta(E_y) = S_y(C)$ or $E_y = R_\theta^{-1}(S_y(C))$. Recall the matrix representation of the ellipse $S_y(C)$ (mentioned earlier), i.e.

$$S_y(C): \mathbf{x}^T \mathbf{N} \mathbf{x} = \rho^2, \quad \mathbf{N} := (\mathbf{N}_y^{-1})^T \mathbf{N}_y^{-1} \in \mathbb{R}_{\text{sym}}^{2 \times 2}. \quad (2.10)$$

Then, (2.9) implies that $\mathbf{x} = \mathbf{R}_\theta^{-1} \tilde{\mathbf{x}} = \mathbf{R}_{-\theta}^T \tilde{\mathbf{x}}$, and by substitution to (2.10), it holds that

$$E_y = R_\theta^{-1}(S_y(C)) = (R_\theta^{-1} \circ S_y)(C): \tilde{\mathbf{x}}^T \mathbf{R}_{-\theta} \mathbf{N} \mathbf{R}_\theta \tilde{\mathbf{x}} = \rho^2. \quad (2.11)$$

Recall that the inverse of an orthogonal matrix equals to the transpose of the matrix, i.e. $\mathbf{M}^{-1} = \mathbf{M}^T$ for $\mathbf{M} \in \mathbb{R}_{\perp}^{n \times n}$, $n \in \mathbb{N}$. By orthogonal decomposition of the symmetric matrix $\mathbf{N} \in \mathbb{R}_{\text{sym}}^{2 \times 2}$ we obtain that $\mathbf{N} = \mathbf{V} \mathbf{D} \mathbf{V}^T$, where $\mathbf{D} := \text{diag}(d_1, d_2) \in \mathbb{R}_{\text{diag}}^{2 \times 2}$ is the spectral matrix of \mathbf{N} (i.e. the diagonal 2×2 matrix of the eigenvalues d_1 and d_2 of \mathbf{N}) and $\mathbf{V} \in \mathbb{R}_{\perp}^{2 \times 2}$ is the orthonormal 2×2 matrix of the eigenvectors corresponding to eigenvalues d_1 and d_2 . Thus, $\mathbf{D} = \mathbf{V}^T \mathbf{N} \mathbf{V}$, and by setting $\mathbf{R}_\theta := \mathbf{V}$ (i.e. we adopt as orthonormal rotation matrix \mathbf{R}_θ the (orthonormal) eigenvalues matrix \mathbf{V}), the relation (2.11) implies the canonical form of $S_y(C)$, i.e. $E_y: \tilde{\mathbf{x}}^T (\rho^{-2} \mathbf{D}) \tilde{\mathbf{x}} = 1$. Hence, the major and minor radius of the ellipse $E_y: (d_1 \tilde{x}/\rho)^2 + (d_2 \tilde{y}/\rho)^2 = 1$ (which are the same for the rotated ellipse $S_y(C)$) are then given by $a = \rho/\sqrt{\max\{d_1, d_2\}}$ and $b = \rho/\sqrt{\min\{d_1, d_2\}}$ respectively.

The eigenvalues d_1 and d_2 of the symmetric matrix \mathbf{N} can be calculated through the roots of the \mathbf{N} 's characteristic polynomial $P_{\mathbf{N}}(d) := |\mathbf{N} - d\mathbb{I}_2|$, $d \in \mathbb{R}$, $i = 1, 2$, where \mathbb{I}_2 being the unitary 2×2 matrix. After some algebra, we derive that

$$d_i = \frac{\sec^2 \omega}{2u^2} \left[(u^2 + 1) + (-1)^{i-1} \sqrt{u^4 + 1 - 2u^2 \cos 2\omega} \right], \quad i = 1, 2, \quad (2.12)$$

with $0 < (u^2 - 1)^2 = u^4 - 2u^2 + 1 < u^4 + 1 - 2u^2 \cos 2\omega$ for every $u > 0$, and therefore, the requested major and minor radius of the ellipse $S_y(C)$ correspond to the eigenvalues d_2 and d_1 respectively (as $d_1 > d_2$), and hence given by (2.6).

Moreover, the non-unitary orthogonal eigenvectors $\mathbf{v}_i = (v_{i;1}, v_{i;2})^T \in \mathbb{R}^{2 \times 1}$ that correspond to its eigenvalues d_i , $i = 1, 2$, i.e. $\mathbf{V} = (\mathbf{v}_1, \mathbf{v}_2) \in \mathbb{R}_{\perp}^{2 \times 2}$, are then calculated through (2.12) and (after some algebra), are found to be

$$\mathbf{v}_i = \left(-\frac{u^2 + 1 - 2u^2 \cos^2 \omega + (-1)^{i-1} \sqrt{u^4 + 1 - 2u^2 \cos 2\omega}}{u^2 \sin 2\omega}, 1 \right)^T, \quad i = 1, 2.$$

Hence, as the major radius a corresponds to the eigenvalue d_2 (shown earlier), the directive angle θ of the ellipse $S_y(C)$ is then of the form $\theta = \angle(\mathbf{e}_1, \mathbf{v}_2) = \arctan(v_{2;2}/v_{2;1})$ and it is given by (2.7).

Notice also the fact that the assumption $\tan \theta < \tan \omega$ yields, through (2.7), that $u^2 + 1 > \sqrt{u^4 + 1 - 2u^2 \cos 2\omega}$, for $\omega > 0$, which cannot hold (as it would then implies that $\cos^2 \omega < 0$). Therefore, it holds that $\tan \theta > \tan \omega$ when $\omega > 0$ while, similarly, $\tan \theta < \tan \omega$ when $\omega < 0$. Hence $|\theta| > |\omega|$. \square

Working similarly for the x -shear of the circle C , the following holds.

Lemma 2.3. *Let C be a circle of radius $\rho > 0$ centered at the origin O (of an o.c.s.). The x -sheared circle $S_x(C)$ corresponds to an ellipse, centered also at the origin O , with analytical expression*

$$S_x(C): u^2 (\sin^2 \omega) x^2 - u^2 (\sin 2\omega) xy + (1 + u^2 \cos^2 \omega) y^2 = \rho^2 u^2 \sin^2 \omega. \quad (2.13)$$

where $u > 0$ and $\omega \in [-\pi, \pi]$ are the corresponding shearing vector's length and angle. The principal radii $0 < b < a$ of $S_x(C)$ are given by

$$a, b = \frac{\sqrt{2} \rho u |\sin \omega|}{\sqrt{u^2 + 1 \pm \sqrt{u^4 + 1 + 2u^2 \cos 2\omega}}}, \quad (2.14)$$

where the minus sign corresponds to the major radius a while the plus sign to the minor radius b . Note that the surface area of $S_x(C)$ adopts the compact form $A = \pi u \rho^2 |\sin \omega|$. Moreover, it holds that $|\omega| > |\theta|$ where θ is the directive angle of the ellipse $S_x(C)$, which is given, through the $S_x(C)$'s major axis slope $\tan \theta$, by

$$\tan \theta = \frac{u^2 \sin 2\omega}{1 + u^2 \cos 2\omega + \sqrt{u^4 + 1 + 2u^2 \cos 2\omega}}. \quad (2.15)$$

Proof. Similarly to the proof of Lemma 2.2, the x -shear transformation S_x , is defined by the matrix relation $\mathbf{x}' = S_x(\mathbf{x}) := \mathbf{N}_x \mathbf{x}$ with transformation matrix \mathbf{N}_x as in (2.3). Thus, we obtain $S_x(C): \mathbf{x}^T (\mathbf{N}_x^{-1})^T \mathbf{N}_x^{-1} \mathbf{x} = \rho^2$, from which we derive the analytical expression (2.13).

The eigenvalues d_1 and d_2 of the symmetric matrix $\mathbf{N} := (\mathbf{N}_x^{-1})^T \mathbf{N}_x^{-1} \in \mathbb{R}_{\text{sym}}^{2 \times 2}$ are calculated (after some algebra) as

$$d_i = \frac{\csc^2 \omega}{2u^2} \left[u^2 + 1 + (-1)^i \sqrt{u^4 + 1 + 2u^2 \cos 2\omega} \right], \quad i = 1, 2, \tag{2.16}$$

with $0 < (u^2 - 1)^2 = u^4 - 2u^2 + 1 < u^4 + 1 + 2u^2 \cos 2\omega$ for every $u > 0$, and therefore the requested major and minor radius of the ellipse $S_x(C)$ are correspond to the eigenvalues d_2 and d_1 respectively (as $d_2 < d_1$), i.e. $a = \rho/\sqrt{d_2}$ and $b = \rho/\sqrt{d_1}$, which are given by (2.14).

Moreover, the non-unitary orthogonal eigenvectors $\mathbf{v}_i = (v_{i;1}, v_{i;2})^T \in \mathbb{R}^{2 \times 1}$ that correspond to its eigenvalues $d_i, i = 1, 2$ are then calculated through (2.16) and (after some algebra), are found to be

$$\mathbf{v}_i = \left(-\frac{1 + 2u^2 \cos^2 \omega + (-1)^{i-1} \sqrt{u^4 + 1 + 2u^2 \cos 2\omega}}{u^2 \sin 2\omega}, 1 \right), \quad i = 1, 2.$$

Hence, as the major radius a corresponds to the eigenvalue d_2 , the directive angle θ of the ellipse $S_x(C)$ is then of the form $\theta = \angle(\mathbf{e}_1, \mathbf{v}_2) = \arctan(v_{2;2}/v_{2;1})$ and it is given by (2.15).

Notice that the assumption $\tan \theta > \tan \omega > 0$ yields, through (2.15), that $u^2 - 1 > \sqrt{u^4 + 1 + 2u^2 \cos 2\omega}$ which cannot hold (as it then yields that $\cos^2 \omega < 0$). Therefore, $\tan \theta < \tan \omega$ for $\omega > 0$, and respectively, $\tan \theta > \tan \omega$ for $\omega < 0$, i.e. $|\theta| < |\omega|$. \square

The following example clarifies the x - and y -shear transformations of a circle, which provides two ellipses having a common radius. It is also the basis for some other examples provided hereafter.

Example 2.1. Let C be a circle of radius $\rho := 10$ centered at the origin O of an o.c.s. spanned by the orthonormal vector base $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2\}$, and its points $P, Q \in C$, with $P(\rho, 0), Q(0, \rho)$, which define C 's orthogonal radii OP and OQ . These radii can be considering as C 's trivially conjugate radii, with $OP \perp OQ$ and $|OP| = |OQ| = \rho = 10$. Let also a third point R with $|OR| = r := 15$ and angle $\angle(\mathbf{e}_1, OR) = \omega := \pi/6 (= 30^\circ)$.

The y -shear transformation S_y which transforms point $P(\rho, 0) \in C$ into R while preserves point $Q \in C$, is the one having shearing vector $\mathbf{u} = \rho^{-1}OR$ (of length $u = |\mathbf{u}| = r/\rho = 3/2$ and angle $\omega = \pi/6$). Indeed, it can be shown through (2.4) that $S_y(Q) = Q$ and $S_y(P) = R$, with u and ω values as above. Therefore, the y -sheared circle corresponds to a concentric ellipse, say $E_y := S_y(C)$, for which OR and OQ are its two conjugate radii. The conjugality of radii OP and OQ is derived from the fact that the y -shear (as well as x -shears) is, in principle, an affine transformation and, as such, it preserves the parallelism on the plane; see Fig. 4 where the circumscribing square frame of the circle C is transformed through S_y into a parallelogram circumscribing $E_y = S_y(C)$.

Working similarly, the x -shear transformation S_x which transforms $Q(\rho, 0) \in C$ into R while preserves point $P \in C$, is the one having the same shearing vector \mathbf{u} as S_y . From (2.3), it holds that $S_x(Q) = R$ and $S_x(P) = P$. Thus, the x -sheared circle corresponds to a concentric ellipse, say $E_x := S_x(C)$, for which OR and OP are its two conjugate radii; see also Fig. 4.

Therefore, the given bundle of the three line segments OP, OQ and OR corresponds to a bundle of three common (conjugate per pair) radii which define the ellipses E_x, E_y and, trivially, the circle C . In particular, we may write: $E_x = E_x(OP, OR), E_y = E_y(OQ, OR)$ and $C = C(OP, OQ)$, meaning that E_x is defined by (its conjugate radii) OP and OQ, E_y is defined by QP and OR , while C is trivially defined by OP and OQ . Figure 4 provides again a clarification of the above discussion.

For the specific example, the analytical expression of ellipse E_y is of the form $E_y: 25x^2 - 18\sqrt{3}xy + 27y^2 = 2700$, due to (2.5) where it was set $u := 3/2$, while its principal radii $0 < b_y < a_y$ are given by $a_y = \frac{5}{2}\sqrt{26 + 2\sqrt{61}} \approx 16.1285$ and $b_y = \frac{5}{2}\sqrt{26 - 2\sqrt{61}} \approx 8.0543$, through (2.6). From (2.7), E_y 's directive angle θ_y is then given by $\theta_y = \arctan \left\{ \frac{\sqrt{3}}{27} (2\sqrt{61} - 1) \right\} \approx 43.1648^\circ$.

The analytical expression of the ellipse E_x is of the form $E_x: 9x^2 - 18\sqrt{3}xy + 43y^2 = 900$, due to (2.13) where was also set $u := 3/2$, while its principal radii $0 < b_x < a_x$ are given by $a_x = \frac{5}{2}(\sqrt{19} + \sqrt{7}) \approx 17.5116$ and $b_x = \frac{5}{2}(\sqrt{19} - \sqrt{7}) \approx 4.2829$, through (2.14). From (2.15), the E_x 's directive angle θ_x is given by $\theta_x = \arctan \left\{ \frac{\sqrt{3}}{27} (2\sqrt{133} - 17) \right\} \approx 21.2599^\circ$.

Figure 4 visualizes exactly Example 2.1 by depicting the circle C together with its x - and y -shears, i.e. the ellipses $E_x = S_x(C)$ and $E_y = S_y(C)$. The S_x and S_y axis shears (of shearing vector $\mathbf{u} = \rho^{-1}OR$) are also illustrated

by presenting the square frame around C and its transformed parallelograms, through S_x and S_y , around the ellipses E_x and E_y respectively. The corresponding foci F_x and F_y are also depicted.

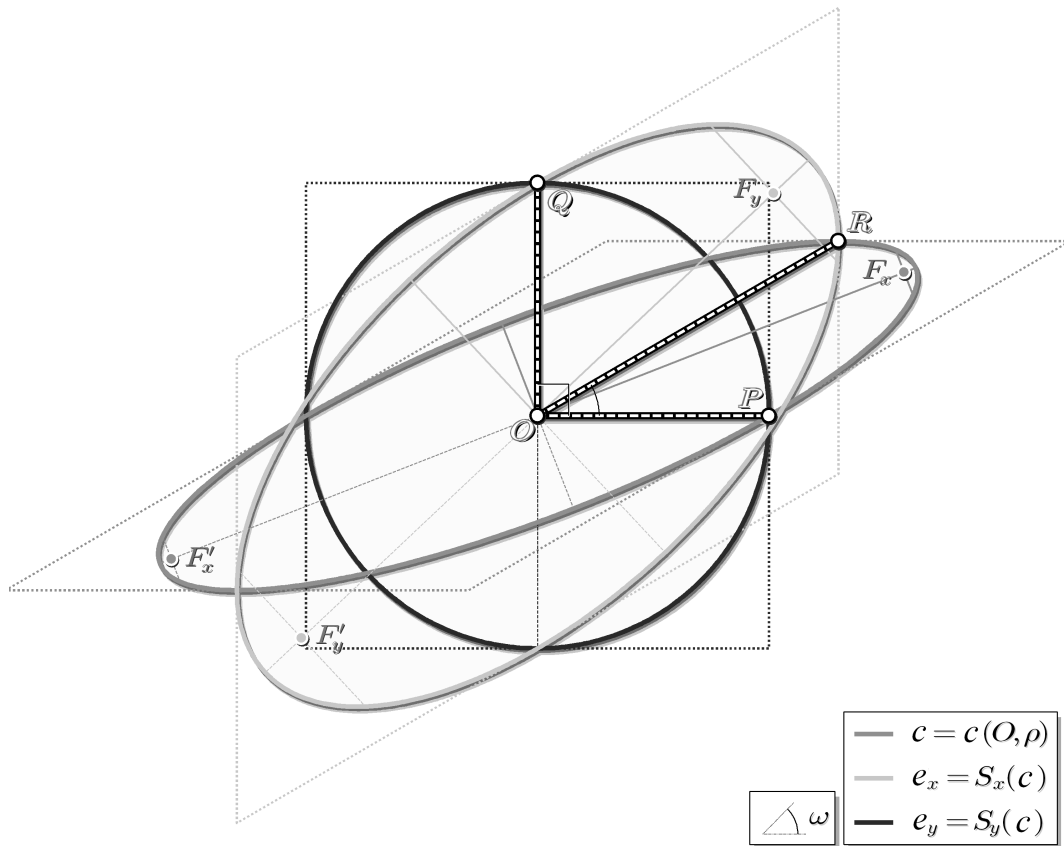


Figure 4. Circle C and its x - and y -shears E_x and E_y .

3. Common Tangential Ellipses

In this section we deliver the main results addressing the problem of finding a common tangential ellipse around three given “mutually conjugate” ellipses which are described as follows:

Definition 3.1. Three concentric and coplanar ellipses shall called *mutually conjugate* (with each other) when each of them is defined by a pair of two conjugate radii taken from a bundle of three given (non-degenerated) line segments, where only two of these segments are permitted to coincide. These line segments as above shall called as the three *mutually conjugate radii* corresponding to the three mutually conjugate ellipses.

Recall Example 2.1 where E_x , E_y and C (all centered at O) are indeed three mutually conjugate ellipses defined by their three given mutually conjugate radii OP , OQ and OR , such that $E_x := E_x(OP, OR)$, $E_y := E_y(OQ, OR)$ and $C := C(OP, OQ)$. Furthermore, we shall extend our investigation, in the sense that we shall derive (not one but) all the common tangential ellipses (of three given mutually conjugate ellipses) that can exist. This Section is divided into two sub-Sections regarding the following cases:

- *The orthogonal case*, in which the existence of the common tangential ellipses of a given circle and two ellipses, mutually conjugate with each other, is investigated, and
- *The general case*, based on the orthogonal one, where the existence of the common tangential ellipses around three given non-circular mutually conjugate ellipses is examined.

Before these sub-Sections we state and prove the following Lemma which is needed for our study. This Lemma investigates the form of a “tangential ellipse” of a given ellipse E , i.e. a concentric (to E) ellipse, say

\bar{E} , which is tangent to E . The ellipses \bar{E} and E are then intersect with each other at two (in total) distinct diametrical (contact) points, on which their corresponding two tangent lines coincide.

Lemma 3.1. Consider a tangential ellipse \bar{E} of a given ellipse E and $\theta \in [-\pi, \pi]$ be a given angle between the major axes of E and \bar{E} . When \bar{E} 's minor radius \bar{b} is given, then its corresponding major radius \bar{a} satisfies the relation

$$A\bar{a}^4 + B\bar{a}^2 + C = 0, \tag{3.1}$$

where

$$A := 4\bar{b}^2\lambda^{-2}(\bar{b}^2 - a^2)\cos^2\theta + 4a^2\bar{b}^2(\lambda^{-2} - 1)\cos^4\theta, \tag{3.2a}$$

$$B := a^2\bar{b}^4(1 - \lambda^{-2})\cos^2 2\theta + 2\bar{b}^2(a^4 - \lambda^{-2}\bar{b}^4)\cos 2\theta + 2\bar{b}^2\left[a^4 + \lambda^{-2}\bar{b}^4 - a^2\bar{b}^2(1 + \lambda^{-2})\right], \tag{3.2b}$$

$$C := a^2\bar{b}^6(1 - \lambda^{-2})\sin^2 2\theta + 4a^2\bar{b}^4(a^2 - \bar{b}^2)\sin^2\theta, \tag{3.2c}$$

with $0 < b < a$ being the E 's given principal radii and λ being its corresponding aspect ratio, i.e. $\lambda := b/a$. The analytical expression of the requested \bar{E} , on an o.c.s. spanned by the principal axes of the given ellipse E , is then formulated by

$$\bar{E}: (\bar{a}^2 \sin^2 \theta + \bar{b}^2 \cos^2 \theta)x^2 - (\bar{a}^2 - \bar{b}^2)(\sin 2\theta)xy + (\bar{a}^2 \cos^2 \theta + \bar{b}^2 \sin^2 \theta)y^2 = \bar{a}^2. \tag{3.3}$$

Proof. We consider the ellipse E centered at the origin O of an o.c.s. which is spanned by its principal axes, i.e. the E 's major and minor axis are form, respectively, the o.c.s.'s horizontal and vertical axis. Hence, the ellipse E (in this o.c.s.), adopts its canonical form

$$E: (x/a)^2 + (y/b)^2 = 1. \tag{3.4}$$

The tangential ellipse \bar{E} of E , which (its major axis) forms a given angle θ with (the major axis of) E , adopts two diametrical points P and P' in common with E . Therefore, \bar{E} can be expressed in the form of the ellipse (centered at the origin O) as in (2.1), and thus (3.3) holds, with $0 < \bar{b} < \bar{a}$ being the \bar{E} 's principal radii. Notice that θ is indeed the directive angle of \bar{E} (with respect to the adopted o.c.s. as above) as it coincides with the given angle θ between the major axes of E and \bar{E} .

Assumed now that the \bar{E} 's minor radius \bar{b} is given, we shall provide in the following the \bar{E} 's major axis \bar{a} , and thus the tangential ellipse \bar{E} of a given ellipse E (in the direction of the given angle θ) can then be calculated easily though (3.3). Let $\lambda := b/a$ and $\bar{\lambda} := \bar{b}/\bar{a}$ be the aspect ratios of the ellipses E and \bar{E} respectively. We consider the contact point $P(x_0, y_0) \in E \cap \bar{E}$ which lies on the o.c.s.'s upper semi-plane, i.e. $y_0 \geq 0$. Solving (3.4) and (3.3), with respect to y , we obtain

$$y_0 = y_E(x_0) := \lambda\sqrt{a^2 - x_0^2}, \quad x_0 \in [-a, a], \text{ and} \tag{3.5a}$$

$$y_0 = y_{\bar{E}}(x_0) := \frac{\frac{1}{2}(\bar{\lambda}^{-2} - 1)x_0 \sin 2\theta + \bar{\lambda}^{-1}\sqrt{\bar{\lambda}^{-2}\bar{b}^2 \cos^2 \theta + \bar{b}^2 \sin^2 \theta - x_0^2}}{\bar{\lambda}^{-2} \cos^2 \theta + \sin^2 \theta}, \tag{3.5b}$$

where, for the latter function $y_{\bar{E}}(x_0)$, it is assumed that $x_0 \in [-t, t]$ with the value $t := \bar{b}^2(\bar{\lambda}^{-2} \cos^2 \theta + \sin^2 \theta)$. Equating the right-hand side of (3.5a) and (3.5b), it holds that

$$\lambda K \sqrt{a^2 - x_0^2} = Lx_0 + \bar{\lambda}^{-1}\bar{b}^2 \sqrt{K - x_0^2}, \tag{3.6}$$

where

$$K := \bar{\lambda}^{-2}\bar{b}^2 \cos^2 \theta + \bar{b}^2 \sin^2 \theta = \bar{b}^2 \left[1 + (\bar{\lambda}^{-2} - 1)\cos^2 \theta\right] > 0 \text{ and} \tag{3.7a}$$

$$L := \frac{1}{2}\bar{b}^2(\bar{\lambda}^{-2} - 1)\sin 2\theta = (K - \bar{b}^2)\tan \theta. \tag{3.7b}$$

In order the curves $y = y_E(x)$ and $y = y_{\bar{E}}(x)$, as defined in (3.5a) and (3.5b), to be tangent to each other at their common point $P(x_0, y_0)$ (and hence to have a common tangent line on P), the derivatives of $y_E = y_E(x)$ and $y_{\bar{E}} = y_{\bar{E}}(x)$ must coincide at $x = x_0$ with x_0 satisfying (3.6). The derivative of (3.6), with respect to $x = x_0$, yields

$$\lambda K \frac{x_0}{\sqrt{a^2 - x_0^2}} = -L + \bar{b}^2 \frac{x_0}{\bar{\lambda} \sqrt{K - x_0^2}}. \tag{3.8}$$

Solving (3.6) and (3.8) in terms of $\sqrt{K-x_0^2}$, we get

$$\frac{\lambda\bar{\lambda}K}{\bar{b}^2}\sqrt{a^2-x_0^2}-\frac{L\bar{\lambda}x_0}{\bar{b}^2}=\sqrt{K-x_0^2}=\frac{\bar{\lambda}^{-1}\bar{b}^2x_0\sqrt{a^2-x_0^2}}{\lambda Kx_0+L\sqrt{a^2-x_0^2}}. \quad (3.9)$$

By multiplication of the left and right side of (3.9), we obtain the squared middle expression of (3.9) of the form

$$K-x_0^2=\frac{(\lambda K\sqrt{a^2-x_0^2}-Lx_0)x_0\sqrt{a^2-x_0^2}}{\lambda Kx_0+L\sqrt{a^2-x_0^2}} \quad \text{or}$$

$$\sqrt{a^2-x_0^2}=\lambda\frac{a^2-K}{L}x_0. \quad (3.10)$$

Moreover, relation (3.8) can be written as

$$\lambda Kx_0+L\sqrt{a^2-x_0^2}=\bar{\lambda}^{-1}\bar{b}^2x_0\sqrt{\frac{a^2-x_0^2}{K-x_0^2}},$$

and applying (3.8) to the left hand side of the above we get

$$\lambda\bar{\lambda}a^2\sqrt{K-x_0^2}=\bar{b}^2\sqrt{a^2-x_0^2}, \quad (3.11)$$

while substituting (3.6) again to the right-hand side of (3.11), we obtain

$$\sqrt{K-x_0^2}=\frac{\bar{b}^2}{L\bar{\lambda}a^2}(a^2-K)x_0. \quad (3.12)$$

Applying (3.10) and (3.12) into (3.6) we have $[Kb^2-(\bar{\lambda}^{-1}\bar{b}^2)^2](a^2-K)=L^2a^2$, and using (3.7b),

$$\lambda^2a^2K-\lambda^2K^2-(\bar{\lambda}^{-1}\bar{b}^2)^2+(\bar{\lambda}^{-1}\bar{b}^2a^{-1})^2K=(K^2-2K\bar{b}^2+\bar{b}^4)\tan^2\theta, \quad (3.13)$$

i.e.

$$K^2b^2(1+\lambda^{-2}\tan^2\theta)-K\left[(\bar{\lambda}^{-1}\bar{b}^2)^2+a^2b^2+2a^2\bar{b}^2\tan^2\theta\right]+a^2\bar{b}^4(\bar{\lambda}^{-2}+\tan^2\theta)=0. \quad (3.14)$$

Finally, by substitution of (3.7a) into the above (3.14) we obtain the biquadratic polynomial relation (3.1) with respect of \bar{E} 's requested major radius \bar{a} . \square

3.1. The orthogonal case

We consider the following lemma which investigates the existence of a c.t.e. of a circle and its two x - and y -shear transformations.

Lemma 3.2. *Let C be a circle of radius $\rho > 0$ centered at the origin O of an o.c.s. The circle C as well as the ellipses E_x and E_y , produced by x - and y -shears of C with the same given shearing vector \mathbf{u} of length $u > 0$ and angle $\omega \in [-\pi, \pi]$, can always adopt a c.t.e. E in the direction of the vector \mathbf{u} (i.e. E 's directive angle is ω). The principal radii $0 < b < a$ of the requested c.t.e. E are then given by $a = \rho\sqrt{u^2+1}$ and $b = \rho$, with E 's foci semi-distance (or linear eccentricity) is being $f = u\rho$, as E is analytically expressed (in the adopted o.c.s.) by*

$$E: \left[(u^2+1)\sin^2\omega + \cos^2\omega\right]x^2 - u^2(\sin 2\omega)xy + \left[(u^2+1)\cos^2\omega + \sin^2\omega\right]y^2 = \rho^2(u^2+1). \quad (3.15)$$

Proof. Consider an orthonormal vector base \mathcal{B} of an o.c.s. of origin O and let S_x and S_y be the x - and y -shear transformations with the same shearing vector (of length $u > 0$ and angle ω). Then, according to Lemmas 2.3 and 2.2, the x - and y -sheared circles $E_x := S_x(C)$ and $E_y := S_y(C)$ are ellipses centered also at O with their corresponding principal radii $0 < b_x < a_x$ and $0 < b_y < a_y$ given by (2.14) and (2.6). The directive angles $\theta_x, \theta_y \in [-\pi, \pi]$ of the tangential ellipses E_x and E_y (with respect to the o.c.s.'s coordinate axes) are then given by (2.15) and (2.7) respectively. Let also \bar{E}_x and \bar{E}_y be the (concentric) tangential ellipses of E_x and E_y respectively,

adopted in the same direction as the common shearing vector (i.e. their directive angles are both ω), and having principal radii $0 < \bar{b}_x < \bar{a}_x$ and $0 < \bar{b}_y < \bar{a}_y$ respectively.

We are shall now investigate the form of tangential ellipses \bar{E}_x and \bar{E}_y of E_x and E_y in the direction of ω , such that both \bar{E}_x and \bar{E}_y are also tangent to the circle C . Thus, their minor radii $\bar{b}_x = \bar{b}_y = \rho$, while their directive angles $\bar{\theta}_x = \bar{\theta}_y = \omega$. In order to apply Lemma 3.1, we consider as angle θ (which is the angle between the major axes of E_x and \bar{E}_x) the difference $\theta := \bar{\theta}_x - \theta_x = \omega - \theta_x$, while for the angle between (the major axes of) E_y and \bar{E}_y we must consider, respectively, $\theta := \bar{\theta}_y - \theta_y = \omega - \theta_y$.

Let $k := \bar{b}_x/b_x$, while $\lambda_x := b_x/a_x$ and $\bar{\lambda}_x := \bar{b}_x/\bar{a}_x$ denote the aspect ratios of E_x and \bar{E}_x respectively. Relation (3.1), through (3.7a), yields

$$k^2 (1 + \lambda_x^{-2} \tan^2 \theta) \left[1 + (\bar{\lambda}_x^{-2} - 1) \cos^2 \theta \right]^2 + \lambda_x^{-2} k^2 (\bar{\lambda}_x^{-2} + \tan^2 \theta) - \left[\bar{\lambda}_x^{-2} k^4 + \lambda_x^{-2} (1 + 2k^2 \tan^2 \theta) \right] \left[1 + (\bar{\lambda}_x^{-2} - 1) \cos^2 \theta \right] = 0,$$

or equivalently,

$$\lambda_x^{-2} (\bar{\lambda}_x^{-2} \cos^2 \theta + \sin^2 \theta) \left\{ \bar{\lambda}_x^{-2} k^4 - \left[(\bar{\lambda}_x^{-2} + \lambda_x^{-2}) \cos^2 \theta + (1 + \bar{\lambda}_x^{-2} \lambda_x^{-2}) \sin^2 \theta \right] k^2 + \lambda_x^{-2} \right\} = 0.$$

and hence we derive the following biquadratic polynomial with respect of k ,

$$\bar{\lambda}_x^{-2} k^4 - \left[(\bar{\lambda}_x^{-2} + \lambda_x^{-2}) \cos^2 \theta + (1 + \bar{\lambda}_x^{-2} \lambda_x^{-2}) \sin^2 \theta \right] k^2 + \lambda_x^{-2} = 0.$$

Substituting $k = \bar{b}_x/b_x = \rho/b_x$ (as $\bar{b}_x = \rho$ were assumed), the above polynomial (3.16) yields

$$\bar{\lambda}_x^{-2} \rho^4 - \left[1 + \bar{\lambda}_x^{-2} \lambda_x^{-2} - (\bar{\lambda}_x^{-2} - 1) (\lambda_x^{-2} - 1) \cos^2 \theta \right] \rho^2 b_x^2 + \lambda_x^{-2} b_x^4 = 0, \tag{3.16}$$

and solving (3.16) with respect to $\bar{\lambda}_x^{-2}$ we get

$$\bar{\lambda}_x^{-2} = b_x^2 \rho^{-2} \frac{b_x^2 \lambda_x^{-2} - \rho^2 - \rho^2 (\lambda_x^{-2} - 1) \cos^2 \theta}{b_x^2 \lambda_x^{-2} - \rho^2 - b_x^2 (\lambda_x^{-2} - 1) \cos^2 \theta}. \tag{3.17}$$

From (2.14) we have

$$b_x^2 \lambda_x^{-2} = \frac{8u^4 \rho^2 \sin^4 \omega}{\delta_x^2 (u^2 + 1 + \sqrt{u^4 + 1 + 2u^2 \cos 2\omega})} = 2\delta_x^{-1} u^2 \rho^2 \sin^2 \omega, \tag{3.18}$$

where

$$\delta_x := u^2 + 1 - \sqrt{u^4 + 1 + 2u^2 \cos 2\omega}. \tag{3.19}$$

and thus, by substitution of (3.18) into (3.17) and then applying b_x as in (2.14), we obtain

$$\bar{\lambda}_x^2 = \frac{\rho^2 \delta_x \left[u^2 - 1 + (u^2 + 1 - \delta_x) (1 - 2\cos^2 \theta) \right]}{2\rho^2 u^2 \delta_x \sin^2 \omega - \rho^2 \delta_x^2 \sin^2 \theta - 4\rho^2 u^2 \sin^2 \omega \cos^2 \theta}. \tag{3.20}$$

Moreover, substituting θ_x from (2.15) to the relation

$$\cos^2 \theta = \frac{1}{1 + \tan^2(\omega - \theta_x)} = \frac{(1 + \tan \omega \tan \theta_x)^2}{(1 + \tan \omega \tan \theta_x)^2 + (\tan \omega - \tan \theta_x)^2},$$

as $\theta := \omega - \theta_x$, we obtain that through (3.19),

$$\cos^2 \theta = \frac{(2u^2 + 2 - \delta_x)^2}{(2u^2 + 2 - \delta_x)^2 + (\tan^2 \omega) (2 - \delta_x)^2}. \tag{3.21}$$

By substitution of the above $\cos^2 \theta$ into (3.20) we obtain (after some algebra) that

$$\bar{\lambda}_x^2 = \frac{\delta_x (u^2 - 1 - \delta_x) \left[2(u^2 + 1) - \delta_x \right]^2 (1 - \sin^2 \omega) + \delta_x (u^2 - 1 + \delta_x) (\delta_x - 2)^2 \sin^2 \omega}{\left[\delta_x (\delta_x - 2) (2u^2 \sin^2 \omega - \delta_x) + 2u^2 \left[2(u^2 + 1) - \delta_x \right]^2 (1 - \sin^2 \omega) \right] (\delta_x - 2) \sin^2 \omega}. \tag{3.22}$$

From (3.19) we have

$$\sin^2 \omega = 1 - \cos^2 \omega = 1 - \frac{1}{2}(1 + \cos 2\omega) = 1 + \frac{1}{4u^2}(\delta_x - 2)(2u^2 - \delta_x), \quad (3.23)$$

as it is easy to see, through (3.19), that $0 < \delta_x < 2$ and $\delta_x < 2u^2$. Substituting (3.23) into (3.22), we finally derive (after a series of simplifications) that $\bar{\lambda}_x^2 = u^2 + 1$, and hence the major radius of \bar{E}_x is given by $\bar{a}_x = \rho\sqrt{u^2 + 1}$, as $\bar{b}_x = \rho$ was assumed.

Working similarly for the case of the tangential ellipse \bar{E}_y we obtain the its aspect ratio

$$\bar{\lambda}_y^{-2} = b_y^2 \rho^{-2} \frac{b_y^2 \lambda_y^{-2} - \rho^2 - \rho^2 (\lambda_y^{-2} - 1) \cos^2 \theta}{b_y^2 \lambda_y^{-2} \sin^2 \theta - \rho^2 + b_y^2 \cos^2 \theta} \quad \text{or} \quad (3.24)$$

$$\bar{\lambda}_y^2 = \frac{\rho^2 \delta_y [u^2 - 1 + (u^2 + 1 - \delta_y)(1 - 2 \cos^2 \theta)]}{2\rho^2 u^2 \delta_y \cos^2 \omega - \rho^2 \delta_y^2 \sin^2 \theta - 4\rho^2 u^2 \cos^2 \omega \cos^2 \theta}, \quad \text{where} \quad (3.25)$$

$$\delta_y := u^2 + 1 - \sqrt{u^4 + 1 - 2u^2 \cos 2\omega}. \quad (3.26)$$

Then, after some algebra, it holds also that $\bar{\lambda}_y^2 = u^2 + 1$, and hence the major radius of \bar{E}_y is given by $\bar{a}_y = \rho\sqrt{u^2 + 1}$, as $\bar{b}_y = \rho$ was also assumed.

Therefore, the principal radii of \bar{E}_x and \bar{E}_y coincide, as $\bar{a}_x = \bar{a}_y = \rho\sqrt{u^2 + 1}$ and $\bar{b}_x = \bar{b}_y = \rho$, and hence the ellipses \bar{E}_x and \bar{E}_y are of the same shape. Moreover, as their directive angles $\bar{\theta}_x$ and $\bar{\theta}_y$ are both assumed to be ω , it is clear that $\bar{E}_x = \bar{E}_y$. We can then denote with $E := \bar{E}_x = \bar{E}_y$ the c.t.e. of E_x , E_y and C . The directive angle θ of E is ω and its principal radii $0 < b < a$ are of the form $a = \rho\sqrt{u^2 + 1}$ and $b = \rho$. The foci semi-distance of E is formulated by $f := \sqrt{a^2 - b^2} = u\rho$. The analytical expression of the c.t.e. E is finally given by setting $\theta := \omega$ into (2.1), and thus (3.15) is finally derived. \square

The following Theorem proves the existence of a c.t.e. of a circle and two ellipses mutually conjugate with each other.

Theorem 3.1. Consider a circle C of radius $\rho > 0$ centered at point O , and two ellipses E_1 and E_2 such that E_1 , E_2 and C correspond to three given mutually conjugate ellipses. Hence, these ellipses are defined by a bundle of three given mutually conjugate radii, say OP , OQ and OR , such that $C = C(OP, OQ)$, $E_1 = E_1(OP, OR)$ and $E_2 = E_2(OQ, OR)$, with $|OP| = |OQ| = \rho$ and $OP \perp OQ$, as C is a circle. These radii are fully determined by the given length $r = |OR|$ and angle $\omega = \angle(OP, OR) \in [-\pi, \pi]$. Then, a common tangential ellipse E , of E_1 , E_2 and C , always exists in the direction of the non-orthogonal radius OR (i.e. the major radius of E is spanned by OR), while point R is one of the E 's foci. The principal radii $0 < b < a$ of E are then given by $a = \sqrt{\rho^2 + r^2}$ and $b = \rho$, while its eccentricity and foci semi-distance are $\varepsilon = r/\sqrt{\rho^2 + r^2}$ and $f = r$ respectively. This c.t.e. E is analytically expressed by

$$E : [(\rho^2 + r^2) \sin^2 \omega + \rho^2 \cos^2 \omega] x^2 - r^2 (\sin 2\omega) xy + [(\rho^2 + r^2) \cos^2 \omega + \rho^2 \sin^2 \omega] y^2 = \rho^2 (\rho^2 + r^2), \quad (3.27)$$

in an o.c.s. spanned by the orthogonal radii OP and OQ .

The two diametrical (common) contact points $T_1(x_1, y_1)$, $x_1 > 0$, and $T'_1(-x_1, -y_1)$ between E_1 and its tangential ellipse E , are then given by

$$x_1 = \sqrt{\rho^2 + r^2 \cos^2 \omega} \quad \text{and} \quad y_1 = \frac{r^2 \sin 2\omega}{2\sqrt{\rho^2 + r^2 \cos^2 \omega}}, \quad (3.28)$$

while their corresponding two common tangent lines t_1 and t'_1 (at points T_1 and T'_1 respectively) are being parallel to OQ (which spans the o.c.s.'s vertical axis $y'Oy$), i.e. $t_1: x = x_1$ and $t'_1: x = -x_1$.

For the two diametrical contact points $T_2(x_2, y_2)$, $y_2 > 0$, and $T'_1(-x_2, -y_2)$ between E_2 and its tangential ellipse E , we have

$$x_2 = \frac{r^2 \sin 2\omega}{2\sqrt{\rho^2 + r^2 \sin^2 \omega}} \quad \text{and} \quad y_2 = \sqrt{\rho^2 + r^2 \sin^2 \omega}, \quad (3.29)$$

while their corresponding two common tangent lines t_2 and t'_2 (at points T_2 and T'_2 respectively) are being parallel to OP (which spans the o.c.s.'s horizontal axis $x'Ox$), i.e. $t_2: y = y_2$ and $t'_2: y = -y_2$.

Finally, for the last two contact points $T_3(x_3, y_3)$ and $T'_3(-x_3, -y_3)$ between circle C and its tangential ellipse E , it holds that

$$x_3 = -\rho \sin \omega \text{ and } y_3 = \rho \cos \omega, \tag{3.30}$$

while their corresponding two tangent lines t_3 and t'_3 (at points T_3 and T'_3 respectively) are being parallel to the non-orthogonal radii OR , as they are given by

$$t_3: y \cos \omega - x \sin \omega = \rho \text{ and } t'_3: x \sin \omega - y \cos \omega = \rho. \tag{3.31}$$

Proof. Consider an o.c.s. with origin O where its horizontal and vertical axes are spanned by the given orthogonal vectors OP and OQ . Let $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2\}$ be the corresponding orthonormal vector base. Hence $P_{\mathcal{B}}(\rho, 0)$ and $Q_{\mathcal{B}}(0, \rho)$. Let S_x be the x -shear transformation with shearing vector $\mathbf{u} := \rho^{-1}OR$, which transforms point $Q \in C$ into R . Indeed, from (2.3) and setting $u := |\mathbf{u}| = r/\rho$, we obtain $S_x(Q) = R$. Notice also that $S_x(P) = P$ (as the horizontal axis $x'Ox$ is invariant under S_x), and hence $P \in C \cap E_1$. Therefore, $S_x(C) = E_1$ as the ellipse E_1 is (by assumption) defined by its two conjugate radii OP and OR . Let also S_y be the y -shear with the same shearing vector \mathbf{u} which (similarly to S_x) transforms point $P \in C$ also into R . Indeed, from (2.4) and setting again $u := |\mathbf{u}| = r/\rho$, we obtain $S_y(P) = R$. Also $S_y(Q) = Q$ (as the vertical axis $y'Oy$ is invariant under S_y), and hence $Q \in C \cap E_2$. Thus $S_y(C) = E_2$ as the ellipse E_2 (by assumption) is defined by its two conjugate radii OQ and OR . From the above discussion, point R is an intersecting point of the ellipses E_1 and E_2 , i.e. $R \in E_1 \cap E_2$.

The given pairs of line segments (OP, OR) and (OQ, OR) are indeed correspond to pairs of conjugate radii for E_1 and E_2 respectively, as these segments are affine transformations (recall S_x and S_y) of the C 's orthogonal radii OP and OQ . As affinity preserves parallelism, the tangent lines of E_1 and E_2 at their points P and Q respectively are parallel to OR , while the tangent lines of E_1 and E_2 at their point R are parallel to OP and OQ respectively (because the tangent lines of the circle C at its points P and Q are, trivially, parallel to its orthogonal radii OQ and OP). Figure 4 clarifies also the above discussion (on which the referred ellipses E_x and E_y correspond to E_1 and E_2 respectively) as the square frame around C is transformed, through S_x and S_y , into the parallelograms around E_1 and E_2 .

Lemma 3.2 can now be applied, where we have to replace E_x and E_y with E_1 and E_2 respectively and set $u := r/\rho$. Therefore, a c.t.e. E of E_1, E_2 and C always exists with directive angle ω (in the adopted o.c.s.), and hence point R lie onto E 's major semi-axis. Moreover, through Lemma 3.2, the foci separation of the c.t.e. E is then $f = 2u\rho$, i.e. $f = 2r = 2|OR|$ (as $u := r/\rho$). Thus, the intersection point $R \in E_1 \cap E_2$ is indeed a focal point of the common tangential E . Moreover, the E 's principal radii are then given by $a = \rho\sqrt{u^2 + 1} = \sqrt{\rho^2 + r^2}$ and $b = \rho$, while its eccentricity $\varepsilon := a^{-1}\sqrt{a^2 - b^2} = u/\sqrt{u^2 + 1} = r/\sqrt{\rho^2 + r^2}$. The c.t.e. E can then be analytically expressed (in the adopted o.c.s.) as (3.15), where we set $u := r/\rho$, and therefore (3.27) holds.

For the calculations of the two contact points $T_1(x_1, y_1)$ and $T'_1(-x_1, -y_1)$ between E_1 and its tangential ellipse E , as well as for their corresponding tangent lines t_1 and t'_1 (at T_1 and T'_1), we consider the following approach. The analytical expression of E_1 is given by (2.13), as $E_1 = S_x(C)$, where we set $u := r/\rho$. Let now $x = x_0$ be a vertical tangent line of E_1 at some point $T_0(x_0, y_0) \in E_1, y_0 > 0$. Then, setting $x := x_0$ into (2.13), a trinomial with respect to y is derived, say $\eta_1 = \eta_1(y; x_0) = 0$. Because we expect $\eta_1(y) = 0$ to have one real (double) root $y = y_0$ (due to the fact that line $x = x_0$ was assumed to be a tangent line of E_1 at T_0), its discriminant must be zero, i.e. $4u^2 \sin^2 \omega (u^2 \rho^2 \cos^2 \omega + \rho^2 - x_0^2) = 0$, or equivalently

$$x_0 = \pm \rho \sqrt{1 + u^2 \cos^2 \omega} = \pm \sqrt{\rho^2 + r^2 \cos^2 \omega},$$

while setting the x_0 value, as above, to the trinomial $\eta(y; x_0) = 0$, its double real roots would then given by

$$y_0 = \pm \frac{u^2 \rho \sin 2\omega}{2\sqrt{1 + u^2 \cos^2 \omega}} = \frac{1}{2} x_0^{-1} r^2 \sin 2\omega.$$

Adopting the positive value for x_0 , from the above relation we obtain

$$x_0 = \sqrt{\rho^2 + r^2 \cos^2 \omega} \text{ and } y_0 = \frac{r^2 \sin 2\omega}{2\sqrt{\rho^2 + r^2 \cos^2 \omega}}. \tag{3.32}$$

Similarly, if $x = x'_0$ is now assumed to be also a vertical tangent line of E at some point $T'_0(x'_0, y'_0) \in E$, then by setting $x := x'_0$ into (3.27), a trinomial with respect also to y is derived, say $\eta'_1 = \eta'_1(y; x'_0) = 0$. As we expect $\eta'_1(y) = 0$ to have also one real (double) root $y = y'_0$ (due to the fact that line $x = x'_0$ was assumed to be a tangent line of E at T'_0), its discriminant must again be zero. The new calculations yields that x'_0 and y'_0 expressions are

exactly the same as x_0 and y_0 in (3.32), i.e. $x'_0 = x_0$ and $y'_0 = y_0$. Therefore, the vertical tangent lines of E_1 and E are coincide at point $T_0 = T'_0$, i.e. the contact point $T_1(x_1, y_1)$ of E_1 and E is indeed given by (3.32), where x_0 and y_0 notations were replaced by x_1 and y_1 respectively, and hence (3.28) holds. The corresponding tangent line t_1 at point T_1 is thus the vertical tangent line $x = x_1 (= x_0 = x'_0)$ which is proved to be a common tangent line between E_1 and E . For clarification see also Fig. 5.

For the two contact points $T_2(x_2, y_2)$, $y_2 > 0$, and $T'_2(-x_2, -y_2)$ between E_2 and E , as well as for their corresponding tangent lines t_2 and t'_2 (at T_2 and T'_2), we consider the analytical expression of E_2 given in (2.5), as $E_2 = S_y(C)$, where we also have to set $u := r/\rho$. Let $y = y_0$ be a horizontal tangent line of E_2 at some point $S_0(\chi_0, \psi_0) \in E_2$, $\psi_0 > 0$. Then, by setting $y := \psi_0$ into (2.5), a trinomial with respect to x is derived, say $\eta_2 = \eta_2(x; \psi_0) = 0$. Because we expect $\eta_2(x) = 0$ to have one real (double) root $x = \chi_0$ (due to the fact that line $y = \psi_0$ was assumed to be a tangent line of E_2 at S_0), its discriminant must be zero, i.e. $4u^2 \cos^2 \omega (u^2 \rho^2 \sin^2 \omega + \rho^2 - y_0^2) = 0$, or equivalently

$$\psi_0 = \pm \rho \sqrt{1 + u^2 \sin^2 \omega} = \pm \sqrt{\rho^2 + r^2 \sin^2 \omega}, \quad (3.33)$$

while adopting the positive value for ψ_0 as above, and setting it to the trinomial $\eta_2(x; \psi_0) = 0$, its double real root is then given by

$$\chi_0 = \frac{u^2 \rho \sin 2\omega}{2\sqrt{1 + u^2 \sin^2 \omega}} = \frac{r^2 \sin 2\omega}{2\sqrt{\rho^2 + r^2 \sin^2 \omega}}. \quad (3.34)$$

Similarly, if $y = \psi'_0$ is assumed to be a horizontal tangent line of E at some point $S'_0(\chi'_0, \psi'_0) \in E$, $\psi'_0 > 0$ then, by setting $y := \psi'_0$ into (3.27), a trinomial with respect also to x is derived, say $\eta'_2 = \eta'_2(x; \psi'_0) = 0$. As we expect the trinomial $\eta'_2(x) = 0$ to have one real (double) root $x = \chi'_0$ (due to the fact that line $y = \psi'_0$ was assumed to be a tangent line of E at S'_0), its discriminant must again be zero. The calculations yield that the values of χ'_0 and ψ'_0 are coincide to the χ_0 and ψ_0 values as in (3.34), i.e. $\chi'_0 = \chi_0$ and $\psi'_0 = \psi_0$. Therefore, the horizontal tangent lines of E_2 and E are coincide at point $S_0 = S'_0$, i.e. the contact point $T_2(x_2, y_2)$, $y_2 > 0$, between E_2 and E is indeed $S_0(\chi_0, \psi_0)$, and hence (3.29) holds. The corresponding tangent line t_2 at point T_2 is then coincide with the horizontal tangent line $y = y_2 (= \psi_0 = \psi'_0)$ which is proved to be a common tangent line of E_2 and E . For clarification see also Fig. 5.

For the last two contact points $T_3(x_3, y_3)$ and $T'_3(-x_3, -y_3)$ between circle C and its tangential ellipse E , it holds that $x_3^2 + y_3^2 = \rho^2$. Moreover, the slope of diameter $T_3T'_3$ is given by $\tan(\omega + \pi/2) = y_3/x_3$. This is due to the fact that ellipse E is a tangential ellipse to C . Hence, the minor axis of E (orthogonal to the E 's major axis which forms an angle ω with o.c.s.'s horizontal axis) is assumed to be equal to the radius of C , i.e. $\rho = b$; see also Fig. 5. The above two relations imply that

$$x_3 = -y_3 \tan \omega \quad \text{and} \quad y_3^2 = \frac{\rho^2}{1 + \tan^2 \omega} = \rho^2 \cos^2 \omega, \quad (3.35)$$

and therefore (3.30) is derived. The corresponding two tangent lines t_3 and t'_3 of C at their contact points T_3 and T'_3 respectively, are given in their usual forms $t_3: x_3x + y_3y = \rho^2$ and $t'_3: x_3x + y_3y = -\rho^2$ (on the adopted o.c.s.), as they are tangent lines of the circle C , and hence by substitution of (3.30) into them, relations (3.31) hold. Therefore, the tangent lines t_3 and t'_3 are parallel to the non-orthogonal radii OR , as their slopes are both $\tan \omega$. \square

Example 3.1. Consider the bundle of three line segments OP , OQ and OR as in Example 2.1, i.e. $\rho = |OP| = |OQ| := 10$ and $r = |OR| := 15$, with $OP \perp OQ$ and $\omega = \angle(OP, OR) := \pi/6 (= 30^\circ)$. These three given line segments correspond to three mutually conjugate radii which define three mutually conjugate ellipses such that $E_1 = E_1(OP, OR)$, $E_2 = E_2(OQ, OR)$ and $E_3 = E_3(OP, OQ)$; see also Fig. 5. Notably, E_3 is a circle of radius $\rho = 10$ as defined by the pair (OP, OQ) of its orthogonal (and equal) conjugate radii. Consider also point O as the origin of an o.c.s. with its horizontal and vertical axes spanned, respectively, by orthogonal radii OP and OQ .

Ellipses E_1 and E_2 can be expressed as the x - and y -shears of circle E_3 , with shearing vector $\mathbf{u} := \rho^{-1}OR$ of length $u := r/\rho = 1.5$. Therefore, E_1 and E_2 are then given in Example 2.1, where E_x and E_y correspond now to E_1 and E_2 respectively, while their principal radii $0 < b_i < a_i$, $i = 1, 2$, and their directive angles θ_i , $i = 1, 2$, are given respectively by the values of a_x, b_x, a_y, b_y and θ_x, θ_y in Example 2.1. Applying now Theorem 3.1, a c.t.e. E of the given three mutually conjugate ellipses E_i , $i = 1, 2, 3$, always exists, and its major axis is spanned by (non-orthogonal common radius) OR , while point R is being one of its foci. The corresponding E 's principal radii are

then given by $a = \sqrt{\rho^2 + r^2} = 5\sqrt{13} \approx 18.0278$ and $b = \rho = 10$, while eccentricity, foci distance, and surface area are given, respectively, by $\varepsilon = \frac{3}{13}\sqrt{13} \approx 0.832$, $f = 30$ and $A = 50\pi\sqrt{13} \approx 566.359$. As E 's focal point, say F , is identical to R , then $F = R(r\cos\omega, r\sin\omega) = R(\frac{15}{2}\sqrt{3}, \frac{15}{2})$ and $F'(-\frac{15}{2}\sqrt{3}, -\frac{15}{2})$ are then correspond to the two diametrical foci of E . The analytical expression of c.t.e. E is given, through (3.27), by $E: 25x^2 - 18\sqrt{3}xy + 43y^2 = 5200$.

The two contact points $T_1(x_1, y_1)$, $y_1 > 0$, and $T'_1(-x_1, -y_1)$ between ellipse E_1 and its tangential E , are calculated through (3.28), i.e. $x_1 = \frac{5}{2}\sqrt{43} \approx 16.394$ and $y_1 = \frac{45}{86}\sqrt{129} \approx 5.943$, with their corresponding two tangent lines t_1 and t'_1 (at their contact points T_1 and T'_1) to be parallel to OQ (which spans o.c.s.'s vertical axis $y'Oy$), i.e. $t_1: x = x_1 = \frac{5}{2}\sqrt{43}$ and $t'_1: x = -x_1 = -\frac{5}{2}\sqrt{43}$.

The other two contact points $T_2(x_2, y_2)$, $y_2 > 0$, and $T'_2(-x_2, -y_2)$ between ellipse E_2 and its tangential E , are calculated through (3.29), i.e. $x_2 = \frac{9}{2}\sqrt{13} \approx 7.7942$ and $y_2 = 25/2 = 12.5$, with their corresponding two tangent lines t_2 and t'_2 (at their contact points T_2 and T'_2) to be parallel to OP (which spans o.c.s.'s horizontal axis $x'Ox$), i.e. $t_2: y = y_2 = 25/2$ and $t'_2: y = -y_2 = -25/2$.

Finally, the last two contact points $T_3(x_3, y_3)$, $y_3 > 0$, and $T'_3(-x_3, -y_3)$ between circle E_3 and its tangential E , are calculated through (3.30), i.e. $x_3 = -5$ and $y_3 = 5\sqrt{3} \approx 8.6603$, with their corresponding two tangent lines t_3 and t'_3 (at their contact points T_3 and T'_3) obtained through (3.30), i.e. $t_3: \sqrt{3}y - x = 20$ and $t'_3: x - \sqrt{3}y = 20$, which are parallel to the non-orthogonal radii OR , as their slopes are both $\tan(\pi/6) = \sqrt{3}/3$.

Figure 5 visualizes exactly Example 3.1 by presenting the three mutually conjugate ellipses $E_1 = E_1(OP, OR)$, $E_2 = E_2(OQ, OR)$ and $E_3 = E_3(OP, OQ)$, with their defining three given mutually conjugate radii as well as their foci points. The c.t.e. E of E_i , $i = 1, 2, 3$, is also presented, together with their six common tangent lines at their six corresponding contact points. The intersection angles of the ellipses E_i , $i = 1, 2, 3$, at their common points P , Q and R , are also denoted, while $\vartheta := \pi/2 - \omega$ and $\varphi := \angle(OP, OQ) = \pi/2$.

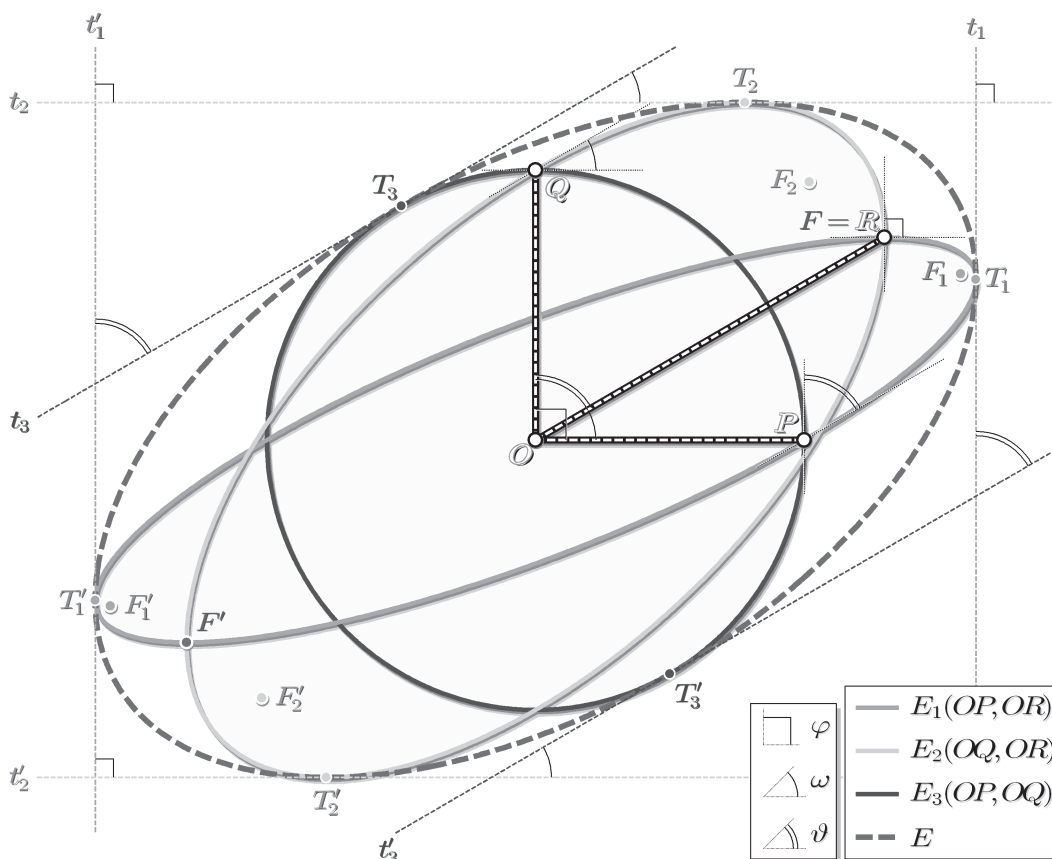


Figure 5. Graphs of the three mutually conjugate ellipses E_i , $i = 1, 2, 3$, of Example 3.1, and their c.t.e. E together with their common tangent lines.

Recall Lemma 3.2 where a c.t.e. E of E_x, E_y and C was obtained. In the following, the number of all existing common tangential ellipses (like E) of the ellipses E_x, E_y and C is investigated.

Lemma 3.3. Let C be a circle of radius $\rho > 0$, centered at the origin O of an o.c.s., while E_x and E_y are the x - and y -shears of C as in Lemma 3.2. There are, at most, two distinct common tangential ellipses, say E and E^* , of E_x , E_y and C , where E is given in (3.27) having directive angle ω (i.e. its major axis is spanned by the shearing vector of length u that transforms C into E_x and E_y), while the second c.t.e. E^* ($\neq E$) have (when it exists) a directive angle θ^* (with respect to OP) such that

$$\frac{\tan \theta^*}{\tan \omega} = -\frac{1+u^2 \cos 2\omega}{1-u^2 \cos 2\omega} = \frac{u^2+1-(u^2-1)\tan^2 \omega}{u^2-1-(u^2+1)\tan^2 \omega}. \quad (3.36)$$

When $\omega \in (0, \pi/2)$ the second c.t.e. E^* (of E_x , E_y and C) exists if and only if $\omega \in (0, \omega_0)$ where

$$\omega_0 := \begin{cases} \frac{1}{2} \arccos \sqrt{2u^{-2} - u^{-4}}, & \text{if } u \geq \sqrt{2}/2, \\ \pi/2, & \text{if } u < \sqrt{2}/2. \end{cases} \quad (3.37)$$

Proof. Consider an o.c.s. with origin at the center O of the given circle C . We assume that E^* is a c.t.e., other than E , i.e. an ellipse centered at O which is also (simultaneously) tangent to E_x , E_y and C . Its principal radii are denoted with $a^* > b^* := \rho$ and its directive angle with θ^* . Therefore, it holds that $E^* = E_x^* = E_y^*$, where E_x^* and E_y^* denote the two tangential ellipses of E_x and E_y respectively in the direction of θ^* and have principal radii $a_x > b_x := \rho$ and $a_y > b_y := \rho$.

The aspect ratio of the tangential ellipse E_x^* ($= E^*$) of E_x , can be obtained through the angle, say θ , between (the major axes of) E_x and its tangential E_x^* , i.e. $\theta := \theta^* - \theta_x$, as we did similarly in the proof of Lemma 3.2 (where had set $\theta := \omega - \theta_x$). Firstly, the relation (2.15) can be expressed in terms of δ_x , as in (3.19). In particular, from (3.19), it holds that

$$\cos^2 \omega = \frac{1}{2}(1 + \cos 2\omega) = \frac{1}{4}u^{-2}(\delta_x - 2)(\delta_x - 2u^2), \quad (3.38)$$

as $0 < \delta_x < 2$ and $\delta_x < 2u^2$, and applying (3.38) into (2.15), we obtain

$$\tan \theta_x = \sqrt{\frac{\delta_x(2u^2 - \delta_x)}{(2 - \delta_x)(2u^2 + 2 - \delta_x)}}, \quad (3.39)$$

as also $\delta_x < 2(u^2 + 1)$. Substituting then (3.39) into

$$\cos^2 \theta = \frac{1}{1 + \tan^2(\theta^* - \theta_x)} = \frac{(1 + \tan \theta^* \tan \theta_x)^2}{(1 + \tan \theta^* \tan \theta_x)^2 + (\tan \theta^* - \tan \theta_x)^2},$$

we obtain (after some algebra) that

$$\cos \theta = \frac{\sqrt{\delta_x(2u^2 - \delta_x)} \sin \theta^* + \sqrt{(2 - \delta_x)(2u^2 + 2 - \delta_x)} \cos \theta^*}{2\sqrt{2u^2 + 2 - \delta_x}}. \quad (3.40)$$

The aspect ratio λ_x^* of E_x^* can then be obtained from (3.20) where $\theta := \theta^* - \theta_x$. In particular, applying (3.23) into (3.20), it holds that

$$\lambda_x^{*2} = \frac{2\delta_x \left[1 - u^2 + (u^2 + 1 - \delta_x)(1 - 2\cos^2 \theta) \right]}{(\delta_x - 2\cos^2 \theta) \left[4u^2 - (\delta_x - 2)(\delta_x - 2u^2) \right] + 2\delta_x^2 \sin^2 \theta},$$

and then, through (3.40), we derive that

$$\lambda_x^{*2} = \frac{2(1 - u^2) + \left[\delta_x(\delta_x - 2u^2 - 2) + 2(u^2 + 1) \right] \cos 2\theta^* + \eta_x \sin 2\theta^*}{2 \left[\delta_x(\delta_x - 2u^2 - 2) + 2(u^2 + 1) \right] \cos^2 \theta^* + \eta_x \sin 2\theta^*}, \quad (3.41)$$

where $\eta_x := \sqrt{\delta_x(2 - \delta_x)(2u^2 - \delta_x)(2u^2 + 2 - \delta_x)}$ or, by substitution of δ_x as in (3.19), $\eta_x = u^2 \sqrt{2(1 - \cos 4\omega)} = u^2 |\sin 2\omega|$.

Similarly to the case of the tangential ellipse E_x^* as above, we now consider as angle θ between (the major axes of) E_y and E_y^* , the difference $\theta := \theta^* - \theta_y$. In particular, from (3.26), it holds

$$\cos^2 \omega = \frac{1}{2}(1 + \cos 2\omega) = \frac{1}{4}u^{-2}\delta_y(2u^2 + 2 - \delta_y), \quad (3.42)$$

as it is easy to see, through (3.26), that $0 < \delta_y < 2(u^2 + 1)$. Applying (3.42) into (2.7), we obtain

$$\tan \theta_y = \sqrt{\frac{(2 - \delta_y)(2u^2 + 2 - \delta_y)}{\delta_y(2u^2 - \delta_y)}}, \tag{3.43}$$

as $0 < \delta_y < 2$ and $\delta_y < 2u^2$ also hold. Substituting then (3.43) into

$$\cos^2 \theta = \frac{1}{1 + \tan^2(\theta^* - \theta_y)} = \frac{(1 + \tan \theta^* \tan \theta_y)^2}{(1 + \tan \theta^* \tan \theta_y)^2 + (\tan \theta^* - \tan \theta_y)^2},$$

we obtain (after some algebra) that

$$\cos \theta = \frac{\sqrt{(2 - \delta_y)(2u^2 + 2 - \delta_y)} \sin \theta^* + \sqrt{\delta_y(2u^2 - \delta_y)} \cos \theta^*}{2\sqrt{u^2 + 1 - \delta_y}}. \tag{3.44}$$

The aspect ratio λ_y^* of E_y^* can then be obtained from (3.25) where $\theta := \theta^* - \theta_y$. In particular, applying firstly (3.42) into (3.25), it holds that

$$\lambda_y^{*2} = \frac{u^2 - 1 + (u^2 + 1 - \delta_y)(1 - 2\cos^2 \theta)}{2(\delta_y - u^2 - 1)\cos^2 \theta + \delta_y(2u^2 + 2 - \delta_y)},$$

and then, through (3.44), we derive that

$$\lambda_y^{*2} = -\frac{2(u^2 - 1) + [\delta_y(\delta_y - 2u^2 - 2) + 2(u^2 + 1)] \cos 2\theta^* - \eta_y \sin 2\theta^*}{2[\delta_y(\delta_y - 2u^2 - 2) + 2(u^2 + 1)] \sin^2 \theta^* + \eta_y \sin 2\theta^*}, \tag{3.45}$$

where $\eta_y := \sqrt{\delta_y(2 - \delta_y)(2u^2 - \delta_y)(2u^2 + 2 - \delta_y)}$.

We assumed earlier that $E^* = E_x^* = E_y^*$ so that ellipse E^* can be another c.t.e. (besides E) of E_x , E_y and C . Thus, the aspect ratios in (3.41) and (3.45) must coincide to the E^* 's aspect ratio $\lambda^* := \lambda_x = \lambda_y$, as the minor radii of E_x^* and E_y^* must be equal with C 's radius ρ as assumed, i.e. as $b_x = b_y = \rho$. Writing the equation $\lambda_x^2 - \lambda_y^2 = 0$ in terms of $\tan \theta^*$, through (3.41) and (3.45), we obtain the trinomial expression

$$A \tan^2 \theta^* + B \tan \theta^* + \Gamma = 0, \tag{3.46}$$

where the coefficients A, B and Γ are given by

$$A := -(\delta_x - 2)(\delta_x - 2u^2) [\delta_y(\delta_y - 2u^2 - 2) + 2(u^2 + 1)], \tag{3.47a}$$

$$B := 2\eta_x [4u^2 - \delta_y(2u^2 + 2 - \delta_x)] - \eta_y [4u^2 - \delta_x(2u^2 + 2 - \delta_x)], \tag{3.47b}$$

$$\Gamma := (\delta_y - 2)(\delta_y - 2u^2) [\delta_x(\delta_x - 2u^2 - 2) + 2(u^2 + 1)]. \tag{3.47c}$$

Notice that the trinomial with respect to $\tan \theta^*$ (3.46) has two, at most, real roots and one of which is expected always to be $\tan \omega$, because in the direction of angle ω the c.t.e. E (of E_x , E_y and C), as in (3.27), always exists (recall Lemma 3.2) with aspect ratio $\lambda = \sqrt{u^2 + 1}$. Therefore, assuming that there exists another tangential ellipse $E^* \neq E$ (of E_x , E_y and C), the trinomial (3.46) implies that

$$\tan \omega = \frac{1}{2A} (-B \pm \sqrt{B^2 - 4A\Gamma}), \quad \tan \theta^* = \frac{1}{2A} (-B \mp \sqrt{B^2 - 4A\Gamma}) \quad \text{or}$$

$$\tan \theta^* = -\frac{B}{A} - \tan \omega. \tag{3.48}$$

By setting $\theta^* := \omega$ into the trinomial (3.46), it holds that $-B \tan \omega = \Gamma + A \tan^2 \omega$, and then using of (3.48) we obtain an alternative form of (3.48), i.e.

$$\tan \theta^* \tan \omega = \frac{\Gamma}{A}. \tag{3.49}$$

Moreover, the coefficients A and Γ , as in (3.47a) and (3.47c), can be simplified in the forms of

$$A = -8u^2(1 - u^2 \cos 2\omega) \cos^2 \omega \quad \text{and} \quad \Gamma = 8u^2(1 + u^2 \cos 2\omega) \sin^2 \omega, \tag{3.50}$$

by substitution of δ_x as in (3.19). Relation (3.49) finally yields (3.36), through (3.50).

When E^* is an ellipse (of direction angle $\theta^* \neq \omega$), the value $\lambda^* := \lambda_x$ would then correspond to aspect ratio of ellipse E^* , and hence $\mathbb{R}_+ \ni \lambda^* \leq 1$, while the value $a^* := a_x = b_x/\lambda_x = \rho/\lambda^*$ provides E^* 's major radius (as ρ is always assumed to be E^* 's minor radius). Recall that always $a_x \geq b_x$ when the values a_x and b_x are real and given by (2.14). Hence, the real-valued ratio $\lambda_x := b_x/a_x$ cannot be greater than 1 or, equivalently, the ratio in (3.41) cannot be > 1 . However, if the (real-valued) ratio as in (3.41) is found to be negative, i.e. $\lambda^{*2} (:= \lambda_x^2) < 0$, then $a^{*2} = \rho^2/\lambda^{*2} < 0$, which is equivalent to the fact that the E^* is shaped now by a hyperbola $E^* : (\tilde{y}/\rho)^2 - (\tilde{x}/a^*)^2 = 1$, $\tilde{x}, \tilde{y} \in \mathbb{R}$ (in some o.c.s.) instead of an ellipse (tangential to E_x, E_y and C).

Therefore, E^* is indeed a c.t.e. of E_x, E_y and C , if and only if (iff) the ratio in (3.41) is equal or lower than 1 or, equivalently, rewriting (3.41) in terms of $\tan \theta^*$, iff

$$\lambda^{*2} = \frac{4 - \delta_x(2u^2 + 2 - \delta_x) + 2\eta_x \tan \theta^* - (2 - \delta_x)(2u^2 - \delta_x) \tan^2 \theta^*}{2\delta_x(\delta_x - 2u^2 - 2) + 4(u^2 + 1) + 2\eta_x \tan \theta^*} \leq 1.$$

By substitution of $\eta_x = u^2 |\sin 2\omega|$ (shown earlier) and δ_x , as in (3.19), the above condition is equivalent to

$$\frac{1 - u^2 + 2u^2 \cos^2 \omega + u^2 |\sin 2\omega| \tan \theta^*}{1 - u^2 + u^2 \cos^2 \omega + u^2 |\sin 2\omega| \tan \theta^* - u^2 \cos^2 \omega \tan^2 \theta^*} \geq 1. \quad (3.51)$$

Otherwise, if the ratio as in (3.51) is negative, E^* is then being a hyperbola which is (simultaneously) tangent and concentric to E_x, E_y and C .

The iff condition (3.51) can be simplified. In particular, when the denominator of the ratio in (3.51) is assumed to be negative, then (3.51) implies that $u^2 \cos^2 \omega (1 + \tan^2 \theta^*) \leq 0$, which cannot hold, and thus the denominator in (3.51) must always be positive. Therefore the condition (3.51) is equivalent to

$$u^2 \cos^2 \omega \tan^2 \theta^* - (\operatorname{sgn} \omega)(\operatorname{sgn} \cos \omega) u^2 \sin 2\omega \tan \theta^* + u^2 \sin^2 \omega - 1 < 0,$$

where $\operatorname{sgn}(\cdot)$ denotes the usual sign operator, or using (3.36),

$$\left\{ 2s \left[(1 + u^2)^2 - 4u^4 \cos^4 \omega \right] + (1 + u^2 + 2u^2 \cos^2 \omega)^2 \right\} u^2 \sin^2 \omega - (1 - u^2 \sin^2 \omega)(1 + u^2 - 2u^2 \cos^2 \omega)^2 < 0, \quad (3.52)$$

where $s := (\operatorname{sgn} \omega) \operatorname{sgn}(\frac{\pi}{2} - |\omega|)$. When $\omega \in (0, \pi/2)$, the condition (3.52) is then reduced, as $s = 1$, to

$$1 - 2u^2 + u^4 \cos^2 2\omega > 0, \quad (3.53)$$

which is equivalent to $\omega \in (0, \omega_0)$, with ω_0 as in the first branch of (3.37), provided that $u \geq \sqrt{2}/2$. However, if $u < \sqrt{2}/2$ is assumed, the condition (3.52) always holds for any angle such that $s = 1$. Hence $\omega \in (0, \pi/2)$. \square

Using the above Lemma 3.3 it can now be shown that, under certain conditions, a second c.t.e. of three mutually conjugate ellipses, with one of them being a circle, also exists.

Theorem 3.2. Consider a circle C of radius $\rho > 0$ centered at O , and two ellipses E_1 and E_2 , all three being mutually conjugate, i.e. E_1, E_2 and C are defined by a bundle of three given line segments (corresponding to their mutually conjugate radii) OP, OQ and OR , such that $E_1 = E_1(OP, OR)$, $E_2 = E_2(OQ, OR)$ and $C = C(OP, OQ)$. Let $r := |OR|$ and $\omega := \angle(OP, OR)$. Then, at most, two common tangential ellipses of E_1, E_2 and C can exist. That is, besides the always existing c.t.e. E (which its major axis is spanned by the non-orthogonal segment OR of the bundle, recall Theorem 3.1), a second c.t.e. E^* ($\neq E$) can also exist, and when it does its major axis forms an angle θ^* with OP (i.e. θ^* is the directive angle of E^* with respect to OP), and is given by

$$\frac{\tan \theta^*}{\tan \omega} = -\frac{\rho^2 + r^2 \cos 2\omega}{\rho^2 - r^2 \cos 2\omega} = -\frac{\rho^2 + r^2 + (\rho^2 - r^2) \tan^2 \omega}{\rho^2 - r^2 + (\rho^2 + r^2) \tan^2 \omega}, \quad (3.54)$$

The second c.t.e. E^* (of E_1, E_2 and C) with $\omega \in (0, \pi/2)$, exists if and only if $\omega \in (0, \omega_0)$ where

$$\omega_0 := \begin{cases} \frac{1}{2} \arccos \left(\rho r^{-2} \sqrt{2r^2 - \rho^2} \right), & \text{if } r \geq \rho \sqrt{2}/2, \\ \pi/2, & \text{if } r < \rho \sqrt{2}/2. \end{cases} \quad (3.55)$$

Equivalently assumed that $\omega \in (0, \pi/2)$, the second c.t.e. E^* exists if and only if $r/\rho \in (0, u_-) \cup (u_+, +\infty)$, where

$$u_{\pm} := (\sec 2\omega)\sqrt{1 \pm \sin 2\omega}. \tag{3.56}$$

The principal radii $0 < b^* < a^*$ of E^* are then given by

$$a^* = \sqrt{\frac{(\rho^2 - r^2)(\rho^4 - r^4 \cos^2 2\omega)}{(\rho^2 - r^2 \cos 2\omega)^2 - 4\rho^2 r^2 \sin^2 \omega}} \text{ and } b^* = \rho, \tag{3.57}$$

while E^* is analytically expressed, on an o.c.s. spanned by orthogonal radii OP and OQ , as

$$E^* : \left[a^{*2}(\rho^2 + r^2 \cos 2\omega)^2 \tan^2 \omega - \rho^2(\rho^2 - r^2 \cos 2\omega)^2 \right] x^2 + \left[a^{*2}(\rho^2 - r^2 \cos 2\omega)^2 + \rho^2(\rho^2 + r^2 \cos 2\omega)^2 \tan^2 \omega \right] y^2 + 2(\rho^2 - a^{*2})(\rho^4 - r^4 \cos^2 2\omega)(\tan \omega)xy = \rho^2 a^{*2} \left[(\rho^2 - r^2 \cos 2\omega)^2 + (\rho^2 + r^2 \cos 2\omega)^2 \tan^2 \omega \right]. \tag{3.58}$$

Proof. Note that $|OP| = |OQ| = \rho$ and $OP \perp OQ$ as C is a circle of radius $\rho > 0$. Consider an o.c.s. with origin at C 's center O , with its horizontal and vertical axes spanned by the orthogonal radii OP and OQ respectively. Let $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2\}$ be the o.c.s.'s orthonormal vector base.

Similar to the (beginning of the) proof of Theorem 3.1, by setting $u := r/\rho$ the ellipses E_x and E_y of Lemma 3.3 are now correspond to the ellipses E_1 and E_2 , while the line segments OP , OQ and OR , indeed, define the three mutually conjugate ellipses as $E_1 = E_1(OP, OR)$, $E_2 = E_2(OQ, OR)$ and $C = C(OP, OQ)$, and hence being three mutually conjugate radii of E_1 , E_2 and C . Moreover, Lemma 3.3 implies that, at most, two common tangential ellipses of E_1 , E_2 and C can exist. Notably, a c.t.e. E in the direction of OR , always exists and it is analytically expressed (in the adopted o.c.s.) by (3.27), while a second c.t.e. $E^* (\neq E)$, can also exist if and only if $\omega = \angle(OP, OR) \in (0, \omega_0)$, with ω_0 as in (3.55), as directly derived from Lemma 3.3 by setting $u := r/\rho$.

Recalling the inequality (3.53), the second c.t.e. $E^* (\neq E)$ can also exist, assumed that $\omega \in (0, \pi/2)$, when $1 - 2u^2 + u^4 \cos^2 2\omega > 0$, which is equivalent (by solving the inequality with respect to u) to $r/\rho = u \in (0, u_-) \cup (u_+, +\infty)$ with u_{\pm} as in (3.56).

The major radius a^* of the second c.t.e. E^* (when E^* exists) is then given by $a^* = b^*/\lambda^* = \rho/\lambda^*$ (as its minor radius $b^* = \rho$), where the E^* 's aspect ratio $\lambda^* (= \lambda_x = \lambda_y)$ is obtained through $(\lambda^*)^{-2} = \ell$ where ℓ being the ratio in (3.51). By substitution of $\tan \theta^*$, as in (3.54), into (3.51) we have

$$\lambda^{*2} = \frac{(u^2 \cos 2\omega - 1)^2 - 4u^2 \sin^2 \omega}{(u^2 - 1)(u^4 \cos^2 2\omega - 1)}, \tag{3.59}$$

and by setting $u := r/\rho$, the major radius $a^* = \rho/\lambda^*$ as in (3.57) is obtained. Moreover, E^* can then be analytically expressed (in the orthonormal vector base \mathcal{B}) by (2.1), where we must set $\omega := \theta^*$, $a := a^*$, $b := b^* = \rho$ and $u := r/\rho$. Hence, (3.58) is finally derived by the substitution of θ^* as in (3.54). \square

The following Example demonstrates a case of non-existent second c.t.e. of three mutually conjugate ellipses with one of them is being a circle. A common tangential hyperbola is calculated instead.

Example 3.2. Recall the bundle of the three given line segments OP , OQ and OR , as given in Example 3.1, which define the three mutually conjugate ellipses $E_1 = E_1(OP, OR)$, $E_2 = E_2(OQ, OR)$ and $E_3 = E_3(OP, OQ)$, with E_3 being a circle of radius $\rho > 0$. Then, a c.t.e. E of these ellipses exists in the direction of the non-orthogonal segment OR having point R as one of its foci; see also Fig. 5. According to Theorem 3.2, a second c.t.e. $E^* (\neq E)$ of E_i , $i = 1, 2, 3$, can exist iff $\omega \in (0, \omega_0)$. However, this is not the case here. Indeed, calculating θ^* as in (3.54) and ω_0 as in (3.55), it holds that $\pi/2 > \pi/6 =: \omega = \angle(OP, OR) > \omega_0 = \frac{1}{2} \arccos(\frac{2}{9}\sqrt{14}) \approx 16.8744^\circ$ (which cannot hold), while $15 =: r > \rho/\sqrt{2} = 5\sqrt{2} \approx 7.0711$, i.e. $30^\circ =: \omega \notin (0, \omega_0) \approx (0, 16.8744^\circ)$. Therefore, the ellipse E is the only c.t.e. of E_i , $i = 1, 2, 3$, that exists. Moreover, the analytical expression (in the adopted o.c.s.) of E^* , as in (3.58), implies, through $\rho := 10$ and $r := 15$, that $E^* : -313x^2 + 306\sqrt{3}xy + 226y^2 + 34000 = 0$, which is clearly a hyperbola as it is expected. This is because the above condition of ω (suggested by Theorem 3.2) does not fulfilled, as $30^\circ \notin (0, 16.8744^\circ)$, which means that the ratio in (3.51) must be < 1 , i.e. E^* 's aspect ratio $\lambda^* \in \mathbb{C}$. Indeed, the above can be confirmed calculating the values of a^* and b^* , as in (3.57), i.e. $a^* = \frac{5}{143}\sqrt{12155}i \approx 3.8549i$ and $b^* = \rho = 10$. From the above expression for E^* we conclude that the canonical form of the hyperbola E^* can be written as $E^* : (\hat{x}/\rho)^2 - (\hat{y}/|a^*|)^2 = 1$, $\hat{x}, \hat{y} \in \mathbb{R}$, as $0 < |a^*| < \rho$, and hence the (major) semi-axis length of hyperbola E^* (which is the minimum distance of E^* from its center) is then ρ while

its imaginary (or minor) semi-axis length is then $|a^*| = \frac{5}{143}\sqrt{12155}$. The directive angle of the hyperbola E^* with respect to OP (i.e. the angle of its (major) axis with respect to the horizontal axis of the adopted o.c.s.) is then given by $\theta^* - \pi/2$ where θ^* as in (3.54), i.e. $\theta^* = \arctan(\frac{17}{3}\sqrt{3}) - \frac{\pi}{2} \approx -5.8175^\circ$. This is so, because in case E^* was an ellipse, the value of $\tan\theta^*$ given in (3.54) corresponds to the slope of E^* 's major axis, while in the present hyperbola case, the value of $\tan\theta^*$ corresponds to the slope of E^* 's imaginary (minor) axis, as E^* is now expressed by $E^* : -(\tilde{x}/|a^*|)^2 + (\tilde{y}/\rho)^2 = 1$ where $0 < |a^*| < \rho$. Thus, due to the orthogonality of the principal axes, we have to subtract $\pi/2$ from the value of θ^* as in (3.54).

Figure 6 visualizes exactly Example 3.2, by presenting the three mutually conjugate ellipses $E_1 = E_1(OP, OR)$, $E_2 = E_2(OQ, OR)$ and $E_3 = E_3(OP, OQ)$ as well as their c.t.e. E , exactly as in Fig. 5, together with their common tangential (c.t.) hyperbola E^* of $E_i, i = 1, 2, 3$.

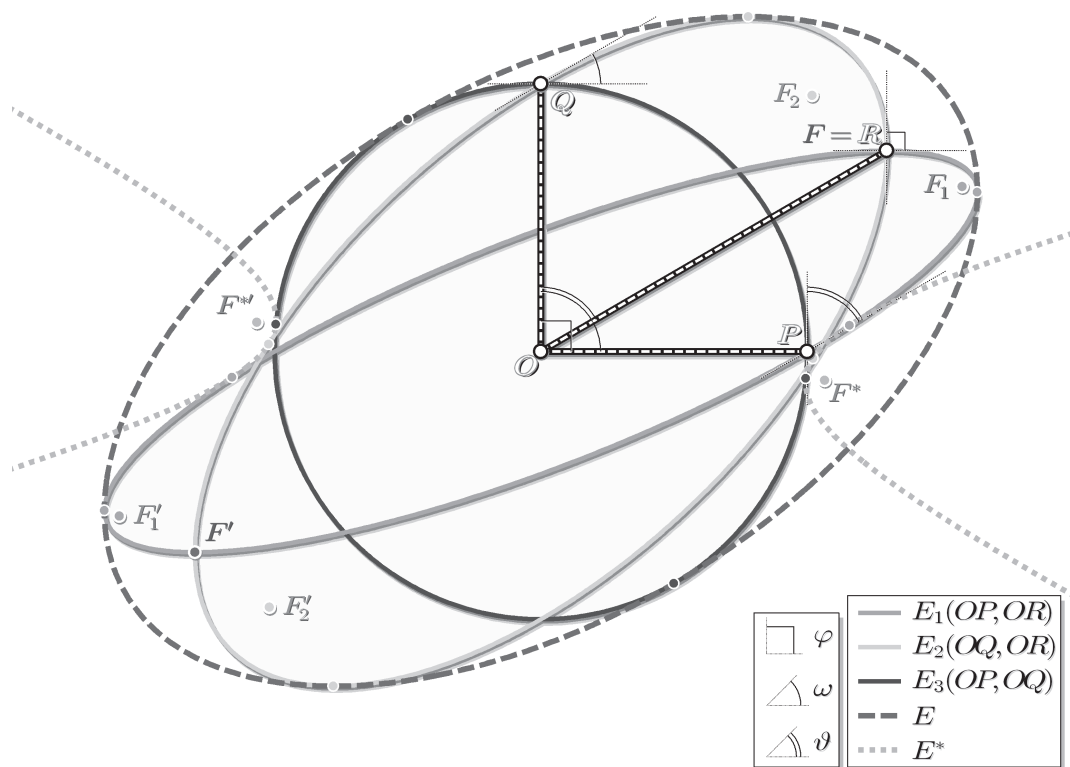


Figure 6. Graphs of the three mutually conjugate ellipses $E_i, i = 1, 2, 3$, as in Example 3.2, together with their c.t. ellipse E and c.t. hyperbola E^* .

The following Example demonstrates the existence of a second c.t.e. of three mutually conjugate ellipses with one of them being a circle.

Example 3.3. Consider a bundle of three given line segments OP, OQ and OR as in Example 3.1, or 3.2, where this time OR 's length is set to be the one-third of the OP 's length, i.e. $r = |OR| := |OP|/3 = 5$. Recall from Example 3.1 that $\rho = |OP| = |OQ| := 10$ with $OP \perp OQ$ and $\omega = \angle(OP, OR) := \pi/6 (= 30^\circ)$. These line segments are considered to be the three mutually conjugate radii that define three mutually conjugate ellipses, i.e. $E_1 = E_1(OP, OR)$, $E_2 = E_2(OQ, OR)$ and $E_3 = E_3(OP, OQ)$. Note that E_3 is a circle of radius $\rho > 0$. Consider also (as in Example 3.1) an o.c.s. spanned by the orthogonal radii OP and OQ . Similarly to the Example 2.1, the analytical expressions of the ellipses E_1 and E_2 are then given by (2.13) and (2.5) respectively, where $S_x(C)$ and $S_y(C)$ correspond now to E_1 and E_2 respectively, with $u := r/\rho = 1/2$, i.e. $E_1 : x^2 - 2\sqrt{3}xy + 19y^2 = 100$ and $E_2 : 17x^2 - 2\sqrt{3}xy + 3y^2 = 300$, while E_3 corresponds to the circle $E_3 : x^2 + y^2 = 100$. A c.t.e. E of $E_i, i = 1, 2, 3$, exists and its major axis is spanned by the non-orthogonal radius OR , while point R being one of its foci; see Fig. 7. The analytical expression (on the adopted o.c.s.) is given, through (3.27), by $E : 17x^2 - 2\sqrt{3}xy + 19y^2 = 2000$. As far as the additional c.t.e. of $E_i, i = 1, 2, 3$, is concerned, it holds that $1/2 = r/\rho \in (0, u_-) \approx (0, 0.732)$, as $\pi/6 =: \omega \in (0, \pi/2)$. Therefore, from Theorem 3.2, we conclude that (unlike Example 3.2) there exist two, in total, common tangential

ellipses, E and E^* , of the three mutually conjugate ellipses E_i , $i = 1, 2, 3$. The existing second c.t.e. E^* has then principal radii $0 < b^* < a^*$ and directive angle θ^* given, through (3.57) and (3.54), by $a^* = \frac{5}{11}\sqrt{77} \approx 11.9659$, $b^* = 10$ and $\theta^* = -\arctan(\frac{3}{7}\sqrt{3}) \approx -36.5868^\circ$, while, through (3.58), its analytical expression is of the form $E^*: 203x^2 + 42\sqrt{3}xy + 225y^2 = 25200$.

The following Fig. 7 is an exact visualization of Example 3.3, by presenting the three mutually conjugate ellipses $E_1 = E_1(OP, OR)$, $E_2 = E_2(OQ, OR)$ and $E_3 = E_3(OP, OQ)$ together with their foci, as well as their two (in total) common tangential ellipses E and E^* . The intersection angles between the ellipses E_i , $i = 1, 2, 3$ at their points P, Q and R are also depicted.

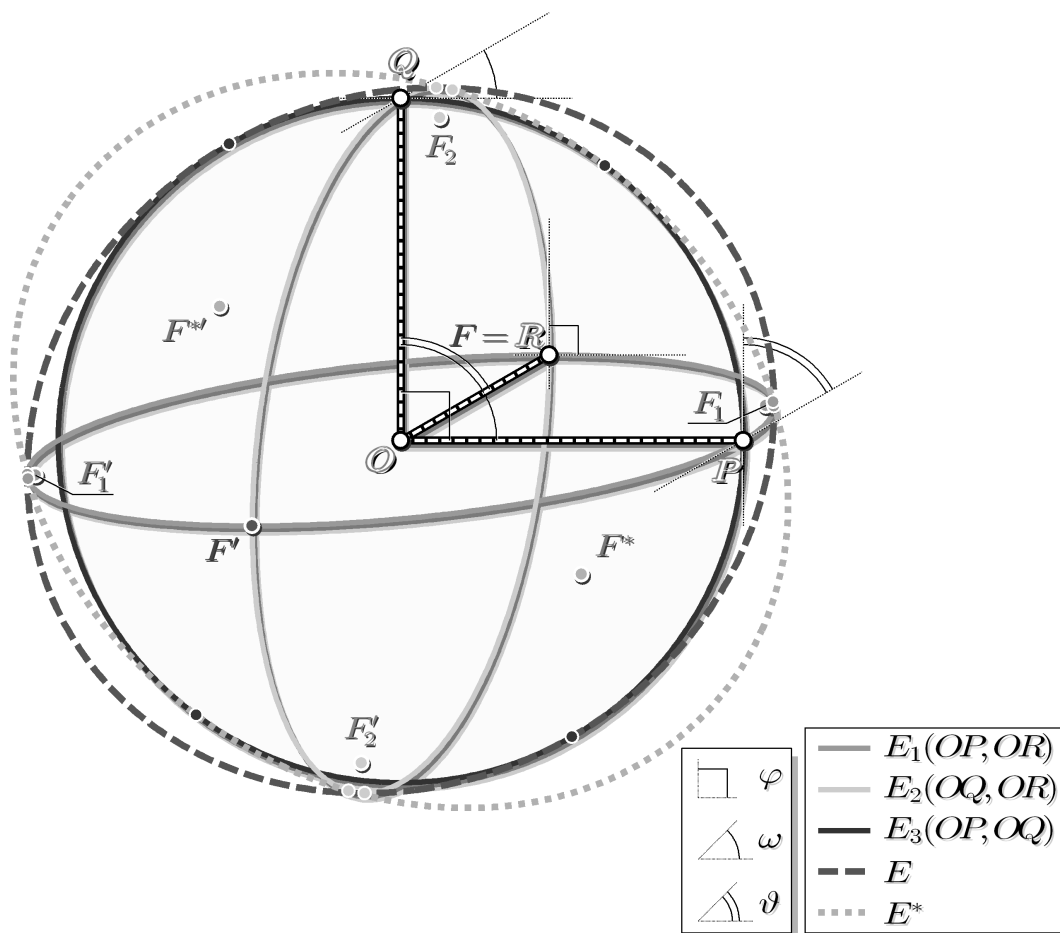


Figure 7. Graphs of the three mutually conjugate ellipses E_i , $i = 1, 2, 3$, as in Example 3.3, together with their two common tangential ellipses E and E^* .

3.2. The general case

We shall now extend the results of the orthogonal case, as discussed in sub-Section 3.1, to the general case of three mutually conjugate ellipses (including the case where none of them is a circle).

From the orthogonal case, investigated in Theorem 3.1, we concluded that there always exists a c.t.e. around three given mutually conjugate ellipses, when one of them is a circle. The tangential ellipse E can then be considered as a “primary” c.t.e., while any other c.t.e. that might exist can be considered as a “secondary” c.t.e. (of E_i , $i = 1, 2, 3$). Having the above in mind, we consider the following definitions.

Definition 3.2. Let E_i , $i = 1, 2, 3$, be three mutually conjugate ellipses. We can always consider a coordinate system in which one of the ellipses is expressed as a circle, and hence two of their three mutually conjugate radii, which define E_i , $i = 1, 2, 3$, are of equal length and orthogonal with each other. A c.t.e. E of E_i , $i = 1, 2, 3$, shall then be called primary when its major semi-axis coincides with the non-orthogonal radius from the corresponding three mutually conjugate radii (defining E_i , $i = 1, 2, 3$).

Definition 3.3. Any other non-primary c.t.e. of E_i , $i = 1, 2, 3$, shall be called as a secondary c.t.e. of E_i , $i = 1, 2, 3$.

Before we proceed with the main Theorem, a special cyclic sum notation \mathfrak{S} is introduced here. In particular, the following form of “multiple” cyclic sums shall be used, i.e.

$$\mathfrak{S}_{\substack{p,q,r \\ \varphi,\vartheta,\omega}} f(p,q,r; \varphi,\vartheta,\omega) := f(p,q,r; \varphi,\vartheta,\omega) + f(q,r,p; \vartheta,\omega,\varphi) + f(r,p,q; \omega,\varphi,\vartheta).$$

The cyclic sum as above refers to a three–termed sum of the variable–depending expression f , where the summation is simultaneously cycle through the three values p,q,r and φ,ϑ,ω , i.e. according to the scheme of indexes’ sequence: $\begin{matrix} p \rightarrow q \rightarrow r \rightarrow p \rightarrow \dots \\ \varphi \rightarrow \vartheta \rightarrow \omega \rightarrow \varphi \rightarrow \dots \end{matrix}$.

The following main Theorem investigates the existence of common tangential ellipses around three mutually conjugate ellipses, and therefore extends Peschka’s Proposition 1.1. It also provides a plane–geometric proof of it.

Main Theorem. Consider three mutually conjugate ellipses E_i , $i = 1, 2, 3$, defined by a bundle of three given line segments which correspond to E_i ’s three mutually conjugate radii. These given radii, say OP , OQ and OR , are determined by their lengths $p, q, r > 0$ respectively, and the two angles $\varphi := \angle(OP, OQ) \in (0, \pi)$ and $\omega := \angle(OP, OR) \in (0, \pi)$. Let also $\vartheta := \angle(OR, OQ) \in (0, \pi) = \varphi - \omega$. Then, there always exists a primary c.t.e. E of E_i , $i = 1, 2, 3$, with their corresponding common tangential lines, between each E_i , $i = 1, 2, 3$, and their c.t.e. E , being parallel to each of the three given conjugate radii. Moreover, a unique secondary c.t.e. E^* ($\neq E$) of E_i , $i = 1, 2, 3$, with $\varphi > \omega$, can exist if and only if

$$\mathfrak{S}_{\substack{p,q,r \\ \varphi,\vartheta,\omega}} p^4 q^2 \sin^2 \varphi (q^2 \sin^2 \varphi - 2r^2 \sin^2 \omega) > 0, \quad (3.60)$$

or when it holds

$$2r^2 (p^2 \sin^2 \omega + q^2 \sin^2 \vartheta) < p^2 q^2 \sin^2 \varphi. \quad (3.61)$$

In case one of the mutually conjugate ellipses is reduced to a circle (and therefore two out of three E_i ’s mutually conjugate radii are orthogonal and have equal length), then the foci of the primary c.t.e. E coincide with the end points of the non–orthogonal diameter of the three mutually conjugate diameters (spanned by the three given mutually conjugate radii) which define E_i , $i = 1, 2, 3$.

Proof. Consider an o.c.s. where its horizontal and vertical axes are spanned by radii OP and OQ' respectively, where $OQ' \perp OP$ and $|OQ'| = |OP|$. Hence, the corresponding orthonormal vector base is given by $\mathcal{B} := \{\mathbf{e}_1 := OP/|OP|, \mathbf{e}_2 := OQ'/|OQ'|\}$, i.e. $P_{\mathcal{B}}(p, 0)$ and $Q'_{\mathcal{B}}(0, p)$, or simply $P(p, 0)$ and $Q'(0, p)$. For point Q it then holds that $Q(q \cos \varphi, q \sin \varphi)$. Let also $E_1 := E_1(OP, OR)$, $E_2 := E_2(OQ, OR)$ and $E_3 := E_3(OP, OQ)$ be three mutually conjugate ellipses defined by their corresponding three mutually conjugate radii OP , OQ and OR . As pointed out in Definition 3.2, a new c.s. can be adopted so that one of the ellipses E_i can be expressed as a circle. Without loss of generality, we may choose the c.s. in which ellipse E_3 is expressed as circle $C(O, p)$. It then holds that $OP \perp OQ$ and $|OP| = |OQ|$, i.e. $\varphi = \pi/2$ and $p = q$. Hence, Theorem 3.1 can be applied in order to derive the primary c.t.e. E of E_i , $i = 1, 2, 3$. Moreover, due to the fact that $0 < \vartheta = \varphi - \omega$, or $\omega < \varphi$, the given radius OR lies “between” OP and OQ , as $\omega \in (0, \varphi) \subset (0, \pi)$; see also Fig. 5 where OR lies between OP and OQ . Therefore, Theorem 3.2 can also be applied in order to derive the secondary (if any) c.t.e. E^* of E_i , $i = 1, 2, 3$.

For the construction of this new c.s., in which E_3 corresponds to circle $C(O, p)$, we consider the following: Let C be a circle of radius p centered at O , and its points $P(p, 0)$ and $Q'(0, p)$. Let $S_x : \mathbf{x}' = \mathbf{N}_{\mathbf{u}} \mathbf{x}$ be a x –shear transformation with shearing vector $\mathbf{u} := p^{-1}OQ$ in the initially adopted o.c.s. (of vector base \mathcal{B}). Then, S_x transforms point the $Q'(0, p)$ into $Q(q \cos \varphi, q \sin \varphi)$. Indeed, substituting the shearing vectors’ length $u := |\mathbf{u}| = q/p$ and its angle $\omega := \varphi$ into (2.3), the x –shear transformation matrix is then given by

$$\mathbf{N}_{\mathbf{u}} = \begin{pmatrix} 1 & u \cos \omega \\ 0 & u \sin \omega \end{pmatrix} = \begin{pmatrix} 1 & \frac{q}{p} \cos \varphi \\ 0 & \frac{q}{p} \sin \varphi \end{pmatrix}, \quad (3.62)$$

and hence it can be easily verified that $S_x(Q') = Q$. Moreover, it holds that $S_x(P) = P$, as the horizontal axis $x'Ox$ (in \mathcal{B}) is an S_x –invariant. Therefore, the x –sheared circle C , i.e. $S_x(C)$, is essentially the ellipse E_3 , as the orthogonal pair of radii (OP, OQ') is transformed into the pair of conjugate radii (OP, OQ) which define ellipse E_3 , as $E_3 = E_3(OP, OQ)$ is assumed, i.e. $S_x(C) = E_3$ with $P \in E_3 \cap E_1$ and $Q \in E_3 \cap E_2$.

Consider now a new vector base, say \mathcal{B}' , in which ellipse E_3 is expressed as circle $C(O, p)$ or, equivalently, $OP \perp_{\mathcal{B}'} OQ$ and $|OP|_{\mathcal{B}'} = |OQ|_{\mathcal{B}'}$. Essentially, the affine transformation S_x (as defined above) corresponds to a

vector base change, from \mathcal{B} to \mathcal{B}' , of the form $\mathbf{x} = \mathbf{N}_{\mathbf{u}}\mathbf{x}'$, in which $P_{\mathcal{B}'}(p, 0)$ and $Q_{\mathcal{B}'}(0, p)$. The length $r' := |OR|_{\mathcal{B}'}$ and angle $\omega' := \angle_{\mathcal{B}'}(OP, OR)$ can now be calculated in order to use them instead of values $r := |OP|$ and $\omega := \angle(OP, OR)$, for the application of Theorems 3.1 and 3.2. The components' vector \mathbf{r} of OR is also given, in the initial vector base \mathcal{B} , by $\mathbf{r} = (r_1, r_2)^T := (r \cos \omega, r \sin \omega)^T \in \mathbb{R}^{2 \times 1}$. When OR is expressed in \mathcal{B}' , its new components' vector, say \mathbf{r}' , is then given by $\mathbf{r}' = (r'_1, r'_2)^T := (r' \cos \omega', r' \sin \omega')^T \in \mathbb{R}^{2 \times 1}$. Applying the vector base change $\mathbf{x} = \mathbf{N}_{\mathbf{u}}\mathbf{x}'$ on the vector \mathbf{r} , we obtain that $\mathbf{r} = \mathbf{N}_{\mathbf{u}}\mathbf{r}'$, or

$$(r'_1, r'_2)^T = \mathbf{r}' = \mathbf{N}_{\mathbf{u}}^{-1}\mathbf{r} = r(\csc \varphi) \left(\sin(\varphi - \omega), \frac{p}{q} \sin \omega \right)^T, \tag{3.63}$$

and thus $r' = |OR|_{\mathcal{B}'} = |\mathbf{r}'|_{\mathcal{B}'} = \sqrt{r_1'^2 + r_2'^2}$, while OR 's corresponding angle ω' is then given by $\omega' = \angle_{\mathcal{B}'}(OP, OR) = \arctan(r'_2/r'_1) \in (-\pi, \pi)$. Therefore, from (3.63), we derive after some algebra, that

$$r' = \frac{r}{q}(\csc \varphi) \sqrt{p^2 \sin^2 \omega + q^2 \sin^2 \vartheta} \quad \text{and} \quad \omega' = \arctan \left(\frac{p \sin \omega}{q \sin \vartheta} \right). \tag{3.64}$$

Recall the mutually conjugate ellipses $E_1 = E_1(OP, OR)$, $E_2 = E_2(OQ, OR)$ and $E_3 = E_3(OP, OQ)$, with $P_{\mathcal{B}'}(p, 0)$, $Q_{\mathcal{B}'}(0, p)$ and $R_{\mathcal{B}'}(r'_1, r'_2)$. Therefore, in vector base \mathcal{B}' , where the ellipse E_3 corresponds to circle $C(O, \rho)$, Theorem 3.1 can be applied, and hence the primary c.t.e. E of E_i , $i = 1, 2, 3$ (as in Definition 3.2), always exists. Recalling Theorem 3.1, the non-orthogonal radii OR coincides with E 's major semi-axis, while the foci of E are essentially the end points of (non-orthogonal) diameter spanned by radius OR . In vector base \mathcal{B}' , Theorem 3.1 shows also that each pair of (diametrical) common tangent lines (t_i, t'_i) between ellipse E_i and its primary c.t.e. E , for $i = 1, 2, 3$, are being parallel to each of the three given mutually conjugate radii OP , OQ and OR . In particular, t_1 and t'_1 are parallel to OQ , t_2 and t'_2 are parallel to OP , while t_3 and t'_3 are parallel to OR . Therefore, as S_x preserves parallelism, we conclude that also in the initially adopted vector base \mathcal{B} the common tangent lines between each E_i and E are parallel to each (of the three) mutually conjugate radii that define E_i , $i = 1, 2, 3$.

Moreover, according to Theorem 3.2 and Definition 3.3, a secondary c.t.e. E^* of E_i , $i = 1, 2, 3$, exists under certain conditions. This secondary tangential E^* —when exists— is also unique, i.e. it is the only non-primary c.t.e. of E_i , $i = 1, 2, 3$. Recall that a second tangential E^* ($\neq E$) can exist, iff (3.53) holds, i.e. iff

$$p^4 - 2p^2r'^2 + r'^4 \cos^2 2\omega' > 0, \tag{3.65}$$

where we have set $u := r'/p$ and $\omega := \omega'$ (as values r and ω in (3.53) correspond now to values r' and ω' in vector base \mathcal{B}'). Thus, applying (3.64) into (3.65) we derive (after some algebra) the equivalent condition

$$p^4 q^2 \sin^2 \varphi (2r^2 \sin^2 \omega - q^2 \sin^2 \varphi) + q^4 r^2 \sin^2 \vartheta (2p^2 \sin^2 \varphi - r^2 \sin^2 \vartheta) + r^4 p^2 \sin^2 \omega (2q^2 \sin^2 \vartheta - p^2 \sin^2 \omega) < 0, \tag{3.66}$$

which can be written in the compact cyclic sum form of (3.60).

Inequality (3.65) holds trivially for every $\omega' \in (0, \pi)$ if $r' < p/\sqrt{2}$, i.e. if $p^4 - 2p^2r'^2 > 0$. That is, when $p > \sqrt{2}r'$ or, equivalently through (3.64), when (3.61) holds, then general condition (iff) (3.65) also holds for every $\omega' \in (0, \pi)$. Therefore, the unique secondary tangential ellipse E^* also exists when (3.61) is assumed. \square

The following Example shows the use of conditions (3.60) and (3.61) for concluding the existence or not of a secondary c.t.e. E^* .

Example 3.4. Recall Example 3.1, where it was shown that there is only one c.t.e. of three mutually conjugate ellipses E_i , $i = 1, 2, 3$, i.e. the primary c.t.e. E (as expected). Setting now $\varphi = \angle(OP, OQ) := \pi/2$ into the special (iff) condition (3.61), as well as $p = q := \rho = 10$, $r := 15$ and $\omega = \angle(OP, OR) := \pi/6$, we derive that (3.61) yields $-35000 > 0$ which cannot hold. As the condition in (3.61) is not an "if and only if" condition, we cannot conclude the non-existence of a secondary c.t.e. E^* . Moreover, by substitution of the above values into the condition as in (3.60), we derive that $223437500 < 0$ which also cannot hold, but this time we can safely conclude that the second c.t.e. does not exist, as (3.60) is an iff condition. The special condition (3.61) can, however, determine the existence of the second c.t.e. E^* of E_i , $i = 1, 2, 3$, as in Example 3.3. Indeed, setting $\varphi := \pi/2$, $\omega := \pi/6$, $p = q := 10$ and $r := 5$ into (3.61), it is obtained that $5000 > 0$ which holds, and hence E^* exists. This also can be confirmed by calculating the general (iff) condition (3.60) which then implies that $-5156250 < 0$ which also holds.

Finally, we point out that the analytical expressions of the two common tangential ellipses E and E^* , can be calculated (in the vector base \mathcal{B}') by setting $r := r'$ and $\omega := \omega'$ both into (3.27) and (3.58) respectively, where r' and ω' are as in (3.64).

Discussion

Consider the problem of finding a concentric common tangent ellipse around three given coplanar and concentric ellipses, say E_i , $i = 1, 2, 3$, each one defined by a pair of two conjugate semi-diameters which are taken from a bundle of three given line segments. This plane-geometric problem was first stated and addressed by G. A. Peschka in [5], in his proof of Pohlke's Fundamental Theorem of Axonometry. However, his proof is based on a parallel projection of an appropriate sphere S onto E_i 's common plane, say \mathbb{E} . Indeed, these *mutually conjugate* ellipses E_i , $i = 1, 2, 3$, correspond to the parallel projections of three maximum circles of the sphere S (lying on three planes orthogonal with each other). According to [4, 5], the common tangential ellipse, say E , of all E_i , $i = 1, 2, 3$, is then the parallel projection of the sphere's contour onto \mathbb{E} . This parallel projection method has been used in literature for the proof of Pohlke's Fundamental Theorem of Axonometry.

With this present paper, the above "Peschka's problem", which is a plane-geometric problem, is addressed exclusively in terms of Analytic Plane Geometry, while it was also thoroughly investigated. Moreover, not only did it was found that a common tangent ellipse (around three given mutually conjugate ellipses, as E_i , $i = 1, 2, 3$) always exists, but also found all the existed concentric and tangent ellipses around three given mutually conjugate ellipses. It was proved that there exist two, in total, c.t.e.: the *primary* one E (which corresponds to the parallel-projected contour of sphere S as above), and the *secondary* one E^* ($\neq E$) which exists under certain conditions. When the secondary c.t.e. E^* does not exist, then a (concentric to E_i) hyperbola exists instead, with its two branches being tangent to all three given mutually conjugate ellipses E_i , $i = 1, 2, 3$. The provided examples and figures were demonstrating these results.

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Affiliations

THOMAS L. TOULIAS

ADDRESS: Avenue Charbo 20, Schaerbeek 1030, Brussels, Belgium.

E-MAIL: th.toulias@gmail.com

GEORGE E. LEFKADITIS

ADDRESS: University of Patras, Department of Architecture, Rio 26504, Patras, Greece.

E-MAIL: gléf@upatras.gr