INTERNATIONAL ELECTRONIC JOURNAL OF GEOMETRY
VOLUME 9 NO. 2 PAGE 36–43 (2016)

The Euler Class in the Simplicial de Rham Complex

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(Communicated by Yusuf Yaylı)

ABSTRACT

We exhibit a cocycle in the simplicial de Rham complex which represents the Euler class. As an application, we construct a Lie algebra cocycle on $L\mathfrak{so}(4)$.

Keywords: *Characteristic classes; simplicial manifold.*
AMS Subject Classification (2010): *Primary:53-XX.*

For any Lie group G, we can define a simplicial manifold $\{NG(*)\}$ and a double complex $\Omega^*(NG(*))$ on it. In classical theory, it is well-known that the cohomology ring of the total complex $\Omega^*(NG)$ is isomorphic to $H^*(BG)$ where BG is a classifying space of G, which is not a manifold in general [2] [5] [6].

In [4], Dupont introduced another double complex $A^{*,*}(NG)$ on NG such that the cohomology ring of its total complex $A^*(NG)$ is also isomorphic to $H^*(BG)$. He used it to construct a homomorphism from $I^*(G)$, the G-invariant polynomial ring over Lie algebra $\mathcal{G}$, to $H^*(BG)$. By using Dupont's method, in [8] the author exhibited cocycles in $\Omega^*(NG)$ which represent the Chern characters. In this paper, we will exhibit cocycles which represent the Euler classes.

Using a cocycle in $\Omega^*(NG)$, we can construct a cocycle in the local truncated complex $[\sigma_{<p}\Omega^*_{\rm loc}(NG)]$ due to Brylinski [3]. Furthermore, we can obtain a Lie algebra cocycle of a free loop group LG. Following Brylinski's idea, we will construct a Lie algebra 2-cocycle on $L\mathfrak{so}(4)$ using a cocycle in $\Omega^4(NSO(4))$.

1. Review of the universal Chern-Weil Theory

In this section we recall the universal Chern-Weil theory following [5]. For any Lie group G, we have simplicial manifolds $N\bar{G}$, NG and simplicial G-bundle $\gamma : N\bar{G} \to NG$ as follows:

$$N\bar{G}(q) = \overbrace{G \times \cdots \times G}^{q+1-times} \ni (g_1, \cdots , g_{q+1})$$

$$NG(q) = \overbrace{G \times \cdots \times G}^{q-times} \ni (h_1, \cdots , h_q) :$$

face operators $\varepsilon_i : NG(q) \to NG(q-1)$

$$\varepsilon_i(h_1, \cdots , h_q) = \begin{cases} (h_2, \cdots , h_q) & i = 0 \\ (h_1, \cdots , h_ih_{i+1}, \cdots , h_q) & i = 1, \cdots , q-1 \\ (h_1, \cdots , h_{q-1}) & i = q. \end{cases}$$

We define $\gamma : N\bar{G} \to NG$ as $\gamma(g_0, \cdots , g_q) = (g_0{g_1}^{-1}, \cdots , g_{q-1}{g_q}^{-1})$.

For any simplicial manifold $X = \{X_*\}$, we can associate a topological space $\parallel X \parallel$ called the fat realization. It is well-known that $\parallel \gamma \parallel$ is the universal bundle $EG \to BG$ [7].

Now we introduce a double complex associated to a simplicial manifold.

Received : *03-10-2015,* ***Accepted :*** *24-08-2016*
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Definition 1.1. For any simplicial manifold $\{X_*\}$ with face operators $\{\varepsilon_*\}$, we define a double complex as follows:

$$\Omega^{p,q}(X) := \Omega^q(X_p)$$

Derivatives are:

$$d' := \sum_{i=0}^{p+1}(-1)^i\varepsilon_i^*, \qquad d'' := (-1)^p \times \text{the exterior differential on } \Omega^*(X_p).$$

For NG and $N\bar{G}$ the following holds [2] [5] [6].

Theorem 1.1. *There exist ring isomorphisms*

$$H(\Omega^*(N\bar{G})) \cong H^*(EG), \qquad H(\Omega^*(NG)) \cong H^*(BG).$$

Here $\Omega^*(N\bar{G})$ and $\Omega^*(NG)$ mean the total complexes.

There is another double complex associated to a simplicial manifold.

Definition 1.2 ([4])**.** A simplicial n-form on a simplicial manifold $\{X_p\}$ is a sequence $\{\phi^{(p)}\}$ of n-forms $\phi^{(p)}$ on $\Delta^p \times X_p$ such that

$$(\varepsilon^i \times id)^*\phi^{(p)} = (id \times \varepsilon_i)^*\phi^{(p-1)} \quad \text{on } \Delta^{p-1} \times X_p.$$

Here ε^i is the canonical i-th face operator of Δ^p.

Let $A^{k,l}(X)$ be the set of all simplicial $(k+l)$-forms on $\Delta^p \times X_p$ which are expressed locally of the form

$$\sum a_{i_1\cdots i_k j_1 \cdots j_l}(dt_{i_1} \wedge \cdots \wedge dt_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l})$$

where $(t_0, t_1, \cdots, t_p)$ are the barycentric coordinates in Δ^p and x_j are the local coordinates in X_p. We define derivatives as:

$$d' := \text{the exterior differential on } \Delta^p$$

$$d'' := (-1)^k \times \text{the exterior differential on } X_p.$$

Then $(A^{k,l}(X), d', d'')$ is a double complex and the following theorem holds.

Theorem 1.2 ([4])**.** *Let $A^*(X)$ denote the total complex of $A^{*,*}(X)$. A map $I_\Delta : A^*(X) \to \Omega^*(X)$ defined as $I_\Delta(\alpha) := \int_{\Delta^p}(\alpha|_{\Delta^p \times X_p})$ induces a natural ring isomorphism $I_\Delta^* : H(A^*(X)) \cong H(\Omega^*(X))$.*

Let $\mathcal{G}$ denote the Lie algebra of G. A connection on a simplicial G-bundle $\pi : \{E_p\} \to \{M_p\}$ is a sequence of 1-forms $\{\theta\}$ on $\{E_p\}$ with coefficients $\mathcal{G}$ such that θ restricted to $\Delta^p \times E_p$ is a usual connection form.

There is a canonical connection $\theta \in A^1(N\bar{G})$ on $\gamma : N\bar{G} \to NG$ defined as follows:

$$\theta|_{\Delta^p \times N\bar{G}(p)} := t_0\theta_0 + \cdots + t_p\theta_p.$$

Here θ_i is defined as $\theta_i = \mathrm{pr}_i^*\bar{\theta}$ where $\mathrm{pr}_i : \Delta^p \times N\bar{G}(p) \to G$ is the projection into the $(i+1)$-th factor of $N\bar{G}(p)$ and $\bar{\theta}$ is the Maurer-Cartan form of G. We obtain also its curvature $\Omega \in A^2(N\bar{G})$ on γ as:

$$\Omega|_{\Delta^p \times N\bar{G}(p)} = d\theta|_{\Delta^p \times N\bar{G}(p)} + \frac{1}{2}[\theta|_{\Delta^p \times N\bar{G}(p)}, \theta|_{\Delta^p \times N\bar{G}(p)}].$$

Let $\mathrm{I}^*(G)$ denote the ring of G-invariant polynomials on $\mathcal{G}$. For $P \in I^k(G)$, we restrict $P(\Omega) \in A^{2k}(N\bar{G})$ to each $\Delta^p \times N\bar{G}(p)$ and apply the usual Chern-Weil theory then we have $I_\Delta(P(\Omega)) \in \Omega^{2k}(NG)$. In this way we have a homomorphism $\mathrm{I}^*(G) \to H(\Omega^*(NG))$ which maps $P \in I^*(G)$ to $[I_\Delta(P(\Omega))]$.

2. The Euler class in the double complex

In this section we exhibit a cocycle in $\Omega^*(NSO(2p))$ which represents the Euler class of the universal bundle $ESO(2p) \to BSO(2p)$. Throughout this section, G means $SO(2p)$.

Recall that the polynomial on $\mathfrak{so}(2p)$ called Pfaffian is defined as follows:

$$\mathrm{Pf}(A, \cdots, A) = \frac{1}{2^{2p}\pi^p p!} \sum_{\tau \in \mathfrak{S}_{2p}} \mathrm{sgn}(\tau) a_{\tau(1)\tau(2)} \cdots a_{\tau(2p-1)\tau(2p)}.$$

Here a_{ij} is a (i, j) entry of $A \in \mathfrak{so}(2p)$.

2.1. The cochain on the edge

We first give the cochain in $\Omega^{2p+1}(N\bar{G}(1))$ which corresponds to the Euler class. This is given by integrating $\mathrm{Pf}\left(\Omega|_{\Delta^1 \times N\bar{G}(1)}\right)$ along Δ^1. Since $\Omega|_{\Delta^1 \times N\bar{G}(1)} = -dt_1 \wedge (\theta_0 - \theta_1) - t_0 t_1 (\theta_0 - \theta_1)^2$, we can see $\mathrm{Pf}\left(\Omega|_{\Delta^1 \times N\bar{G}(1)}\right)$ is equal to

$$\frac{1}{2^{2p}\pi^p p!} \sum_{\tau \in \mathfrak{S}_{2m}} \mathrm{sgn}(\tau)((-dt_1 \wedge (\theta_0 - \theta_1) - t_0 t_1 (\theta_0 - \theta_1)^2)_{\tau(1)\tau(2)}$$
$$\cdots (-dt_1 \wedge (\theta_0 - \theta_1) - t_0 t_1 (\theta_0 - \theta_1)^2)_{\tau(2p-1)\tau(2p)}).$$

We set:

$$\bar{P}^k_\tau := (\theta_0 - \theta_1)^2_{\tau(1)\tau(2)} \cdots (\theta_0 - \theta_1)^2_{\tau(2k-3)\tau(2k-2)} (\theta_0 - \theta_1)_{\tau(2k-1)\tau(2k)}$$
$$(\theta_0 - \theta_1)^2_{\tau(2k+1)\tau(2k+2)} \cdots (\theta_0 - \theta_1)^2_{\tau(2p-1)\tau(2p)}.$$

Then the following equation holds.

$$\int_{\Delta^1} \mathrm{Pf}\left(\Omega|_{\Delta^1 \times N\bar{G}(1)}\right) = (-1)^p \frac{1}{2^{2p}\pi^p p!} \left(\int_0^1 (t_0 t_1)^{p-1} dt_1\right) \sum_{\tau \in \mathfrak{S}_{2p}} \sum_{k=1}^{p} \mathrm{sgn}(\tau) \bar{P}^k_\tau.$$

Now we obtain the cochain in $\Omega^{2p-1}(NG(1))$.

Proposition 2.1. *The cochain μ_p in $\Omega^{2p-1}(NG(1))$ which corresponds to the Euler class is given as follows:*

$$\mu_1 = (-1)^p \frac{1}{2^{2p}\pi^p p!} \frac{1}{{}_{2p-1}C_{p-1} \cdot p} \sum_{\tau \in \mathfrak{S}_{2p}} \sum_{k=1}^{p} \mathrm{sgn}(\tau) P^k_\tau.$$

Here P^k_τ is defined as:

$$P^k_\tau := (h^{-1}dh)^2_{\tau(1)\tau(2)} \cdots (h^{-1}dh)^2_{\tau(2k-3)\tau(2k-2)} (h^{-1}dh)_{\tau(2k-1)\tau(2k)} (h^{-1}dh)^2_{\tau(2k+1)\tau(2k+2)} \cdots (h^{-1}dh)^2_{\tau(2p-1)\tau(2p)}.$$

Proof. This follows from the equation $\int_0^1 (t_0 t_1)^{p-1} dt_1 = \frac{1}{{}_{2p-1}C_{p-1} \cdot p}$ and $\gamma^* \sum_{\tau \in \mathfrak{S}_{2p}} \mathrm{sgn}(\tau) P^k_\tau = \sum_{\tau \in \mathfrak{S}_{2p}} \mathrm{sgn}(\tau) \bar{P}^k_\tau$. □

As a special case of Proposition 3.1, we obtain the following theorem.

Theorem 2.1. *In the case of $G = SO(2)$, the cocycle $E_{1,1}$ in $\Omega^2(NG)$ which represents the Euler class of $ESO(2) \to BSO(2)$ is given as follows:*

$$E_{1,1} = \frac{1}{4\pi}(-(h^{-1}dh)_{12} + (h^{-1}dh)_{21}) \quad \in \Omega^1(SO(2)).$$

If we write an element h in $SO(2)$ as

$$h = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

then the equation

$$h^{-1}dh = \begin{pmatrix} 0 & -d\theta \\ d\theta & 0 \end{pmatrix}$$

holds, so we obtain

$$E_{1,1} = \frac{1}{4\pi}(2d\theta) = \frac{d\theta}{2\pi}.$$

2.2. The cochain in $\Omega^p(NG(p))$

In $\Omega^p(N\bar{G}(p))$, $\Omega|_{\Delta^p \times N\bar{G}(p)}$ is equal to $-\sum_{i=1}^{p} dt_i \wedge (\theta_0 - \theta_i) - \sum_{0\le i<j\le p} t_i t_j (\theta_i - \theta_j)^2$, so the cochain $\int_{\Delta^p} \mathrm{Pf}(\Omega|_{\Delta^p \times N\bar{G}(p)})$ in $\Omega^p(N\bar{G}(p))$which corresponds to the Euler class is given as follows:

$$\frac{1}{2^{2p}\pi^p p!} \sum_{\tau \in \mathfrak{S}_{2p}} \mathrm{sgn}(\tau)\big(-\sum_{i=1}^{p} dt_i \wedge (\theta_0 - \theta_i)\big)_{\tau(1)\tau(2)} \cdots \big(-\sum_{i=1}^{p} dt_i \wedge (\theta_0 - \theta_i)\big)_{\tau(2p-1)\tau(2p)}.$$

Now

$$dt_i \wedge (\theta_0 - \theta_i) = dt_i \wedge \{(\theta_0 - \theta_1) + (\theta_1 - \theta_2) + \cdots + (\theta_{i-1} - \theta_i)\}$$

and for any differential forms α, β, γ and any integer $0 \le k, l, x \le p$, the equation $\alpha \wedge (dt_i \wedge (\theta_x - \theta_{x+1})_{\tau(2k-1)\tau(2k)}) \wedge \beta \wedge (dt_j \wedge (\theta_x - \theta_{x+1})_{\tau(2l-1)\tau(2l)}) \wedge \gamma = -\alpha \wedge (dt_j \wedge (\theta_x - \theta_{x+1})_{\tau(2k-1)\tau(2k)}) \wedge \beta \wedge (dt_i \wedge (\theta_x - \theta_{x+1})_{\tau(2l-1)\tau(2l)}) \wedge \gamma$ holds, so the terms of these forms cancel with each other in $\mathrm{Pf}(\Omega|_{\Delta^p \times N\bar{G}(p)})$.

We set:

$$\varphi_s := h_1 \cdots h_{s-1} dh_s h_s^{-1} \cdots h_1^{-1}.$$

Then we can check that $\gamma^* \varphi_s = g_1(\theta_{s-1} - \theta_s) g_1^{-1}$ hence we obtain the following proposition.

Proposition 2.2. *The cochain μ_p in $\Omega^p(NG(p))$ which corresponds to the Euler class is given as follows:*

$$\mu_p = \frac{(-1)^{\frac{p(p+1)}{2}}}{2^{2p}\pi^p (p!)^2} \sum_{\sigma \in \mathfrak{S}_p} \sum_{\tau \in \mathfrak{S}_{2p}} \mathrm{sgn}(\tau)\mathrm{sgn}(\sigma)(\varphi_{\sigma(1)})_{\tau(1)\tau(2)} \cdots (\varphi_{\sigma(p)})_{\tau(2p-1)\tau(2p)}.$$

Using Proposition 3.1 and Proposition 3.2, we obtain the cocycle which represents the Euler class of $ESO(4) \to BSO(4)$ in $\Omega^4(NSO(4))$.

Theorem 2.2. *In the case of $G = SO(4)$, the cocycle which represents the Euler class of $ESO(4) \to BSO(4)$ in $\Omega^4(NG)$ is the sum of the following $E_{1,3}$ and $E_{2,2}$:*

$$\begin{array}{ccc}
0 & & \\
\uparrow{\scriptstyle d''} & & \\
E_{1,3} \in \Omega^3(SO(4)) & \xrightarrow{d'} & \Omega^3(SO(4) \times SO(4)) \\
 & & \uparrow{\scriptstyle d''} \\
 & & E_{2,2} \in \Omega^2(SO(4) \times SO(4)) \xrightarrow{d'} 0
\end{array}$$

$$E_{1,3} = \frac{1}{192\pi^2} \sum_{\tau \in \mathfrak{S}_4} \mathrm{sgn}(\tau)\big((h^{-1}dh)_{\tau(1)\tau(2)} (h^{-1}dh)^2_{\tau(3)\tau(4)} + (h^{-1}dh)^2_{\tau(1)\tau(2)} (h^{-1}dh)_{\tau(3)\tau(4)}\big)$$

$$E_{2,2} = \frac{-1}{64\pi^2} \sum_{\tau \in \mathfrak{S}_4} \mathrm{sgn}(\tau)\big((h_1^{-1}dh_1)_{\tau(1)\tau(2)} (dh_2 h_2^{-1})_{\tau(3)\tau(4)} + (dh_2 h_2^{-1})_{\tau(1)\tau(2)} (h_1^{-1}dh_1)_{\tau(3)\tau(4)}\big).$$

2.3. The cocycle in $\Omega^{p+q}(NG(p-q))$

Repeating the same argument in section 3.2, we obtain a cocycle in $\Omega^{p+q}(NG(p-q))$.
We set:

$$R_{ij} := (\varphi_i + \varphi_{i+1} + \cdots + \varphi_{j-1})^2 \qquad (1 \le i < j \le p-q+1).$$

Theorem 2.3. *The cocycle in $\Omega^{p+q}(NG(p-q))$ $(0 \le q \le p-1)$ which represents the Euler class of $ESO(2p) \to BSO(2p)$ is*

$$\sum_{\sigma \in \mathfrak{S}_{p-q}, \tau \in \mathfrak{S}_{2p}} \sum (T^{\tau,\sigma}_{p,q}(R_{i_1 j_1})_{\tau(1)\tau(2)}(\varphi_{\sigma(1)})_{\tau(3)\tau(4)} \cdots (R_{i_q j_q})_{\tau(2p-3)\tau(2p-2)}(\varphi_{\sigma(p-q)})_{\tau(2p-1)\tau(2p)})$$

where R_{ij} $(1 \le i < j \le p-q+1)$ are put q-times between $\varphi_{\sigma(l)}$ and $\varphi_{\sigma(l+1)}$ or the edge in $\varphi_{\sigma(1)} \cdots \varphi_{\sigma(p-q)}$ permitting overlaps and $\sum$ means the sum of all such forms. $T^{\tau,\sigma}_{p,q}$ is defined as:

$$T^{\tau,\sigma}_{p,q} = \mathrm{sgn}(\tau)\mathrm{sgn}(\sigma)\frac{(-1)^{p+\frac{(p-q)(p-q-1)}{2}}}{2^{2p}\pi^p p!}\left(\int_{\Delta^{p-q}} \prod_{i<j}(t_{i-1}t_{j-1})^{r_{ij}} dt_1 \wedge \cdots \wedge dt_{p-q}\right)$$

where r_{ij} means the number of R_{ij} in each form.

Theorem 2.4. *In the case of $G = SO(6)$, the cocycle which represents the Euler class in $\Omega^6(NG)$ is the sum of the following $E_{1,5}, E_{2,4}$ and $E_{3,3}$:*

$$\begin{array}{ccccccc}
0 & & & & & & \\
\uparrow{\scriptstyle d''} & & & & & & \\
E_{1,5} \in \Omega^5(G) & \xrightarrow{d'} & \Omega^5(NG(2)) & & & & \\
 & & \uparrow{\scriptstyle d''} & & & & \\
 & & E_{2,4} \in \Omega^4(NG(2)) & \xrightarrow{d'} & \Omega^4(NG(3)) & & \\
 & & & & \uparrow{\scriptstyle d''} & & \\
 & & & & E_{3,3} \in \Omega^3(NG(3)) & \xrightarrow{d'} & 0
\end{array}$$

$$E_{1,5} = \frac{-1}{2^6 \cdot 180\pi^3} \sum_{\tau \in \mathfrak{S}_6} \mathrm{sgn}(\tau)\big((h^{-1}dh)^2_{\tau(1)\tau(2)}(h^{-1}dh)_{\tau(3)\tau(4)}(h^{-1}dh)_{\tau(5)\tau(6)} + (h^{-1}dh)_{\tau(1)\tau(2)}(h^{-1}dh)^2_{\tau(3)\tau(4)}(h^{-1}dh)_{\tau(5)\tau(6)} + (h^{-1}dh)_{\tau(1)\tau(2)}(h^{-1}dh)_{\tau(3)\tau(4)}(h^{-1}dh)^2_{\tau(5)\tau(6)}\big)$$

$$E_{2,4} = \frac{1}{2^6 \cdot 6 \cdot 4!\pi^3} \sum_{\tau \in \mathfrak{S}_6} \mathrm{sgn}(\tau) \cdot$$

$$\Big((h_1^{-1}dh_1)_{\tau(1)\tau(2)}(dh_2h_2^{-1})_{\tau(3)\tau(4)} \cdot \Big(2h_1^{-1}dh_1h_1^{-1}dh_1 + 2dh_2h_2^{-1}dh_2h_2^{-1} + h_1^{-1}dh_1dh_2h_2^{-1} + dh_2h_2^{-1}h_1^{-1}dh_1\Big)_{\tau(5)\tau(6)}$$

$$+(h_1^{-1}dh_1)_{\tau(1)\tau(2)}\Big(2h_1^{-1}dh_1h_1^{-1}dh_1 + 2dh_2h_2^{-1}dh_2h_2^{-1} + h_1^{-1}dh_1dh_2h_2^{-1} + dh_2h_2^{-1}h_1^{-1}dh_1\Big)_{\tau(3)\tau(4)}(dh_2h_2^{-1})_{\tau(5)\tau(6)}$$

$$+\Big(2h_1^{-1}dh_1h_1^{-1}dh_1 + 2dh_2h_2^{-1}dh_2h_2^{-1} + h_1^{-1}dh_1dh_2h_2^{-1} + dh_2h_2^{-1}h_1^{-1}dh_1\Big)_{\tau(1)\tau(2)} \cdot (h_1^{-1}dh_1)_{\tau(3)\tau(4)}(dh_2h_2^{-1})_{\tau(5)\tau(6)}$$

$$-(dh_2h_2^{-1})_{\tau(1)\tau(2)}(h_1^{-1}dh_1)_{\tau(3)\tau(4)}\cdot$$
$$\Big(2h_1^{-1}dh_1h_1^{-1}dh_1+2dh_2h_2^{-1}dh_2h_2^{-1}+h_1^{-1}dh_1dh_2h_2^{-1}+dh_2h_2^{-1}h_1^{-1}dh_1\Big)_{\tau(5)\tau(6)}$$
$$-(dh_2h_2^{-1})_{\tau(1)\tau(2)}\Big(2h_1^{-1}dh_1h_1^{-1}dh_1+2dh_2h_2^{-1}dh_2h_2^{-1}$$
$$+h_1^{-1}dh_1dh_2h_2^{-1}+dh_2h_2^{-1}h_1^{-1}dh_1\Big)_{\tau(3)\tau(4)}(h_1^{-1}dh_1)_{\tau(5)\tau(6)}$$
$$-\Big(2h_1^{-1}dh_1h_1^{-1}dh_1+2dh_2h_2^{-1}dh_2h_2^{-1}+h_1^{-1}dh_1dh_2h_2^{-1}+dh_2h_2^{-1}h_1^{-1}dh_1\Big)_{\tau(1)\tau(2)}$$
$$\cdot(dh_2h_2^{-1})_{\tau(3)\tau(4)}(h_1^{-1}dh_1)_{\tau(5)\tau(6)}\Big).$$

$$E_{3,3}=\frac{1}{2^6\cdot 6^2\pi^3}\sum_{\tau\in\mathfrak{S}_6}\mathrm{sgn}(\tau)\cdot$$
$$\Big((h_1^{-1}dh_1)_{\tau(1)\tau(2)}(dh_2h_2^{-1})_{\tau(3)\tau(4)}(h_2dh_3h_3^{-1}h_2^{-1})_{\tau(5)\tau(6)}$$
$$-(dh_2h_2^{-1})_{\tau(1)\tau(2)}(h_1^{-1}dh_1)_{\tau(3)\tau(4)}(h_2dh_3h_3^{-1}h_2^{-1})_{\tau(5)\tau(6)}$$
$$-(h_1^{-1}dh_1)_{\tau(1)\tau(2)}(h_2dh_3h_3^{-1}h_2^{-1})_{\tau(3)\tau(4)}(dh_2h_2^{-1})_{\tau(5)\tau(6)}$$
$$+(h_2dh_3h_3^{-1}h_2^{-1})_{\tau(1)\tau(2)}(h_1^{-1}dh_1)_{\tau(3)\tau(4)}(dh_2h_2^{-1})_{\tau(5)\tau(6)}$$
$$+(dh_2h_2^{-1})_{\tau(1)\tau(2)}(h_2dh_3h_3^{-1}h_2^{-1})_{\tau(3)\tau(4)}(h_1^{-1}dh_1)_{\tau(5)\tau(6)}$$
$$-(h_2dh_3h_3^{-1}h_2^{-1})_{\tau(1)\tau(2)}(dh_2h_2^{-1})_{\tau(3)\tau(4)}(h_1^{-1}dh_1)_{\tau(5)\tau(6)}\Big).$$

3. The cocycle in a local truncated complex

We recall the filtered local simplicial de Rham complex due to Brylinski [3].

Definition 3.1 ([3]). The filtered local simplicial de Rham complex $F^p\Omega^{*,*}_{\mathrm{loc}}(NG)$ over a simplicial manifold NG is defined as follows:

$$F^p\Omega^{r,s}_{\mathrm{loc}}(NG)=\begin{cases}\varinjlim_{1\in V\subset G^r}\ \Omega^s(V) & \text{if } s\geq p\\ 0 & \text{otherwise.}\end{cases}$$

Let $F^p\Omega^*(NG)$ be a filtered complex

$$F^p\Omega^{r,s}(NG)=\begin{cases}\Omega^s(NG(r)) & \text{if } s\geq p\\ 0 & \text{otherwise}\end{cases}$$

and $[\sigma_{<p}\Omega^*(NG)]$ a truncated complex

$$[\sigma_{<p}\Omega^{r,s}(NG)]=\begin{cases}0 & \text{if } s\geq p\\ \Omega^s(NG(r)) & \text{otherwise.}\end{cases}$$

Then there is an exact sequence:

$$0\to F^p\Omega^*(NG)\to\Omega^*(NG)\to[\sigma_{<p}\Omega^*(NG)]\to 0$$

which induces a boundary map $\beta: H^l(NG,[\sigma_{<p}\Omega^*_{\mathrm{loc}}])\to H^{l+1}(NG,[F^p\Omega^*_{\mathrm{loc}}])$.

Let $\mu_1+\cdots+\mu_p$, $\mu_{p-q}\in\Omega^{p+q}(NG(p-q))$ be a cocycle in $\Omega^{2p}(NG)$. Using this cocycle, we can construct a cocycle η in $[\sigma_{<p}\Omega^*_{\mathrm{loc}}(NG)]$ in the following way.

We take a contractible open set $U \subset G$ containing 1. Using the same argument in [5], we can construct mappings $\{\sigma_l : \Delta^l \times U^l \to U\}_{0\leq l}$ inductively with the following properties:
(1) $\sigma_0(pt) = 1$;
(2)

$$\sigma_l(\varepsilon^j(t_0,\cdots,t_{l-1});h_1,\cdots,h_l) = \begin{cases}\sigma_{l-1}(t_0,\cdots,t_{l-1};\varepsilon_j(h_1,\cdots,h_l)) & \text{if } j\geq 1\\ h_1\cdot\sigma_{l-1}(t_0,\cdots,t_{l-1};h_2,\cdots,h_l) & \text{if } j=0.\end{cases}$$

We define mappings $\{f_{m,q} : \Delta^q \times U^{m+q-1} \to G^m\}$ as

$$f_{m,q}(t_0,\cdots,t_q;h_1,\cdots,h_{m+q-1}) := (h_1,\cdots,h_{m-1},\sigma_q(t_0,\cdots,t_q;h_m,\cdots,h_{m+q-1})).$$

A $(2p-m-q)$-form $\beta_{m,q}$ on U^{m+q-1} is defined as $\beta_{m,q} = (-1)^m \int_{\Delta^q} f^*_{m,q}\mu_m$. Then we define the cochain η as the sum of following η_l on U^{2p-1-l} for $0 \leq l \leq p-1$:

$$\eta_l := \sum_{m+q=2p-l,\ p\geq m\geq 1} \beta_{m,q}.$$

Theorem 3.1 ([3][8]). *$\eta := \eta_0 + \cdots + \eta_{p-1}$ is a cocycle in $[\sigma_{<p}\Omega^*_{\text{loc}}(NG)]$ whose cohomology class is mapped to $[\mu_1 + \cdots + \mu_p]$ in $H^{2p}(NG,[F^p\Omega^*_{\text{loc}}])$ by a boundary map $\beta : H^{2p-1}(NG,[\sigma_{<p}\Omega^*_{\text{loc}}]) \to H^{2p}(NG,[F^p\Omega^*_{\text{loc}}])$.*

Proof. See [8]. □

4. Construction of a Lie algebra cocycle

For any Lie group G, let $C^\infty_{loc}(G^p,\mathbb{R})$ denote the group of germs at $(1,\cdots,1)$ of smooth functions $G^p \to \mathbb{R}$ and $H^p_{loc}(G,\mathbb{R})$ denote the cohomology group of the following complex:

$$\cdots \to C^\infty_{loc}(G^p,\mathbb{R}) \xrightarrow{\delta:=\sum_{i=0}^{p+1}(-1)^i\varepsilon_i^*} C^\infty_{loc}(G^{p+1},\mathbb{R}) \to \cdots$$

Brylinski constructed a natural cochain map $\phi : C^p_{loc}(G,\mathbb{R}) \to C^p(\mathcal{G},\mathbb{R})$ as follows:

$$\phi(c)(\xi_1,\cdots,\xi_p) :=$$

$$[\frac{\partial^p}{\partial y_1\cdots\partial y_p}\sum_{\rho\in\mathfrak{S}_p}\text{sgn}(\rho)c(\exp(y_{\rho(1)}\xi_{\rho(1)}),\cdots,\exp(y_{\rho(p)}\xi_{\rho(p)}))]_{y_i=0}$$

where $C^p(\mathcal{G},\mathbb{R})$ is the space of smooth alternating multilinear maps $\mathcal{G} \to \mathbb{R}$ and $\xi_i \in \mathcal{G}$. For example, if we take $\delta c \in C^\infty_{loc}(G^2,\mathbb{R})$ and set $X_{\rho(i)} := \exp(y_{\rho(i)}\xi_{\rho(i)})$ then

$$\phi(\delta c)(\xi_1,\xi_2) = [\frac{\partial^2}{\partial y_1\partial y_2}\sum_{\rho\in\mathfrak{S}_2}\text{sgn}(\rho)(\delta c)(X_{\rho(1)},X_{\rho(2)})]_{y_i=0}$$

$$= [\frac{\partial^2}{\partial y_1\partial y_2}\sum_{\rho\in\mathfrak{S}_2}\text{sgn}(\rho)(c(X_{\rho(2)}) - c(X_{\rho(1)}X_{\rho(2)}) + c(X_{\rho(1)}))]_{y_i=0}$$

$$= [\frac{\partial^2}{\partial y_1\partial y_2}(-c(X_1X_2 - X_2X_1))]_{y_i=0} = (d(\phi(c)))(\xi_1,\xi_2).$$

Let LU be the free loop space of a contractible open set $U \subset SO(4)$ containing 1 and $\text{ev} : LU \times S^1 \to U$ be the evaluation map, i.e. for $\gamma \in LU$ and $\theta \in S^1$, $\text{ev}(\gamma,\theta)$ is defined as $\gamma(\theta)$. Then $\int_{S^1}\text{ev}^*$ maps $\eta_1 \in \Omega^1(U^2)$ to a cochain in $\Omega^0(LU^2)$. This cochain defines a cohomology class in local cohomology group $H^2_{\text{loc}}(LSO(4),\mathbb{R})$. So as an application of Theorem 3.2, we can obtain a cocycle in $\phi(\int_{S^1}\text{ev}^*\eta_1) \in C^2(L\mathfrak{so}(4),\mathbb{R})$.
Now we compute this cocycle. We define:

$$a := \int_{S^1}\text{ev}^*\int_{\Delta^2} f^*_{1,2}E_{1,3}, \quad b := \int_{S^1}\text{ev}^*\int_{\Delta^1} f^*_{2,1}E_{2,2}, \quad c := \int_{S^1}\text{ev}^*\eta_1$$

then $c(\gamma_1,\gamma_2) = a(\gamma_1,\gamma_2) + b(\gamma_1,\gamma_2)$ for $\gamma_1,\gamma_2 \in LU$. Recall that

$$f_{1,2}(t_0,t_1,t_2;\gamma_1(\theta),\gamma_2(\theta)) = \sigma_2(t_0,t_1,t_2;\gamma_1(\theta),\gamma_2(\theta))$$

$$f_{2,1}(t_0,t_1,t_2;\gamma_1(\theta),\gamma_2(\theta)) = (\gamma_1(\theta),\sigma_1(t_0,t_1;\gamma_2(\theta))).$$

In this case we can take:

$$\gamma_i(\theta) = \exp(y_i\xi_i(\theta))$$

$$\sigma_1(t_0,t_1;\exp(y_2\xi_2(\theta))) := \exp(t_1y_2\xi_2(\theta))$$

$$\sigma_2(t_0,t_1,t_2;\exp(y_1\xi_1(\theta)),\exp(y_2\xi_2(\theta))) := \exp((1-t_0)y_1\xi_1(\theta))\exp(t_2y_2\xi_2(\theta))$$

where $\xi_i \in L\mathfrak{so}(4)$. By observing the coefficient of y_1y_2, we see $\phi(a(\gamma_1,\gamma_2)) = 0$.

We define a map $\beta_{\gamma_1,\gamma_2} : S^1 \times \Delta^1 \to SO(4) \times SO(4)$ as follows:

$$\beta_{\gamma_1,\gamma_2}(\theta;t_0,t_1) := (\gamma_1(\theta),\sigma_1(t_0,t_1;\gamma_2(\theta))).$$

Then $b(\gamma_1,\gamma_2) = \int_{S^1\times\Delta^1}\beta^*_{\gamma_1,\gamma_2}E_{2,2}$ and up to $O(|y_1|^2)$ and $O(|y_2|^2)$,

$$\frac{\partial\beta_{\gamma_1,\gamma_2}}{\partial\theta} = \left(y_1\frac{\partial\xi_1(\theta)}{\partial\theta}, t_1y_2\frac{\partial\xi_2(\theta)}{\partial\theta}\right), \quad \frac{\partial\beta_{\gamma_1,\gamma_2}}{\partial t_1} = (0, y_2\xi_2(\theta))\,.$$

Therefore

$$[\frac{\partial^2}{\partial y_1\partial y_2}b(\gamma_1,\gamma_2)]_{y_i=0} = \frac{-1}{128\pi^2}\sum_{\tau\in\mathfrak{S}_4}\mathrm{sgn}(\tau)\int_0^1\left(\frac{\partial\xi_1(\theta)}{\partial\theta}\right)_{\tau(1)\tau(2)}\xi_2(\theta)_{\tau(3)\tau(4)}d\theta.$$

Now we obtain the following theorem.

Theorem 4.1. *There exists a Lie algebra 2-cocycle α on $L\mathfrak{so}(4)$ which is expressed as follows:*

$$\alpha(\xi_1,\xi_2) := \frac{-1}{128\pi^2}\sum_{\tau\in\mathfrak{S}_4}\Bigg(\mathrm{sgn}(\tau)\cdot$$

$$\int_0^1\left(\left(\frac{\partial\xi_1(\theta)}{\partial\theta}\right)_{\tau(1)\tau(2)}\xi_2(\theta)_{\tau(3)\tau(4)} - \left(\frac{\partial\xi_2(\theta)}{\partial\theta}\right)_{\tau(1)\tau(2)}\xi_1(\theta)_{\tau(3)\tau(4)}\right)d\theta\Bigg).$$

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