On The Inverse Sum In Degree Index and Co Index

Gülistan KAYA GÖK

1 Hakkari University, Department of Mathematics Education, Hakkari, TURKEY

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Abstract. The inverse sum in degree index of G is specified the degrees $d_i$ and $d_j$. Some bounds are found for inverse sum in degree index in this study. Also, some definitions and relations are obtained in terms of degrees.

Keywords: Inverse sum in degree index, co index.

Derece Endeksinde ve Ko Endeksinde Ters Toplam


Anahtar Kelimeler: Derece endeksinde ters toplam, Eş endeks.

1. INTRODUCTION

Let G be a simple, connected graph on the vertex set $V(G)$ and the edge set $E(G)$. For $v_i \in V(G)$, the degree of vertex $v_i$ denoted by $d_i$, the maximum degree is denoted by $\Delta$ and the minimum degree is denoted by $\delta$.

The inverse sum in degree matrix $[ISI](G)$ of graphs is defined as

$$ [ISI]_{ij} = \begin{cases} d_i + d_j & \text{if } i \text{ adjacent to } j \\ \frac{d_i d_j}{d_i + d_j} & \text{otherwise} \end{cases} $$

The eigenvalues of $[ISI](G)$ are denoted by $\delta^+$. New bounds for these eigenvalues are reported in terms of the degrees.

The Inverse Sum In Degree Index ($ISI$) index of G is defined as

$$ \sum_{v_i, v_j \in E(G)} \frac{d_i + d_j}{d_i d_j} $$

(See [2] for details.)

In this study, different bounds are set using the Estrada index and Zagreb index for ISI index.
The Estrada index of graph $G$ is explained as

$$EE(G) = \sum_{i=1}^{n} e^{\lambda_i}$$

where $\lambda$ is the eigenvalue of adjacency matrix of $G$. ([1], [10])

The Zagreb co index of $G$ is described in [5], [7] as

$$\overline{Z}_1(G) = \sum_{v_i,v_j \notin E(G)} (d_G(i) + d_G(j)).$$
$$\overline{Z}_2(G) = \sum_{v_i,v_j \notin E(G)} (d_G(i)d_G(j)).$$

The Harmonic index of $G$ is specified in [8] as

$$H(G) = \sum_{v_i,v_j \in E(G)} \frac{2}{d_i + d_j}.$$

Considering these topological indices, Estrada inverse sum in degree index and inverse sum in degree co index are defined. Indeed, some inequalities are obtained concerned with these indices. (See [6] for more details deal with this topic.)

2. PRELIMINARIES

In this section, some lemmas and theorems that are needed in main results will be given.

**Lemma 2.1.** [9]

Let $M = (m_{ij})$ be an $n \times n$ irreducible nonnegative matrix and $\lambda_1(M)$ be the greatest eigenvalue with $R_i(M) = \sum_{j=1}^{m} m_{ij}$. Then,

$$(\min R_i(M): 1 \leq i \leq n) \leq \lambda_1(M) \leq (\max R_i(M): 1 \leq i \leq n)$$

**Lemma 2.2.** [4]

If $G$ is a simple connected graph and $\lambda_1(G)$ is the spectral radius , then

$$\lambda_1(G) \leq \max(\sqrt{m_i m_j}: 1 \leq i,j \leq n, v_i, v_j \in E)$$
Theorem 2.1. [3]

If \( a_i, b_i \in R^+ \), \( 1 \leq i \leq n \), then

\[
\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left( \sum_{i=1}^{n} a_i b_i \right)^2 \leq \frac{n^2}{4} (M_1 M_2 - m_1 m_2)^2
\]

where \( M_1 = \max_{1 \leq i \leq n} a_i, M_2 = \max_{1 \leq i \leq n} b_i, m_1 = \min_{1 \leq i \leq n} a_i, m_2 = \min_{1 \leq i \leq n} b_i. \)

Theorem 2.2. [5]

Let \( G \) be a graph with \( n \) vertices and \( m \) edges. Then,

\[
Z_1(\overline{G}) = Z_1(G) + n(n-1)^2 - 4m(n-1)
\]

\[
\overline{Z}_1(G) = 2m(n-1) - Z_1(G) = \overline{Z}_1(\overline{G}).
\]

Lemma 2.3. [7]

If \( G \) is a regular graph, then

\[
Z_1(G) \geq 4m^2/n,
\]

\[
Z_2(G) \geq 4m^3/n^2,
\]

\[
\overline{Z}_1(G) \leq -4m^2/n + 2m(n-1),
\]

\[
\overline{Z}_2(G) \leq 2m^2\left(1 - \frac{2m}{n^2} - \frac{1}{n}\right).
\]

3. MAIN RESULTS

3.1. Inverse Sum In Degree Index and Estrada Inverse Sum In Degree Index

Firstly, a relation is given for the largest eigenvalue of ISI matrix including degrees in this subsection. After, an inequality is obtained for ISI index using this relation. In addition, Estrada inverse sum in degree index is defined and some relations are found in terms of degrees and vertices.

Theorem 3.1.1. Let \( G \) be graph on \( n \) vertices and \( m \) edges. Then,

\[
ISI(G) \geq \sqrt{(Z_2(G) H(G))^2 - \frac{n^2}{4} \left(\Delta^2 - \delta^2\right)^2}.
\]

Proof. Let choose \( a_k = d_i d_j, b_k = \frac{1}{d_i + d_j}, M_1 = \Delta^2, m_1 = \delta^2, M_2 = \frac{1}{2\delta}, m_2 = \frac{1}{2\Delta}. \)

By Theorem 2.1, it is seen that
\[
\sum_{v_i \in V \cap E(G)} (d_i d_j)^2 \sum_{v_j \in V \cap E(G)} \left( \frac{1}{d_i + d_j} \right)^2 - \left( \sum_{v_i \in V \cap E(G)} \frac{d_i d_j}{d_i + d_j} \right)^2 \leq \frac{n^2}{4} \left( \frac{\Delta^2}{2\delta} - \frac{\delta^2}{2\Delta} \right).
\]

If necessary organizing is applied, this inequality is obtained as follows:

\[
\left( \sum_{v_i \in V \cap E(G)} (d_i d_j) \right)^2 \left( \sum_{v_j \in V \cap E(G)} \left( \frac{1}{d_i + d_j} \right)^2 - \frac{n^2}{4} \left( \frac{\Delta^2}{2\delta} - \frac{\delta^2}{2\Delta} \right) \right) \leq \left( \sum_{v_i \in V \cap E(G)} \frac{d_i d_j}{d_i + d_j} \right)^2.
\]

Putting the definitions in the above inequality, it gets

\[
(Z_2(G))^2 (H(G))^2 - \frac{n^2}{4} \left( \frac{\Delta^3 - \delta^3}{2\Delta \delta} \right) \leq (ISI(G))^2.
\]

Hence,

\[
ISI(G) \geq \sqrt{(Z_2(G) H(G))^2 - \frac{n^2}{4} \left( \frac{\Delta^3 - \delta^3}{2\Delta \delta} \right)}.
\]

**Lemma 3.1.1.** For a simple connected graph of ISI(G),

\[
\gamma_1^+ \leq \frac{\Delta}{n^{1/n} (d_1^n + \Delta)(d_n^n + \Delta)}
\]

where \( \Delta \) is the maximum degree of \( G \).

**Proof.** Let \( D(G)^{-1} ISI(G) D(G) = Q(G) \) and \( X = (x_1, x_2, ..., x_n)^T \) be an eigen vector of \( Q(G) \) corresponding to an eigen value \( \gamma^+ \). Also, \( x_i = 1 \) and \( 0 < x_k \leq 1 \) for every \( k \). Let \( x_j = \max_k (x_k : v_i v_k \in E \text{ is adjacent to } k) \). It is known that \( (Q(G)) X = \gamma_1^+ (G) X \). If \( i \_th \) equation from above equation is taken, then \( \gamma_1^+ (G) x_i = \sum_k \left( \frac{d_i d_k}{d_i + d_k} \right) x_k = \left( d_i \sum_k \frac{d_k}{d_i + d_k} \right) x_k \). By the Aritmetic-Geometric mean inequality, it gives...
Using the Lemma 2.1,

\[ \gamma_1^+(G) \leq \frac{d_i \Delta}{n^{1/n}(d_i^n + \Delta)} \]

The \( j \)-th equation of same equation has

\[ \gamma_j^+(G) \leq \frac{d_j \Delta}{n^{1/n}(d_j^n + \Delta)} \]

From Lemma 2.2, it is expressed that

\[ \gamma_1^+(G) \leq \sqrt{\left( \frac{d_i \Delta}{n^{1/n}(d_i^n + \Delta)} \right) \left( \frac{d_j \Delta}{n^{1/n}(d_j^n + \Delta)} \right)} \]

Hence,

\[ \gamma_1^+ \leq \frac{\Delta}{n^{1/n} \sqrt{(d_i^n + \Delta)(d_j^n + \Delta)}} \]

Since, \( \Delta = d_1 \geq d_2 \geq \ldots \geq d_n = \delta \), it is clear that

\[ \gamma_1^+ \leq \frac{\Delta}{n^{1/n} \sqrt{(\Delta^n + \Delta)(\delta^n + \Delta)}} \]

**Definition 3.1.1.** Let \( G \) be a graph and \( \gamma_1^+ \geq \gamma_2^+ \geq \ldots \geq \gamma_n^+ \) be eigenvalues of inverse sum in degree matrix of \( G \). Estrada inverse sum in degree index is defined as

\[ E_{ist} = \sum_{j=1}^{n} e^{\gamma_j^+}. \]

**Theorem 3.1.2.** Let \( G \) be a graph with \( n \) vertices and \( E_{ist} \) be an Estrada inverse sum in deg index. Then,

\[ E_{ist} \geq e^K + \frac{(n - 1)}{e^{1/n - 1}} \]
where \( K = \frac{\Delta}{n^{1/n} \sqrt{(\Delta n + \Delta)(\Delta n + \Delta)}} \).

**Proof.** \( E_{\text{isti}} = \sum_{j=1}^{n} e^{\gamma_j^+} \geq e^{\gamma_1^+} + (n - 1) \left( \prod_{j=2}^{n} e^{\gamma_j^+} \right)^{1/n-1} \) using the Aritmetric-Geometric mean inequality. Since, \( \sum_{i=1}^{n} e^{\gamma_j^+} = 0 \) then \( E_{\text{isti}} \geq e^{\gamma_1^+} + \frac{(n-1)}{e^{1/n-1}}. \) It is known that \( \gamma_j^+ \leq K. \) Hence,

\[
E_{\text{isti}} \geq e^K + \frac{(n-1)}{e^{1/n-1}}.
\]

**Theorem 3.1.3.** Let \( G \) be a graph with \( n \) vertices and \( E_{\text{isti}} \) be an Estrada inverse sum in deg. index. Then,

\[
E_{\text{isti}} \leq \sqrt{-Kn \sum_{k=0}^{\infty} \frac{2^k}{k!}}
\]

**Proof.** It is easy to see that \( \frac{1}{n} \sum_{j=1}^{n} (e^{\gamma_j^+})^2 \geq \left( \frac{1}{n} \sum_{j=1}^{n} e^{\gamma_j^+} \right)^2. \) On the other hand, \( \frac{1}{n} \sum_{j=1}^{n} e^{2\gamma_j^+} \geq \frac{1}{n^2} E_{\text{isti}}^2. \) Hence,

\[
n \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j=1}^{n} (2\gamma_j^+)^k \geq E_{\text{isti}}^2
\]

and thus,

\[
E_{\text{isti}}^2 \leq n \sum_{k=0}^{\infty} \frac{2^k}{k!} \sum_{j=1}^{n} (\gamma_j^+)^k.
\]

Knowing that \( \gamma_1^+ \geq \cdots \geq \gamma_n^+ \) and \( \gamma_1^+ \leq K, \) it is obtained that

\[
E_{\text{isti}}^2 \leq n \sum_{k=0}^{\infty} \frac{2^k}{k!} n. K^k.
\]

It is clear that the equality holds

\[
E_{\text{isti}} \leq \sqrt{n^2 \sum_{k=0}^{\infty} \frac{(2K)^k}{k!}}.
\]

**Theorem 3.1.4.** Let \( G \) be a graph with \( n \) vertices and \( E_{\text{isti}} \) be an Estrada inverse sum in degree index. Then,

\[
E_{\text{isti}} \leq \sqrt{e^{2K} - 2e^K + e^K}.
\]
Proof. \((E_{ist} - e^{\gamma_1})^2 = \left(\sum_{j=1}^{n} e^{\gamma_j^+}\right)^2 - 2\left(\sum_{j=1}^{n} e^{\gamma_j^+} e^{\gamma_1^+}\right) + e^{2\gamma_1^+}\)
\[
\leq \left(\sum_{j=1}^{n} e^{\gamma_j^+}\right)^2 - 2ne^{\gamma_1^+} \left(\prod_{j=1}^{n} e^{\gamma_j^+}\right)^{1/n} + e^{2\gamma_1^+}.
\]

Since \(\left(\sum_{j=1}^{n} e^{\gamma_j^+}\right)^2 = \left(\sum_{k \geq 0}^{\infty} \frac{1}{k!} \sum_{j=1}^{n} (\gamma_j^+)^k\right)^2 \leq \left(\sum_{k \geq 0}^{\infty} \frac{1}{k!} \left(\sum_{j=1}^{n} \gamma_j^+\right)^k\right)^2\) and \(\sum_{j=1}^{n} \gamma_j^+ = 0\), then
\[
(E_{ist} - e^{\gamma_1^+})^2 \leq -2ne^{\gamma_1^+} + e^{2\gamma_1^+} = e^{\gamma_1^+} [e^{\gamma_1^+} - 2n].
\]

The inequality states that
\[
E_{ist} \leq \sqrt{e^{\gamma_1^+} (e^{\gamma_1^+} - 2n)} + e^{\gamma_1^+}.
\]

In the sequel, Theorem 3.1.2 says that
\[
E_{ist} \leq \sqrt{e^K (e^K - 2n)} + e^K.
\]

3.2 Inverse Sum In Degree Co Index

In this subsection, \(ISI\) co index is described and different bounds are yielded concerned with Zagreb co indices, the vertices and the edges.

**Definition 3.2.1.**

Let \(G\) be a simple, connected graph. \(ISI\) co index is defined as follows:
\[
ISI(\bar{G}) = \sum_{v \neq j \in E(G)} \frac{d_G(i)d_G(j)}{d_G(i) + d_G(j)}
\]

**Theorem 3.2.1.** Let \(ISI(\bar{G})\) be the complement of inverse sum in degree index. If \(G\) is regular then
\[
ISI(\bar{G}) \leq \frac{(n-1)^2 \left[\binom{n}{2} - m\right] - (n-1) \left(-\frac{4m^2}{n} + 2m(n-1)\right) + \left(2m^2 \left(1 - \frac{2m}{n^2} - \frac{1}{n}\right)\right)}{2(n-1) \left[\binom{n}{2} - m\right] + \frac{4m^2}{n} - \left(2m(n-1)\right)}.
\]

**Proof.** By the definition of \(ISI(\bar{G})\); \(ISI(\bar{G}) = \sum_{v \neq j \in E(G)} \frac{d_G(i)d_G(j)}{d_G(i) + d_G(j)}\). Since \(d_G(i) = (n-1 - d_i)\) and \(d_G(j) = (n-1 - d_j)\), then
\[ ISI (\overline{G}) = \sum_{v_i, v_j \in E(\overline{G})} \frac{(n - 1 - d_i) \cdot (n - 1 - d_j)}{(n - 1 - d_i) + (n - 1 - d_j)} \]

\[ = \sum_{v_i, v_j \in E(\overline{G})} \frac{(n - 1)^2 - (n - 1)(d_i + d_j) + d_id_j}{2(n - 1) - (d_i + d_j)} \]

\[ \leq \frac{\sum_{v_i, v_j \in E(\overline{G})}(n - 1)^2 - (n - 1)(d_i + d_j) + d_id_j}{\sum_{v_i, v_j \in E(\overline{G})}2(n - 1) - (d_i + d_j)} \]

Because, \( G \) has \( \left( \begin{array}{c} n \\ 2 \end{array} \right)-m \) edges. By Theorem 2.2, it is stated that

\[ ISI (\overline{G}) \leq \frac{(n - 1)^2\left( \begin{array}{c} n \\ 2 \end{array} \right) - m} {2. (n - 1).\left( \begin{array}{c} n \\ 2 \end{array} \right) - m} - (n - 1)Z_1 (\overline{G}) + Z_2 (\overline{G}) \]

Since \( Z_1 (\overline{G}) = Z^{-1}_1 (G) \), then

\[ ISI (\overline{G}) \leq \frac{(n - 1)^2\left( \begin{array}{c} n \\ 2 \end{array} \right) - m} {2. (n - 1).\left( \begin{array}{c} n \\ 2 \end{array} \right) - m} - (n - 1)Z^{-1}_1 (G) + Z^{-1} (G) \]

Using the Lemma 2.3, it is concluded that

\[ ISI (\overline{G}) \leq \frac{(n - 1)^2\left( \begin{array}{c} n \\ 2 \end{array} \right) - m} {2. (n - 1).\left( \begin{array}{c} n \\ 2 \end{array} \right) - m} + \frac{4m^2}{n} - 2m(n - 1) \]

\textbf{Corollary 3.2.1.} Let \( ISI(\overline{G}) \) be the complement of inverse sum in degree index. If \( G \) is regular then,

\[ \overline{ISI} (\overline{G}) \leq \frac{(n - 1)m - (n - 1).\left( \begin{array}{c} n \\ 2 \end{array} \right) - m} {2. (n - 1)m + \frac{4m^2}{n} - (2m(n - 1))} \]

\textbf{Proof.} Applying similar steps to the Theorem 3.2.1, it is obtained that

\[ \overline{ISI} (\overline{G}) = \sum_{v_i, v_j \in E(\overline{G})} \frac{d(\overline{G})(i) \cdot d(\overline{G})(j)}{d(\overline{G})(i) + d(\overline{G})(j)} \]

\[ = \sum_{v_i, v_j \in E(\overline{G})} \frac{(n - 1)^2 - (n - 1)(d_i + d_j) + d_id_j}{2(n - 1) - (d_i + d_j)} \]
\[
\leq \frac{(n-1)^2 m - (n-1)Z^-_1 (G) + Z^-_2 (G)}{2(n-1)m - Z^-_1 (G)}.
\]

By Lemma 2.3, it is resulted that

\[
\overline{ISI} (\overline{G}) \leq \frac{(n-1)^2 m - (n-1)\left(\frac{4m^2}{n} + 2m(n-1)\right) + \left(2m^2 \left(1 - \frac{2m}{n^2} - \frac{1}{n}\right)\right)}{2(n-1)m + \frac{4m^2}{n} - (2m(n-1))}
\]

4. CONCLUSION

In this study, the inverse sum in degree index is expanded, the Estrada inverse sum in degree index is defined and some bounds are found deal with these indices. In the sequel, inverse sum in degree co index is described and some inequalities are obtained in terms of the edges and vertices.

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6. REFERENCES