*Karadeniz Fen Bilimleri Dergisi*, 9(1), 94-102, 2019. DOI: 10.31466/kfbd.542566

**KFBD**

**Karadeniz Fen Bilimleri Dergisi**
The Black Sea Journal of Sciences
ISSN (Online): 2564-7377

Araştırma Makalesi / Research Article

# Generalized $\left(\frac{G'}{G}\right)$ - Expansion Method for Some Soliton Wave Solution of the Coupled Potential Korteweg–de Vries (KdV) equation

İbrahim Enam İNAN

Firat University, Faculty of Education, 23119 Elazig, Turkey

**Corresponding Author:** ieinan@yahoo.com

**Received:** 20.03.2019
**Accepted:** 29.04.2019

**Abstract**

In this article, some soliton wave solutions of the coupled potential KdV equation have been found using the generalized (G '/ G) - expansion method. For this equation, hyperbolic function solutions, trigonometric function solutions and rational function solutions have been obtained. It was seen that the solutions provided the equation using Mathematica 11.2 In addition, the graphic performances of some solutions are given.

**Keywords:** The coupled potential KdV-equation, Soliton wave solutions, Generalized $\left(\frac{G'}{G}\right)$- expansion method.

# Potansiyel KdV- denklem çiftinin bazı soliton dalga çözümleri için Genelleştirilmiş $\left(\frac{G'}{G}\right)$- açılım metodu

**Öz**

Bu makalede, genelleştirilmiş (G '/ G) – açılım metodu kullanılarak potansiyel KdV denklem çiftinin bazı soliton dalga çözümleri bulunmuştur. Bu denklem için hiperbolik fonksiyon çözümleri, trigonometrik fonksiyon çözümleri ve rasyonel fonksiyon çözümleri elde edilmiştir. Çözümlerin Mathematica 11.2 kullanılarak denklemi sağladığı görülmüştür. Ayrıca, bazı çözümlerin grafik performansları verilmiştir.

**Anahtar Kelimeler:** Potansiyel KdV- denklem çifti, Soliton dalga çözümleri, Genelleştirilmiş $\left(\frac{G'}{G}\right)$- açılım metodu.

## 1. Introduction

Nonlinear partial differential equations (NPDEs) have an important place in applied sciences. There are some analytical methods for solving these equations in the literature (Bock and Kruskal, 1979; Malfliet, 1992; Chuntao, 1996; Cariello and Tabor, 1989; Fan, 2000a; Clarkson, 1989). In addition to these methods, there are many methods of solving such equations by using an auxiliary equation. By using these methods, partial differential equations are converted to ordinary differential equations and the solutions of partial differential equations are found with the help of these ordinary differential equations. Some of these methods are given in (Fan, 2000b; Elwakil et al.,2002; Chen and Zhang, 2004; Fu et al.,2001; Shen and Pan, 2003; Chen and Hong-Qing, 2004; Chen et al.,2004; Chen and Yan, 2006; Wang et al., 2008; Guo and Zhou, 2010; Lü et al., 2010; Li et al., 2010; Manafian, 2016; Khater, 2015; Manafian et al., 2017; Yan, 2001).

We used the generalized $\left(\frac{G\prime}{G}\right)$- Expansion Method for finding the some soliton wave solution of the coupled potential KdV-equation. This method is given in the second chapter.

## 2. Analysis of Method

The method will be introduced briefly. Consider a general partial differential equation of two variables,

$$Q(u, u_t, u_x, u_{xx}, \dots) = 0, \tag{1}$$

Using the wave variable $u(x,t) = u(\xi)$, $\xi = x - \mu t$ the Eq.(1) turns into an ordinary differential equation,

$$Q'(u', u'', u''', \dots) = 0 \tag{2}$$

here $\mu$ is constant. With this conversion, we obtain a nonlinear ordinary differential equation for $u(\xi)$. We can express the solution of Eq.(2) as below,

$$u(\xi) = \sum_{k=0}^{m} d_k \Phi(\xi)^k + \sum_{k=1}^{m} e_k \Phi(\xi)^{-k} \tag{3}$$

where $m$ is a positive integer is found as the result of balancing the highest order linear term and the highest order nonlinear term found in the equation, the coefficients $d_k$ and $e_k$ are constants. $\Phi(\xi) = \left(\frac{G\prime}{G}\right)$ satisfies the following ordinary differential equation,

$$k_1 G G'' - k_2 G G' - k_3 (G')^2 - k_4 G^2 = 0. \tag{4}$$

Substituting solution (3) into Eq. (2) yields a set of algebraic equation for $\left(\frac{G\prime}{G}\right), \left(\frac{G\prime}{G}\right)^{-k}$, then, all coefficients of $\left(\frac{G\prime}{G}\right), \left(\frac{G\prime}{G}\right)^{-k}$, have to vanish. Then, $d_k$, $e_k$ ,$k_1, k_2, k_3, k_4$ and $\mu$ constants are found. The special solutions of Eq. (4) are as follows, (Manafian et al., 2017).

1. if $k_2 \neq 0, f = k_1 - k_3$ and $s = k_2^2 + 4k_4(k_1 - k_3) > 0$, then

$$\Phi(\xi) = \frac{k_2}{2f} + \frac{\sqrt{s}}{2f} \frac{C_1 sinh\left(\frac{\sqrt{s}}{2k_1}\xi\right) + C_2 cosh\left(\frac{\sqrt{s}}{2k_1}\xi\right)}{C_1 cosh\left(\frac{\sqrt{s}}{2k_1}\xi\right) + C_2 sinh\left(\frac{\sqrt{s}}{2k_1}\xi\right)}$$

2. if $k_2 \neq 0, f = k_1 - k_3$ and $s = k_2^2 + 4k_4(k_1 - k_3) < 0$, then

$$\Phi(\xi) = \frac{k_2}{2f} + \frac{\sqrt{-s}}{2f} \frac{-C_1 sin\left(\frac{\sqrt{-s}}{2k_1}\xi\right) + C_2 cos\left(\frac{\sqrt{-s}}{2k_1}\xi\right)}{C_1 cos\left(\frac{\sqrt{-s}}{2k_1}\xi\right) + C_2 sin\left(\frac{\sqrt{-s}}{2k_1}\xi\right)}$$

3. if $k_2 \neq 0, f = k_1 - k_3$ and $s = k_2^2 + 4k_4(k_1 - k_3) = 0$ , then

$$\Phi(\xi) = \frac{k_2}{2f} + \frac{C_2}{C_1 + C_2\xi}$$

4. if $k_2 = 0, f = k_1 - k_3$ and $g = fk_4 > 0$ , then

$$\Phi(\xi) = \frac{\sqrt{g}}{f} \frac{C_1 sinh\left(\frac{\sqrt{g}}{k_1}\xi\right) + C_2 cosh\left(\frac{\sqrt{g}}{k_1}\xi\right)}{C_1 cosh\left(\frac{\sqrt{g}}{k_1}\xi\right) + C_2 sinh\left(\frac{\sqrt{g}}{k_1}\xi\right)}$$

5. if $k_2 = 0, f = k_1 - k_3$ and $g = fk_4 < 0$ , then

$$\Phi(\xi) = \frac{\sqrt{-g}}{f} \frac{-C_1 sin\left(\frac{\sqrt{-g}}{k_1}\xi\right) + C_2 cos\left(\frac{\sqrt{-g}}{k_1}\xi\right)}{C_1 cos\left(\frac{\sqrt{-g}}{k_1}\xi\right) + C_2 sin\left(\frac{\sqrt{-g}}{k_1}\xi\right)}$$

6. if $k_4 = 0$ and $f = k_1 - k_3$, then

$$\Phi(\xi) = \frac{C_1 k_2^2 exp\left(\frac{-k_2}{k_1}\xi\right)}{fk_1 + C_1 k_1 k_2 exp\left(\frac{-k_2}{k_1}\xi\right)}$$

7. if $k_2 \neq 0$ and $f = k_1 - k_3 = 0$, then

$$\Phi(\xi) = -\frac{k_4}{k_2} + C_1 exp\left(\frac{k_2}{k_1}\xi\right)$$

8. if $k_1 = k_3,\ k_2 = 0$ and $f = k_1 - k_3 = 0$, then

$$\Phi(\xi) = C_1 + \frac{k_4}{k_1}\xi$$

9. if $k_3 = 2k_1,\ k_2 = 0$ and $k_4 = 0$, then

$$\Phi(\xi) = -\frac{1}{C_1 + \left(\frac{k_3}{k_1} - 1\right)\xi}$$

## 3.Application

The coupled potential KdV equation were considered as follows (Yan, 2001),

$$u_t - u_{xxx} - 3uu_{xx} + 3vu_{xx} - 3u_x^2 - 3u^2u_x + 6uvu_x - 3v^2u_x = 0$$
$$v_t - v_{xxx} + 3uv_{xx} - 3vv_{xx} - 3v_x^2 - 3u^2v_x + 6uvv_x - 3v^2v_x = 0 \quad (5)$$

If $u(x,t) = u(\xi)$, $\xi = x - \mu t$ conversion is used, the (5) equation becomes the following ordinary differential equation,

$$-\mu u' - u''' - 3uu'' + 3vu'' - 3(u')^2 - 3u^2u' + 6uvu' - 3v^2u' = 0$$
$$-\mu v' - v''' + 3uv'' - 3vv'' - 3(v')^2 - 3u^2v' + 6uvv' - 3v^2v' = 0 \quad (6)$$

When balancing $u'''$ with $vu''$ and $uvu'$, $v'''$ with $uv''$ and $uvv'$ then gives $m_1 = 1$ and $m_2 = 1$. The solutions are as follows,

$$u(\xi) = d_0 + d_1\Phi(\xi) + e_1\Phi(\xi)^{-1}$$
$$v(\xi) = f_0 + f_1\Phi(\xi) + g_1\Phi(\xi)^{-1} \quad (7)$$

If Eq. (7) is substituted in Eq. (6), we have a system of algebraic equations for $d_0, d_1, e_1, f_0, f_1, g_1, k_1, k_2, k_3, k_4$ and $\mu$. These algebraic equations system are as follows

$$-\mu e_1 - 3d_0^2e_1 - 3e_1^2 - 3d_1e_1^2 + 6d_0e_1f_0 - 3e_1f_0^2 + 6e_1^2f_1 - 6e_1f_1g_1 + 3d_1g_1^2 + \frac{3d_0e_1k_2}{k_1} - \frac{3f_0e_1k_2}{k_1} - \frac{6d_0e_1f_1k_2}{k_1} + \frac{6f_0e_1f_1k_2}{k_1} + \frac{6d_0d_1g_1k_2}{k_1} - \frac{6f_0d_1g_1k_2}{k_1} - \frac{e_1k_2^2}{k_1^2} + \frac{3e_1f_1k_2^2}{k_1^2} + \frac{3d_1g_1k_2^2}{k_1^2} + \frac{\mu e_1k_3}{k_1} + \frac{3d_0^2e_1k_3}{k_1} + \frac{6e_1^2k_3}{k_1} + \frac{3d_1e_1^2k_3}{k_1} - \frac{6d_0e_1f_0k_3}{k_1} + \frac{3e_1f_0^2k_3}{k_1} - \frac{6e_1^2f_1k_3}{k_1} + \frac{6e_1g_1f_1k_3}{k_1} - \cdots \quad (8)$$

If the system is solved, the coefficients are found as

***Case 1.***

$e_1 = 0, f_0 = d_0, k_4 = 0, k_2 \neq 0, k_1 \neq 0, k_3 \neq 0,\ g_1 = 0, f_1 = d_1, k_3 = \frac{1}{2}(2k_1 - d_1 k_1),$
$\mu = -\frac{k_2^2}{k_1^2}, d_1 \neq 0$

***Solution 1.***

$$u(x,t) = d_0 + \frac{k_2}{k_1} + \frac{\left(Sinh\left[\frac{\sqrt{k_2^2}\left(k_1^2 x + k_2^2 t\right)}{2k_1^3}\right]C_1 + Cosh\left[\frac{\sqrt{k_2^2}\left(k_1^2 x + k_2^2 t\right)}{2k_1^3}\right]C_2\right)\sqrt{k_2^2}}{\left(Cosh\left[\frac{\sqrt{k_2^2}\left(k_1^2 x + k_2^2 t\right)}{2k_1^3}\right]C_1 + Sinh\left[\frac{\sqrt{k_2^2}\left(k_1^2 x + k_2^2 t\right)}{2k_1^3}\right]C_2\right)k_1} \tag{9}$$

$$v(x,t) = d_0 + \frac{k_2}{k_1} + \frac{\left(Sinh\left[\frac{\sqrt{k_2^2}\left(k_1^2 x + k_2^2 t\right)}{2k_1^3}\right]C_1 + Cosh\left[\frac{\sqrt{k_2^2}\left(k_1^2 x + k_2^2 t\right)}{2k_1^3}\right]C_2\right)\sqrt{k_2^2}}{\left(Cosh\left[\frac{\sqrt{k_2^2}\left(k_1^2 x + k_2^2 t\right)}{2k_1^3}\right]C_1 + Sinh\left[\frac{\sqrt{k_2^2}\left(k_1^2 x + k_2^2 t\right)}{2k_1^3}\right]C_2\right)k_1} \tag{10}$$

***Case 2.***

$e_1 = 0, f_0 = d_0, f_1 = d_1, g_1 = 0, k_2 = 0, k_3 = \frac{1}{2}(2k_1 - d_1 k_1), k_1(-k_1 + k_3) \neq 0, \mu =$
$-\frac{2d_1 k_4}{k_1}$

***Solution 2.***

$$u(x,t) = d_0 + \frac{\sqrt{2}\left(-\mathrm{Sin}\left[\frac{\sqrt{-d_1 k_1 k_4}(x k_1 + 2t d_1 k_4)}{\sqrt{2}k_1^2}\right]C_1 + \mathrm{Cos}\left[\frac{\sqrt{-d_1 k_1 k_4}(x k_1 + 2t d_1 k_4)}{\sqrt{2}k_1^2}\right]C_2\right)\sqrt{-d_1 k_1 k_4}}{\left(\mathrm{Cos}\left[\frac{\sqrt{-d_1 k_1 k_4}(x k_1 + 2t d_1 k_4)}{\sqrt{2}k_1^2}\right]C_1 + \mathrm{Sin}\left[\frac{\sqrt{-d_1 k_1 k_4}(x k_1 + 2t d_1 k_4)}{\sqrt{2}k_1^2}\right]C_2\right)k_1} \tag{11}$$

$$v(x,t) = d_0 + \frac{\sqrt{2}\left(-\mathrm{Sin}\left[\frac{\sqrt{-d_1 k_1 k_4}(x k_1 + 2t d_1 k_4)}{\sqrt{2}k_1^2}\right]C_1 + \mathrm{Cos}\left[\frac{\sqrt{-d_1 k_1 k_4}(x k_1 + 2t d_1 k_4)}{\sqrt{2}k_1^2}\right]C_2\right)\sqrt{-d_1 k_1 k_4}}{\left(\mathrm{Cos}\left[\frac{\sqrt{-d_1 k_1 k_4}(x k_1 + 2t d_1 k_4)}{\sqrt{2}k_1^2}\right]C_1 + \mathrm{Sin}\left[\frac{\sqrt{-d_1 k_1 k_4}(x k_1 + 2t d_1 k_4)}{\sqrt{2}k_1^2}\right]C_2\right)k_1} \tag{12}$$

$$u(x,t) = d_0 + \frac{\sqrt{2}\left(\mathrm{Sinh}\left[\frac{\sqrt{d_1 k_1 k_4}(x k_1 + 2t d_1 k_4)}{\sqrt{2}k_1^2}\right]C_1 + \mathrm{Cosh}\left[\frac{\sqrt{d_1 k_1 k_4}(x k_1 + 2t d_1 k_4)}{\sqrt{2}k_1^2}\right]C_2\right)\sqrt{d_1 k_1 k_4}}{\left(\mathrm{Cosh}\left[\frac{\sqrt{d_1 k_1 k_4}(x k_1 + 2t d_1 k_4)}{\sqrt{2}k_1^2}\right]C_1 + \mathrm{Sinh}\left[\frac{\sqrt{d_1 k_1 k_4}(x k_1 + 2t d_1 k_4)}{\sqrt{2}k_1^2}\right]C_2\right)k_1} \tag{13}$$

$$v(x,t) = d_0 + \frac{\sqrt{2}\left(\text{Sinh}\left[\frac{\sqrt{d_1k_1k_4}(xk_1+2td_1k_4)}{\sqrt{2}k_1^2}\right]C_1+\text{Cosh}\left[\frac{\sqrt{d_1k_1k_4}(xk_1+2td_1k_4)}{\sqrt{2}k_1^2}\right]C_2\right)\sqrt{d_1k_1k_4}}{\left(\text{Cosh}\left[\frac{\sqrt{d_1k_1k_4}(xk_1+2td_1k_4)}{\sqrt{2}k_1^2}\right]C_1+\text{Sinh}\left[\frac{\sqrt{d_1k_1k_4}(xk_1+2td_1k_4)}{\sqrt{2}k_1^2}\right]C_2\right)k_1} \quad (14)$$

***Case 3.***

$e_1 = g_1,\ f_0 = d_0,\ k_2 = 0, e_1 \neq 0,\ d_1 = 0, k_4 = 0,\ f_1 = 0,\ k_1(-k_1 + k_3) \neq 0, \mu = -\frac{3(e_1k_1-e_1k_3)}{k_1}$

***Solution 3.***

$$u(x,t) = d_0 + e_1(-x - C_1 + 3e_1t) \quad (15)$$

$$v(x,t) = d_0 + e_1(-x - C_1 + 3e_1t) \quad (16)$$

***Case 4.***

$e_1 = 0,\ f_0 = d_0,\ f_1 = d_1, k_2 = 0, g_1 = 0,\ k_1 = k_3,\ k_1k_4 \neq 0,\ \ \mu = -\frac{3d_1k_4}{k_1}, d_1 \neq 0$

***Solution 4.***

$$u(x,t) = d_0 + d_1(C_1 + \frac{k_4(k_1x+3d_1k_4t)}{k_1^2}) \quad (17)$$

$$v(x,t) = d_0 + d_1(C_1 + \frac{k_4(k_1x+3d_1k_4t)}{k_1^2}) \quad (18)$$

***Case 5.***

$d_1 = 0,\ f_1 = 0,\ g_1 = e_1,\ k_2 = 0,\ k_4 = 0,\ k_1(-k_1 + k_3) \neq 0, -d_0e_1 + e_1f_0 \neq 0,\ \ \mu = -\frac{3(d_0^2k_1+e_1k_1-2d_0f_0k_1+f_0^2k_1-e_1k_3)}{k_1}$

***Solution 5.***

$$u(x,t) = d_0 + e_1(-x - C_1 - 3(d_0^2 - e_1 - 2d_0f_0 + f_0^2)t) \quad (19)$$

$$v(x,t) = 3e_1^2t + f_0 - e_1(x + C_1 + 3d_0^2t - 6d_0f_0t + 3f_0^2t) \quad (20)$$

## 4. Explanations and Graphical Representations of The Obtained Some Solutions

The graphical performance of found some solutions are demonstrated Figs. 1-4. These figures have the following physical explanations

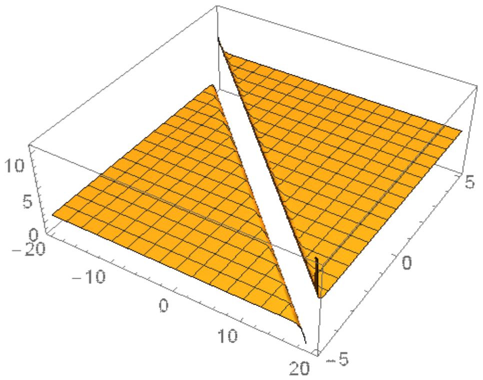


**Figure 1.** The 3 Dimensional surfaces of Eq. (9) for $k_1 = 1, k_2 = 2, d_0 = 1, C_1 = 2, C_2 = 3$.

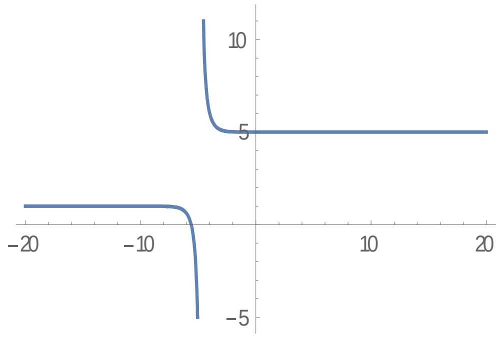


**Figure 2.** The 2 Dimensional surfaces of Eq. (9) for $k_1 = 1, k_2 = 2, d_0 = 1, C_1 = 2, C_2 = 3$ and $t = 1$.

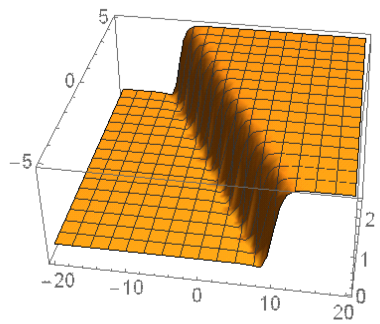


**Figure 3.** The 3 Dimensional surfaces of Eq. (11) for $k_1 = 1, k_4 = 2, d_0 = 1, d_1 = 1, C_1 = 3, C_2 = 2$.

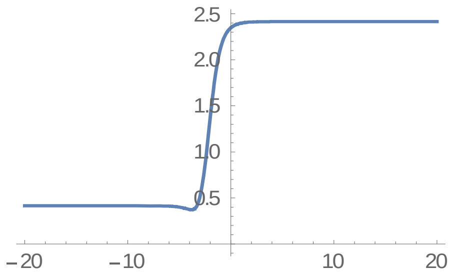


**Figure 4.** The 2 Dimensional surfaces of Eq. (11) for $k_1 = 1, k_4 = 2, d_0 = 1, d_1 = 1, C_1 = 3, C_2 = 2$ and $t = 1$.

The coupled potential KdV-equation: The shapes of Eqs.(9)-(11) are represented in Figures 1-4 within the intervals $-20 \leq x \leq 20, -5 \leq t \leq 5$.

## 5. Conclusions

The Generalized $\left(\frac{G\prime}{G}\right)$- Expansion Method were used for some soliton wave solution the coupled potential KdV-equation. Some nonlinear partial differential equations were solved by this method. It can be solved similarly in a number of nonlinear partial differential equations.

## References

Bock, T.L. and Kruskal, M.D. (1979). "A two-parameter Miura transformation of the Benjamin-Ono equation", *Physics Letters A*, 74 , 173-176.

Cariello, F. and Tabor, M. (1989)."Painleve expansions for nonintegrable evolution equations", *Physica D*, 39, 77-94.

Chen, H. T. and Hong-Qing, Z. (2004). "New double periodic and multiple soliton solutions of the generalized (2+1)-dimensional Boussinesq equation", *Chaos Solitons and Fractals*, 20, 765-769.

Chen, Y., Wang, Q. and Li, B. (2004). "Jacobi elliptic function rational expansion method with symbolic computation to construct new doubly periodic solutions of nonlinear evolution equations", *Zeitschrift für Naturforschung A*, 59, 529-536.

Chen, Y. and Yan, Z. (2006) ."The Weierstrss elliptic function expansion method and its applications in nonlinear wave equations", *Chaos Solitons and Fractals*, 29, 948-964.

Chuntao, Y. (1996). "A simple transformation for nonlinear waves", *Physics Letters A*, 224, 77-84.

Chen, H. and Zhang H., (2004). "New multiple soliton solutions to the general Burgers-Fisher equation and the Kuramoto-Sivashinsky equation", *Chaos Solitons and Fractals*, 19, 71-76.

Clarkson, P.A. (1989). "New similarity solutions for the modified boussinesq equation", *Journal of Physics A*: *Mathematical and General*, 22, 2355-2367.

Elwakil, S. A., El-labany, S.K., Zahran, M.A. and Sabry, R. (2002). "Modified extended tanh-function method for solving nonlinear partial differential equations", *Physics Letters A*, 299, 179-188.

Fan, E. ($2000_a$). "Two new application of the homogeneous balance method", *Physics Letters A*, 265, 353-357.

Fan, E. ($2000_b$). "Extended tanh-function method and its applications to nonlinear equations", *Physics Letters A*, 277, 212-218.

Fu, Z., Liu S. And Zhao, Q. (2001). "New Jacobi elliptic function expansion and new periodic solutions of nonlinear wave equations", *Physics Letters A*, 290, 72-76.

Guo, S. and Zhou, Y. (2010) ."The extended $\left(\frac{G\prime}{G}\right)$ -exnsion method and its applications to the Whitham-Broer-Kaup-like equations and coupled Hirota-Satsuma KdV equations", *Applied Mathematics and Computation*, 215, 3214-3221.

Khater, M.M.A. (2015). "Extended Exp($-\varphi(\xi)$)-Expansion Method for Solving the Generalized Hirota-Satsuma Coupled KdV System", *Global Journal of Science Frontier Research: F Mathematics and Decision Sciences*, 15, 7, Version 1.0.

Li, L., Li, E. and Wang, M. (2010)."The $\left(\frac{G\prime}{G}, \frac{1}{G}\right)$-expansion method and its application to travelling wave solutions of the Zakharov equations", *Applied Mathematics-A Journal of Chinese Universities*, *25*, 454-462.

Lü, H. L., Liu, X. Q. and Niu, L. (2010) ."A generalized $\left(\frac{G\prime}{G}\right)$ -expansion method and its applications to nonlinear evolution equations", *Applied Mathematics and Computation*, 215, 3811-3816.

Malfliet, W. (1992)."Solitary wave solutions of nonlinear wave equations", *American Journal of Physics*, 60, 650-654.

Manafian, J. (2016). "Optical soliton solutions for Schrödinger type nonlinear evolution equations by the tan $\left(\frac{\Phi(\varphi)}{2}\right)$ – expansion Method", *Optik*, 127, 4222-4245.

Manafian, J., Aghdaei, M.F., Khalilian, M. and Jeddi, R.S. (2017) ."Application of the generalized $G'/G$-expansion method for nonlinear PDEs to obtaining soliton wave solution", *Optik*, 135, 395–406.

Shen, S. and Pan, Z. (2003). "A note on the Jacobi elliptic function expansion method", *Physics Letters A*, 308, 143-148.

Wang, M., Li, X. and Zhang, J. ( 2008) ."The $\left(\frac{G'}{G}\right)$-expansion method and travelling wave solutions of nonlinear evolutions equations in mathematical physics", *Physics Letters A*, 372, 417-423.

Yan, Z. (2001)."New explicit travelling wave solutions for two new integrable coupled nonlinear evolution equations", *Physics Letters A*, 292, 100-106.