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On Submanifolds in a Riemannian Manifold with Golden Structure

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Abstract. A golden Riemannian structure (J, g) on a Riemannian manifold is given by a tensor field J of type $(1, 1)$ satisfying the golden section relation $J^2 = J + I$, and a pure Riemannian metric g, that is a metric satisfying $g(JX, Y) = g(X, JY)$. We investigate some fundamental properties of the induced structure on submanifolds immersed in golden Riemannian manifolds. We obtain effective relations for some induced structures on submanifolds of codimension 2. We also construct an example on submanifold of a golden Riemannian manifold.

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1. Introduction

The theory of submanifolds is an interesting topic in the study of differential geometry. It has the origin in the study of geometry of plane curves and surfaces initiated by Fermat. Since then, it has been evolving in different directions of differential geometry and mechanics, especially. It is still an active research field playing an important role in the development of modern differential geometry. Ahmad M et. al. ([1–3, 8]) studied submanifolds of almost r-para-contact Riemannian manifold endowed with semi-symmetric and quater symmetric connections. Hretcanu [14] studied submanifolds of almost product Riemannian manifolds. CR-submanifolds of LP-Sasakian manifolds were studied by Ahmad, Ozgur and Haseeb [18].

Crasmareanu and Hretcanu [6] constructed the golden structure on a differentiable Riemannian manifold $(\overline{M}, g)$ as a particular case of polynomial structure [12] based on golden ratio. Gezer et. al [10] investigated the integrability conditions of golden Riemannian structures. The Golden structure was also studied in [5, 9, 11, 17, 19]. Hretcanu and Crasmareanu [15] also defined metallic structure as a generalization of golden structure. Blaga and Hretcanu [16] studied submanifolds of metallic manifolds. Hretcanu and Crasmareanu [7] studied some properties of invariant submanifolds in a Riemannian manifold with golden structure. Poyraz and Erol [19] studied the hypersurface of a Golden Riemannian manifold. Hretcanu [13] studied submanifolds of Riemannian manifold with golden structure. Bahadur and Uddin [4] studied slant submanifolds of golden Riemannian manifolds.

Motivated by above studies in this paper, we study submanifold of a golden Riemannian manifold.The paper is organized as follows:

In section 2, we define golden Riemannian manifolds.

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In section 3, we establish several properties of induced structure (P, g, ξ, u, a) on the submanifold immersed in golden Riemannian manifold. In last section, we construct an example of golden Riemannian structure on Euclidean space and its submanifolds.

2. Golden Riemannian Manifolds

In this section, we give a brief information of golden Riemannian manifolds.

Definition 2.1. [6] Let (M, g) be a Riemannian manifold. A golden structure on $(\overline{M}, g)$ is a non-null tensor J of type (1,1) which satisfies the equation

$$J^2 = J + I, \tag{2.1}$$

where I is the identity transformation. We say that the metric g is J-compatible if

$$g(JX, Y) = g(X, JY) \tag{2.2}$$

for all X,Y vector fields on $\overline{M}$. If we substitute JX into X in (2.2), then we have

$$g(JX, JY) = g(JX, Y) + g(X, Y).$$

The Riemannian metric (2.2) is called J-compatible and $(\overline{M}, J, g)$ is called a Golden Riemannian manifold.

Proposition 2.2. *[6] A golden structure on the manifold M has the power*

$$J^n = F_n J + F_{n-1} I \tag{2.3}$$

for any integer n, where (F_n) is the Fibonacci sequence.

Using an explicit expression for the Fibonacci sequence namely the Binet's formula

$$F_n = \frac{J^n - (1 - J)^n}{\sqrt{5}},$$

we obtain a new form for the equality (2.3) *as*

$$J^n = (\frac{\phi^n - (1-\phi)^n}{\sqrt{5}})J + (\frac{\phi^{n-1} - (1-\phi)^{n-1}}{\sqrt{5}})I.$$

The straight forward computations yield:

Proposition 2.3. *[6] (i) The eigen values of a golden structure J are the golden ratio ϕ and $1 - \phi$.*
(ii) A golden structure J is an isomorphism on the tangent space T_xM of the manifold M for every $x \in M$.
(iii) It follows that J is invertible and its inverse $\widehat{J} = J^{-1}$ satisfies

$$\widehat{\phi}^2 = -\widehat{\phi} + 1.$$

3. Properties of Induced Structure on Submanifolds in Golden Riemannian Manifolds

Let us consider that M is an n-dimensional submanifold of codimension r, isometrically immersed in an $(n + r)$-dimensional golden Riemannian manifold $(\overline{M}, \overline{g}, J)$ with $n, r \in N$.

We denote by T_xM the tangent space of M in a point $x \in M$ and by $T_x^{\perp}M$ the normal space of M in x. Let i be the differential of the immersion $i : M \to \overline{M}$. The induced Riemannian metric g on M is given by $g(X, Y) = g(iX, iY)$ for every $X, Y \in \chi(M)$.

We consider a local orthonormal basis $N_1, N_2, ..., N_r$ of the normal space $T_x^{\perp}M$. We assume that the indices α, β, γ run over the range $1, 2, ..., r$.

For every $X \in T_xM$ the vector fields $J(iX)$ and $J(N_\alpha)$ can be decomposed in tangential and normal components as follows:

$$J(i(X)) = i(P(X)) + \sum_{\alpha=1}^{r} u_\alpha(X) N_\alpha, \tag{3.1}$$

$$J(N_\alpha) = (\xi_\alpha) + \sum_{\beta=1}^{r} a_{\alpha\beta} N_\beta, \tag{3.2}$$

where P is a $(1, 1)$ tensor field on M, $\xi \in \xi(M)$, u_α are $1 - forms$ on M and $(a_{\alpha\beta})_r$ is an $r \times r$ matrix of smooth real functions on M.

Proposition 3.1. *[7] The structure $\sum = (P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ induced on the submanifold M by the golden Riemannian structure (g, J) on $\overline{M}$ satisfies the following equalities:*

$$P^2(X) = P(X) + X - \sum_\alpha u_\alpha(X)\xi_\alpha,$$

$$u_\alpha(P(X)) = u_\alpha(X) - \sum_\beta a_{\alpha\beta} u_\beta(X),$$

$$a_{\alpha\beta} = a_{\beta\alpha},$$

$$u_\beta(\xi_\alpha) = \delta_{\alpha\beta} + a_{\alpha\beta} - \sum_\gamma a_{\alpha\gamma} a_{\gamma\beta},$$

$$P(\xi_\alpha) = \xi_\alpha - \sum_\beta a_{\alpha\beta}\xi_\beta,$$

$$u_\alpha(X) = g(X, \xi_\alpha),$$

$$g(PX, Y) = g(X, PY),$$

$$g(PX, PY) = g(X, PY) + g(X, Y) + \sum_\alpha u_\alpha(X)u_\alpha(Y)$$

for every $X, Y \in \chi(M)$, where $\delta_{\alpha\beta}$ is the Kronecker delta.

Definition 3.2. A submanifold M in a manifold $\overline{M}$ endowed with structural tensor field J (i.e J is a tensor field on $\overline{M}$) is called invariant with respect to J if $J(T_x) \subset T_x(M)$ for every $x \in M$.

Remark 3.3. The induced structure $\sum = (P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ on the submanifold M by the golden Riemannian structure (g, J) is invariant if and only if $u_\alpha = 0$ (equivalently $\xi_\alpha = 0$) for every $\alpha \in (1, ..., r)$.

The Gauss and Weingarten formula are

$$\overline{\nabla}_X Y = \nabla_x Y + \sum_{\alpha=0}^{r} h_\alpha(X, Y)N_\alpha, \tag{3.3}$$

$$\overline{\nabla}_X N_\alpha = -A_\alpha X + \nabla^\perp_X N_\alpha. \tag{3.4}$$

If $\{N_1, ..., N_r\}$ and $\{N'_1, ..., N'_r\}$ are two local orthogonal basis on a normal space $T^\perp_x$, then the decomposition of N'_α in the basis $\{N_1, ..., N_r\}$ is the following

$$N'_\alpha = \sum_{\gamma=1}^{r} k^\gamma_\alpha N_\gamma$$

for any $\alpha \in \{1, ..., r\}$, where (k^γ_α) is an $r \times r$ orthogonal matrix and we have

$$u'_\alpha = \sum_\gamma k^\gamma_\alpha u_\gamma, \xi'_\gamma = \sum_\gamma k^\gamma_\alpha \xi_\gamma \text{ and } a'_{\alpha\beta} = \sum_\gamma k^\gamma_\alpha a_{\gamma\delta} k^\delta_\beta.$$

Thus, if $\xi_1, \xi_2, ..., \xi_r$ are linearly independent vector fields, then $\xi'_1, \xi'_2, ..., \xi'_r$ are also linearly independent.
We know that $a_{\alpha\beta}$ is symmetric in α and β, under a suitable transformation, we can find that $a_{\alpha\beta}$ can be reduced to $a'_{\alpha\beta} = \lambda_\alpha\delta_{\alpha\beta}$, where $\lambda_\alpha(\alpha \in \{1, .., r\}$ are eigen values of the matrix $(a_{\alpha\beta})_r$ and in this case we have $u'_\beta(\xi_\alpha) = \delta_{\alpha\beta}(1 + \lambda_\alpha - \lambda_\alpha\lambda_\beta)$ and from this we obtain $u'_\alpha(\xi_\alpha) = (1 + \lambda_\alpha - \lambda^2_\alpha)$.

Remark 3.4. If M is a non-invariant n-dimensional submanifold of codimension r, immersed in a golden Riemannian manifold $(\overline{M}, \overline{g}, J)$ so that the tangent vector fields $\xi_1, \xi_2, ..., \xi_r$ are linearly independent, then from Proposition 3.1 we obtain

$$\|\xi_\alpha\|^2 = 1 + a_{\alpha\alpha} - \sum_\gamma a^2_{\alpha\gamma}$$

and, for $\alpha \neq \beta$, we have

$$\sum_\gamma a_{\alpha\gamma} a_{\gamma\beta} = a_{\alpha\beta}.$$

For the normal connection $\nabla^{\perp}_X N_\alpha$, we have the decomposition

$$\nabla^{\perp}_X N = \sum_{\beta=1}^{r} l_{\alpha\beta}(X)N_\beta \tag{3.5}$$

for every $X \in \chi(M)$.

Therefore, we obtain an $r \times r$ matrix $(l_{\alpha\beta}(X))_r$ of $1 - form$ on M. From $\overline{g}(N_\alpha, N_\beta) = \delta_{\alpha\beta}$, we get

$$\overline{g}(\nabla^{\perp}_x, N_\beta) + \overline{g}(N_\alpha, \nabla^{\perp}_X N_\beta) = 0$$

which is equivalent with

$$\overline{g}(\sum_\gamma l_{\alpha\gamma}(X)N_\gamma, N_\beta) + \overline{g}(N_\alpha, \sum_\gamma l_{\beta\gamma}(X)N_\gamma) = 0$$

for any $X \in \chi(M)$. Thus, we obtain

$$l_{\alpha\beta} = -l_{\beta\alpha}$$

for any $\alpha, \beta \in \{1, ..r\}$.

Theorem 3.5. *Let M is an n-dimensional submanifold of codimension r in a golden Riemannian manifold with structure $(\overline{M}, \overline{g}, J)$. If the structure J is parallel with respect to the Levi - Civita connection $\overline{\nabla}$ defined on $\overline{g}$, then the induced structure $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ induced on M by the structure J has the following properties:*

$$(\nabla_X P)(Y) = \sum_\alpha [g(A_\alpha X, Y)\xi_\alpha + u_\alpha(Y)A_\alpha X], \tag{3.6}$$

$$(\nabla_X u_\alpha)(Y) = \sum_\beta [h_\beta(X, Y)a_{\beta\alpha} - u_\beta(Y)l_{\alpha\beta}(X)] - h_\alpha(X, PY), \tag{3.7}$$

$$\nabla_X \xi_\alpha = -P(A_\alpha X) + \sum_\beta a_{\alpha\beta}A_\beta X + \sum_\beta l_{\alpha\beta}(X)\xi_\beta, \tag{3.8}$$

$$X(a_{\alpha\beta}) = -u_\alpha(A_\beta X) - u_\beta(A_\alpha X) + \sum_\gamma [l_{\alpha\gamma(X)}a_{\gamma\beta} + l_{\beta\gamma}(X)a_{\alpha\gamma},] \tag{3.9}$$

for any $X \in \chi(M)$.

Proof. Using (3.1) and $\overline{\nabla}J = 0$, we obtain

$$J(\overline{\nabla}_X Y) = \overline{\nabla}_X(PY) + \overline{\nabla}_X \sum_\alpha (u_\alpha(Y))N_\alpha$$

$$J(\overline{\nabla}_X Y) = \overline{\nabla}_X PY + \sum_\alpha [u_\alpha(Y)\overline{\nabla}_X N_\alpha + N_\alpha \overline{\nabla}_X(u_\alpha(Y))].$$

Using (3.3) and (3.4), we obtain

$$J[\nabla_X Y + \sum_{\alpha=1}^{r} h_\alpha(X, Y)N_\alpha] = \nabla_X(PY) + \sum_{\alpha=1}^{r} h_\alpha(X, PY)N_\alpha +$$

$$\sum_\alpha [u_\alpha(Y)(-A_\alpha X + \nabla^{\perp}_X N_\alpha) + N_\alpha(\nabla_X u_\alpha(Y) + \sum_{\beta=1}^{r} h_\beta(X, u(Y))N_\beta].$$

Using (3.1), (3.2) and (3.5), we obtain

$$\sum_{\alpha=1}^{r} h_\alpha(X, Y)\xi_\alpha + \sum_{\alpha=1}^{r} h_\alpha(X, Y) \sum_{\beta=1}^{r} a_{\alpha\beta}N_\beta$$

$$= (\nabla_X P)(Y) + \sum_{\alpha=1}^{r} h_\alpha(X, PY)N_\alpha - \sum_\alpha u_\alpha(Y)A_\alpha X +$$

$$\sum_\alpha u_\alpha(Y)[\sum_\beta l_{\alpha\beta}(X)N_\beta] + \sum_\alpha (\nabla_X u_\alpha)(Y)N_\alpha.$$

Comparing tangential and normal components, we obtain (3.6) and (3.7).

Using (3.2) and $\overline{\nabla} J = 0$, we obtain

$$J(\overline{\nabla}_X(N_\alpha)) = \overline{\nabla}_X \xi_\alpha + \overline{\nabla}_X \sum_\beta a_{\alpha\beta} N_\beta.$$

Using (3.3), (3.4) and (3.5), we obtain

$$-P(A_\alpha X) - \sum_\alpha u_\alpha(A_\alpha X) N_\alpha + \sum_\beta l_{\alpha\beta}(X)\xi_\beta + \sum_\beta l_{\alpha\beta}(X) \sum_\gamma a_{\beta\gamma} N_\gamma =$$

$$\nabla_X \xi_\alpha + \sum_\alpha h_\alpha(X, \xi_\alpha) N_\alpha + \sum_\beta X(a_{\alpha\beta}) N_\beta - \sum_\beta a_{\alpha\beta} A_\beta X + \sum_\beta a_{\alpha\beta} \sum_\gamma l_{\beta\gamma}(X) N_\gamma.$$

Thus, identifying the tangential part and respectively the normal part in the last equality, we obtain (3.8) and (3.9). □

Definition 3.6. If we have the equality $N_P(X, Y) - 2\sum_\alpha du_\alpha(X, Y)\xi_\alpha = 0$ for any $X, Y \in \chi(M)$, then the $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ induced structure on submanifold M in a golden Riemannian manifold $(\overline{M}, \overline{g}, J)$ is said to be normal.

Remark 3.7. The compatibility condition $\overline{\nabla} J = 0$, where $\overline{\nabla}$ is Levi-Civita connection with respect to the metric $\overline{g}$ implies the integrability of the structure J which is equivalent with the vanishing of the Nijenhuis torsion tensor field of J:

$$N_J(X, Y) = [JX, JY] + J^2[X, Y] - J[JX, Y] - J[X, JY].$$

For this assumption, we must have the next general lemma:

Lemma 3.8. *We suppose that we have golden structure J on a manifold $\overline{M}$ and linear connection D with the torsion T. If N_J is Nijenhuis torsion tensor field of J, then we obtain*

$$N_J(X, Y) = (D_{JX}J)(Y) - (D_{JY}J)(X) - T[JX, JY] - JT(X, Y)-$$

$$T(X, Y) + J(D_Y J)(X) + J(T(JX, Y)) - J(D_X J)(Y) + T(X, JY).$$

Proof. *From the definition of the torsion T follows that*

$$[X, Y] = D_X Y - D_Y X - T(X, Y) \tag{3.10}$$

and from this we get

$$[JX, JY] = D_{JX} JY - D_{JY} JX - T(JX, JY), \tag{3.11}$$

$$[JX, Y] = D_{JX} Y - D_Y JX - T(JX, Y) \tag{3.12}$$

and

$$[X, JY] = D_X JY - D_{JY} X - T(X, JY). \tag{3.13}$$

Using relations $(D_X J)(Y) = D_X JY - J(D_X Y)$ and (2.1) and replacing the relations (3.10), (3.11), (3.12) and (3.13) in the formula of Nijenhuis tensor field of J, we obtain

$$N_J(X, Y) = D_{JX} JY - D_{JY} JX - T(JX, JY) + (J + I)[X, Y]-$$

$$J[D_{JX} Y - D_Y JX - T(JX, Y)] - J[D_X JY - D_{JY} X - T(X, JY)]$$

$$N_J(X, Y) = (D_{JX} J)(Y) + J(D_{JX} Y) - (D_{JY} J)(X) - J(D_{JY} X) - T[JX, JY]+$$

$$J(D_X Y) - J(D_Y X) - JT(X, Y) + D_X Y - D_Y X - T(X, Y) - J(D_{JX} Y)+$$

$$J((D_Y J)(X) + J(D_Y X)) + J(D_{JY} X) + T(X, JY)$$

$$N_J(X, Y) = (D_{JX} J)(Y) - (D_{JY} J)(X) - T(JX, JY) - JT(X, Y)-$$

$$T(X, Y) + J(D_Y J)(X) + J(T(JX, Y)) - J(D_X J)(Y) + T(X, JY).$$

Proposition 3.9. *Let M be a submanifold of codimension r in a golden Riemannian manifold $(\overline{M}, \overline{g}, J)$. If the induced structure $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ on M is normal and the normal connection $\nabla^\perp$ on M vanishes identically (i.e. $l_{\alpha\beta} = 0$), then we obtain the equality*

$$\sum_\alpha g(X, \xi_\alpha)(PA_\alpha - A_\alpha P)(Y) = \sum_\alpha g(Y, \xi_\alpha)(PA_\alpha - A_\alpha P)(X)$$

for any $X, Y \in \chi(M)$.

Proof. From the definition 3.6, we have

$$N_P(X,Y) - 2\sum_{\alpha} du_\alpha(X,Y)\xi_\alpha = 0$$

for any $X, Y \in \chi(M)$. Then we have

$$\sum_{\alpha} g(X,\xi_\alpha)(PA_\alpha - A_\alpha P) - \sum_{\alpha} g(Y,\xi_\alpha)(PA_\alpha - A_\alpha P)(X)$$
$$+ \sum_{\alpha,\beta}[g(X,\xi_\beta)l_{\alpha\beta}(X) - g(Y,\xi_\beta)l_{\alpha\beta}(Y)]\xi_\alpha = 0$$

for any $X, Y \in \chi(M)$.

Also, given that normal connection $\nabla^\perp$ of M vanishes identically (*i.e.* $l_{\alpha\beta} = 0$), we obtain

$$\sum_{\alpha} g(X,\xi_\alpha)(PA_\alpha - A_\alpha P)(Y) = \sum_{\alpha} g(Y,\alpha)(PA_\alpha - A_\alpha P)(X).$$

□

Proposition 3.10. *Under the assumption of last result, Proposition 3.9 does not depend on the choice of a basis in the normal space $T_x^\perp(M)$ for any $x \in M$.*

Proof. If $\{N'_\alpha\}$ is another basis in $T_x^\perp(M)$, then we have

$$N'_\alpha = \sum_{\beta} O_{\alpha\beta}N_\alpha, \tag{3.14}$$

where $(O_{\alpha\beta})_r$ is an orthogonal matrix.

From the condition $\overline{\nabla}_X N'_\alpha = 0$, we obtain

$$\overline{\nabla}_X N'_\alpha = \sum_{\beta} O_{\alpha\beta}\overline{\nabla}_X N_\beta + \sum_{\beta} \overline{\nabla}_X O_{\alpha\beta}N_\beta$$
$$\sum_{\beta} X(O_{\alpha\beta})N_\alpha = 0 \tag{3.15}$$

for any $X \in M$.

$\{N_\beta\}$ is linearly independent set, then

$$O_{\alpha\beta} = constant.$$

On the other hand,

$$\overline{\nabla}_X N'_\alpha = -A'_\alpha X$$

and

$$\overline{\nabla}_X N'_\alpha = \sum_{\beta} \overline{\nabla}_X(O_{\alpha\beta})N_\beta - \sum_{\beta} O_{\alpha\beta}A_\beta X. \tag{3.16}$$

Thus, from the relations (3.14), (3.15) and (3.16), we obtain

$$-A'_\alpha X = \sum_{\beta} \overline{\nabla}_X(O_{\alpha\beta})N_\beta - \sum_{\beta} O_{\alpha\beta}O_{\beta\alpha},$$
$$A'_\alpha X = \sum_{\beta} O_{\alpha\beta}A_\beta X.$$

Therefore, we have

$$J(N'_\alpha) = i_*\xi_\alpha + \sum_{\beta} a'_{\alpha\beta}N'_\beta.$$

Using (3.14), we obtain

$$J(N'_\alpha) = i_*\xi'_\alpha + \sum_{\beta}\sum_{\gamma} a'_{\alpha\beta}O_{\beta\gamma}N_\gamma. \tag{3.17}$$

Using equality (3.2) and (3.14), we get

$$J(N'_\alpha) = \sum_{\beta} O_{\alpha\beta}\xi_\beta + \sum_{\beta}\sum_{\gamma} O_{\alpha\beta}a_{\beta\gamma}N_\gamma. \tag{3.18}$$

From (3.17) and (3.18), we obtain

$$\xi'_\alpha = \sum_\beta O_{\alpha\beta}\xi_\beta \tag{3.19}$$

and

$$\sum_\beta\sum_\gamma a'_{\alpha\beta}O_{\beta\gamma} = \sum_\beta\sum_\gamma O_{\alpha\beta}a_{\beta\gamma}.$$

On the basis $\{N'_1, ..., N'_r\}$, the condition of Proposition 3.10 becomes

$$\sum_\alpha g(X,\xi'_\alpha)(PA'_\alpha - A'_\alpha P)(Y) = \sum_\alpha g(Y,\xi'_\alpha)(PA'_\alpha - A'_\alpha P)(X).$$

From (3.16) and (3.19), we get

$$\sum_\alpha g(X, O_{\alpha\beta}\xi\beta)(PO_{\alpha\gamma}A_\gamma - O_{\alpha\gamma}A_\gamma P)(Y) - \sum_\alpha g(Y, O_{\alpha\beta}\xi_\beta)(PO_{\alpha\gamma}A_\gamma - O_{\alpha\gamma}A_\gamma P)(X)$$

$$= \sum_\alpha O_{\alpha\beta}O_{\alpha\gamma}[g(X,\xi_\beta)(PA_\gamma - A_\gamma P)(Y) - g(Y,\xi_\beta)(PA_\gamma - A_\gamma P)] = 0.$$

From the orthogonality of the matrix $(O_{\alpha\beta})$, it follows that

$$\sum_\alpha [g(X,\xi_\alpha)(PA_\alpha - A_\alpha P)(Y) - g(Y,\xi_\alpha)(PA_\alpha - A_\alpha P)(X)] = 0.$$

Therefore, the Proposition 3.9 does not depend on the choice of a basis in the normal space $T^\perp_x(M)$ for any $x \in M$. □

Lemma 3.11. *Let M be a submanifold in a golden Riemannian manifold $(\overline{M}, \overline{g}, J)$. Let $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ be the induced structure on M. Then*

$$g((PA_\alpha - A_\alpha P)(X), Y)$$

on M is skew-symmetric for any $X, Y \in \chi(M)$.

Proof.

$$g(PA_\alpha X - A_\alpha PX, Y) = g(PA_\alpha X, Y) - g(A_\alpha PX, Y)$$
$$g(PA_\alpha X - A_\alpha PX, Y) = g(X, A_\alpha PY) - g(PA_\alpha Y, X)$$
$$g(PA_\alpha X - A_\alpha PX, Y) = -g((PA_\alpha - A_\alpha P)(Y), X).$$

So, $g((PA_\alpha - A_\alpha P)(X), Y)$ is skew-symmetric. □

Proposition 3.12. *Let M be a submanifold of codimension r $(r \geq 2)$ in a golden Riemannian manifold $(\overline{M}, \overline{g}, J)$ and structure J is parallel to Levi-Civita connection $\overline{\nabla}$ defined on $\overline{M}$ with $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ induced structure on M by J. If the normal connection $\nabla^\perp$ vanishes identically on the normal bundle $T^\perp(M)$ (i.e. $l_{\alpha\beta} = 0$,) then the tangent vector fields $\{\xi_1, \xi_2, ..., \xi_r\}$ are linearly independent if and only if the determinant of the matrix $(I_r + A - A^2)$ does not vanish in any $x \in M$, (where I_r is the $r \times r$ identity matrix).*

Proof. Let $k_1, ..., k_r$ be the real number with the properties that

$$k_1\xi_1 + k_2\xi_2 + ... + k_r\xi_r = 0 \tag{3.20}$$

in any point $x \in M$.
From the equality (3.6), we obtain

$$g(\xi_\alpha, \xi_\beta) = u_\beta(\xi_\alpha) = \delta_{\alpha\beta} + a_{\alpha\beta} - \sum_\gamma a_{\alpha\gamma}a_{\gamma\beta}. \tag{3.21}$$

Multiplying the equality (3.20) by ξ_α { for any $\alpha \in (1, 2..., r)$ } and using the equality (3.21), we obtain

$$\begin{cases} k_1(1 + a_{11} - \sum_\gamma a_{1\gamma}a_{\gamma 1}) + k_2(a_{12} - \sum_\gamma a_{1\gamma}a_{\gamma 2}) + ... + k_r(a_{1r} - \sum_\gamma a_{1\gamma}a_{\gamma r}) = 0 \\ k_1(a_{21} - \sum_\gamma a_{2\gamma}a_{\gamma 2}) + k_2(1 + a_{22} - \sum_\gamma a_{2\gamma}a_{\gamma 2}) + ... + k_r(a_{2r} - \sum_\gamma a_{r\gamma}a_{\gamma r}) = 0 \\ ------------------------------------ \\ k_1(a_{r1} - \sum_\gamma a_{r\gamma}a_{\gamma 1}) + k_2(a_{r2} - \sum_\gamma a_{r\gamma}a_{\gamma 2}) + ... + k_r(1 + a_{rr} - \sum_\gamma a_{r\gamma}a_{\gamma r}) = 0. \end{cases}$$

This linear system of equations has the unique solution $k_1 = k_2 = ... = k_r = 0$ if and only if it does not have a vanishing determinant. Furthermore, the determinant of the linear system of equations is the determinant of the following matrix

$$I_r + \begin{pmatrix} a_{11} & a_{12} & a_{13}.....a_{1r} \\ a_{21} & a_{22} & a_{23}.....a_{2r} \\ . & . & ...... \\ . & . & ..... \\ a_{r1} & a_{r2} & a_{r3}.....a_{rr} \end{pmatrix} - \begin{pmatrix} a_{11} & a_{12}.....a_{1r} \\ a_{21} & a_{22}.....a_{2r} \\ . & . \\ . & . \\ a_{r1} & a_{r2}.....a_{rr} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12}.....a_{1r} \\ a_{21} & a_{22}.....a_{2r} \\ . & . \\ . & . \\ a_{r1} & a_{r2}.....a_{rr} \end{pmatrix},$$

that is determinant of the matrix

$$I_r + A - A^2.$$

□

Lemma 3.13. *Let M be an n-dimensional submanifold of co-dimension 2 in a golden Riemannian manifold $(\overline{M}, \overline{g}, J)$, with the normal induced structure $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta}))$ and structure J is parallel to Levi-Civita connection $\overline{\nabla}$. If the normal connection $\nabla^\perp$ vanishes identically (i.e. $l_{\alpha\beta}$) then the following equation is hold good*

$$g(Y, \xi_1)(PA_1 - A_1P)(X) + g(Y, \xi_2)(PA_2 - A_2X)(X) + g((PA_1 - AP_1)(X), Y)\xi_1$$
$$+g((PA_2 - A_2P)X, Y) = 0 \tag{3.22}$$

for any $X, Y \in \chi(M)$.

Proof. By virtue of Lemma 3.11 we obtain

$$g(X, \xi_1)(PA_1 - A_1P)(Y) + g(X, \xi_2)(PA_2 - A_2P)(Y)$$
$$= g(Y, \xi_1)(PA_1 - A_1P)(X) + g(Y, \xi_2)(PA_2 - A_2P)(X)$$

for any $X, Y \in \chi(M)$.

Multiplying by $Z \in \chi(M)$ we have

$$g(X, \xi_1)g((PA_1 - A_1P)(Y), Z) + g(X, \xi_2)g((PA_2 - A_2P)(Y), Z)$$
$$= g(Y, \xi_1)g((PA_1 - A_1P)(X), Z) + g(Y, \xi_2)g((PA_2 - A_2P)(X), Z) \tag{3.23}$$

for any $X, Y, Z \in \chi(M)$.
Inverting Y by Z in the last equality we obtain

$$g(X, \xi_1)g((PA_1 - A_1P)Z, Y) + g(X, \xi_2)g((PA_2 - A_2P)Z, Y)$$
$$= g(Z, \xi_1)g((PA_1 - A_1P)X, Y) + g(Z, \xi_2)g((PA_2 - A_2P)(X), Y). \tag{3.24}$$

Adding equalities (3.23) and (3.24) we obtain

$$g(X, \xi_1)g((PA_1 - A_1P)Z, Y) + g(X, \xi_2)g((PA_2 - A_2P)Z, Y)$$
$$+g(X, \xi_1)g((PA_1 - A_1P)Y, Z) + g(X, \xi_2)g((PA_2 - A_2P)Y, Z)$$
$$= g(Y, \xi_1)g((PA_1 - A_1P)X, Z) + g(Y, \xi_2)g((PA_2 - A_2P)X, Z)$$
$$+g(Z, \xi_1)g((PA_1 - A_1P)X, Y) + g(Z, \xi_2)g((PA_2 - A_2P)X, Y).$$

By property of skew-symmetry, we obtain

$$g([g(Y, \xi_1)((PA_1 - A_1P)(X), Z) + (g(Y, \xi_2)(PA_2 - A_2P)(X), Z)$$
$$+g((PA_1 - PA_1)X, Y)\xi_1) + g((PA_2 - A_2P)X, Y)\xi_2], Z) = 0$$

for any $Z \in \chi(M)$. Thus we obtain equality (3.22) . □

Lemma 3.14. *Let M be an n-dimensional submanifold of codimension 2 in a golden Riemannian manifold $(\overline{M}, \overline{g}, J)$, with the normal induced structure $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ and structure J is parallel to Levi - Civita connection $\nabla^\perp$ vanishes identically (i.e., $l_{\alpha\beta} = 0$) and $\sigma \neq 0$, then the following equations are good*

$$(PA_1 - A_1P)\xi_1 = 0,$$

$$(PA_2 - A_2P)\xi_2 = 0,$$

$$(PA_1 - A_1P)\xi_2 = 0,$$

$$(PA_2 - A_2P)\xi_1 = 0.$$

Proof. With $X = Y = \xi_1$ in equality (3.22)

$$g(\xi_1, \xi_1)(PA_1 - A_1P)(\xi_1) + g(\xi_1, \xi_2)(PA_2 - A_2P)(\xi_1)$$

$$+g((PA_1 - A_1P)(\xi_1), \xi_1) + g((PA_2 - A_2P)(\xi_1), \xi_1)\xi_2 = 0.$$

Using $g(\xi_1, \xi_1) = a + \sigma \neq 0$, $g(\xi_1, \xi_2) = 0$

$$g((PA_1 - A_1P)\xi_1, \xi_1) = -g((PA_1 - A_1P)\xi_1, \xi_2)$$

$$g((PA_1 - A_1P)\xi_1, \xi_2) = 0$$

$$(PA_1 - A_1P)\xi_1 = 0.$$

With $X = Y = \xi_2$, in equality (3.22), we obtain

$$g(\xi_2, \xi_2)(PA_1 - A_1P)\xi_2 + g(\xi_1, \xi_2)(PA_2 - A_2P)\xi_1$$

$$+g((PA_1 - A_1P)\xi_1, \xi_2)\xi_1 + g((PA_2 - A_2P)\xi_1, \xi_1)\xi_2 = 0.$$

Using that $g(\xi_2, \xi_2) = b + \sigma \neq 0$, $g(\xi_1, \xi_2) = 0$,

$$(PA_2 - A_2P)\xi_2 = 0.$$

If we put $X = \xi_1$ and $Y = \xi_2$ in equality (3.22), we obtain

$$g(\xi_2, \xi_1)(PA_1 - A_1P)\xi_1 + g(\xi_2, \xi_2)(PA_2 - A_2P)\xi_1$$

$$+g((PA_1 - A_1P)\xi_1, \xi_2)\xi_2 + g((PA_2 - A_2P)\xi_1, \xi_2)\xi_2 = 0.$$

Using that $g(\xi_2, \xi_2) = b + \sigma \neq 0$, $g(\xi_1, \xi_2) = 0$, we obtain

$$(PA_2 - A_2P)\xi_1 = 0.$$

Again $X = \xi_2$ and $Y = \xi_1$, we obtain

$$g(\xi_1, \xi_1)(PA_1 - A_1P)\xi_2 + g(\xi_2, \xi_2)(PA_2 - A_2P)\xi_2$$

$$+g((PA_1 - A_1P)\xi_2, \xi_1)\xi_1 + g((PA_2 - A_2P)\xi_2, \xi_1)\xi_2 = 0.$$

Using that $g(\xi_1, \xi_1) = a + \sigma \neq 0$, $g(\xi_1, \xi_2) = 0$, we obtain

$$(PA_1 - A_1P)\xi_2 = 0.$$

□

Proposition 3.15. *Let M be an n-dimensional submanifold of codimension 2 in a golden Riemannian manifold $(\overline{M}, \overline{g}, J)$, with the normal induced structure $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ and structure J is parallel to Levi - Civita connection $\nabla^\perp$ vanishes identically (i.e., $l_{\alpha\beta} = 0$) and $\sigma \neq 0$, and trace $A = 0$. Then P commutes with the Weingartan operator A_α ($\alpha \in \{1, 2\}$), thus the following relations take place*

$$(i)(PA_1 - A_1P)(X) = 0, \tag{3.25}$$

$$(ii)(PA_2 - A_2P)(X) = 0 \tag{3.26}$$

$\forall X \in \chi(M)$

Proof.

$$g((PA_\alpha - A_\alpha P), \xi_\beta) = g(PA_\alpha X, \xi_\beta) - g((A_\alpha P)X, \xi_\beta)$$

$$g((PA_\alpha - A_\alpha P)X, \xi_\beta) = -[g(PA_\alpha \xi_\beta, X) - g((A_\alpha P)\xi_\beta, X)$$

$$g((PA_\alpha - A_\alpha P)X, \xi_\beta) = -g((PA_\alpha - A_\alpha P)\xi_\beta, X),$$

where $\alpha\,\beta \in \{1, 2\}$, from the last Lemma

$$(PA_\alpha - A_\alpha P)\xi_\beta = 0$$

for any $\alpha\,\beta \in \{1, 2\}$.

$$g((PA_1 - A_1 P)X, \xi_\beta) = -g((PA_1 - A_1 P)\xi_\beta, X)$$

$$(PA_1 - A_1 P)X = 0.$$

Similarly

$$(PA_2 - A_2 P)X = 0$$

for any $\alpha\,\beta \in \{1, 2\}$.

In the following we assume that M is an n-dimensional submanifold of codimension 2 in golden Riemannian manifold $(\overline{M}, \overline{g}, J)$ with induced structure $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_2)$ on M $(\alpha, \beta \in \{1, 2\})$. We suppose that the normal connection vanishes identically, thus $(l_{\alpha\beta} = 0)$. In these conditions, the relations of Proposition 3.1 have the following forms:

$$P^2 X = P(X) + X - u_1(X)\xi_1 - u_2(X)\xi_2, \tag{3.27}$$

and

$$u_1(PX) = u_1(X) - a_{11}u_1(X) - a_{12}u_2(X),$$

$$u_2(PX) = u_2(X) - a_{21}u_1(X) - a_{22}u_2(X),$$

$$u_1(\xi_1) = 1 + a_{11} - a_{11}^2 - a_{12}^2,$$

$$u_2(\xi_2) = 1 + a_{22} - a_{12}^2 - a_{22}^2,$$

$$u_1(\xi_1) = u_2(\xi_1) = a_{21} - a_{21}(a_{11} + a_{22}),$$

$$P(\xi_1) = \xi_1 - a_{11}\xi_1 - a_{12}\xi_2,$$

$$P(\xi_2) = \xi_2 - a_{21}\xi_1 - a_{22}\xi_2,$$

$$g(PX, PY) = g(X, PY) + g(X, Y) + u_1(X)u_1(Y) + u_2(X)u_2(Y)$$

for any $X, Y \in \chi(M)$. We denote by $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

Furthermore, from Theorem 3.5 under the assumption that the normal connection $\nabla^\perp$ vanishes identically (*i.e.* $l_{\alpha\beta}$), we obtain

$$(\nabla_X P)(Y) = g(A_1 X, Y)\xi_1 + g(A_2 X, Y)\xi_2 + g(Y, \xi_1)A_1 X + g(Y, \xi_2)A_2 X,$$

$$(\nabla_X u_1)(Y) = -g(A_1 X, PY) + a_{11}g(A_1 X, Y) + a_{21}g(A_2 X, Y),$$

$$(\nabla_X u_2)(Y) = -g(A_2 X, PY) + a_{12}g(A_1 X, Y) + a_{22}g(A_2 X, Y),$$

$$\nabla_X \xi_1 = -P(A_1 X) + a_{11}A_1 X + a_{12}A_2 X,$$

$$\nabla_X \xi_2 = -P(A_2 X) + a_{21}A_1 X + a_{22}A_2 X,$$

$$X(a_{12}) = -2u_1(A_1 X),$$

$$X(a_{22}) = -2u_2(A_2 X).$$

□

Remark 3.16. A simplifier assumption for these relations is $a_{11} + a_{22} = 0$. Thus, *trace* $A = 0$. Under this assumption, if we denote $a_{11} = -a_{22} = a$ $a_{12} = a_{21} = b$ and $1 - a^2 - b^2 = \sigma$, from the relations , we easily see that

$$u_1(\xi_1) = u_2(\xi_2) = a + \sigma \Leftrightarrow g(\xi_1, \xi_1) = g(\xi_2, \xi_2) = a + \sigma,$$
$$u_1(\xi_2) = u_2(\xi_1) = b,$$
$$u_1(PX) = (1 - a)u_1(X) - bu_2(X),$$
$$u_2(PX) = (1 - a)u_2(X) - bu_1(X),$$

and

$$P(\xi_1) = (1 - a)\xi_1 - b\xi_2, \tag{3.28}$$
$$P(\xi_2) = (1 - a)\xi_2 - b\xi_1. \tag{3.29}$$

Proposition 3.17. *Let M ba a submanifold of codimension 2 in a golden Riemannian manifold $(\overline{M}, \overline{g}, J)$ and structure J is parallel to Levi - Civita connection $\overline{\nabla}$ defined on $\overline{M}$ with the normal induced structure $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_2)$. If the normal connection $\nabla^\perp$ vanishes identically, that is $l_{\alpha\beta} = 0$, trace $A = 0$ and $\sigma \neq 0$, then the following relations occurs :*

$$(a + \sigma)A_1\xi_1 + bA_1\xi_2 = h_1(\xi_1, \xi_1)\xi_1 + h_1(\xi_1, \xi_2)\xi_2, \tag{3.30}$$
$$(a + \sigma)A_1\xi_2 + bA_1\xi_1 = h_1(\xi_1, \xi_2)\xi_1 + h_1(\xi_2, \xi_2)\xi_2, \tag{3.31}$$
$$(a + \sigma)A_2\xi_1 + bA_2\xi_2 = h_2(\xi_1, \xi_1)\xi_1 + h_2(\xi_1, \xi_2)\xi_2, \tag{3.32}$$
$$(a + \sigma)A_2\xi_2 + bA_2\xi_1 = h_2(\xi_1, \xi_2)\xi_1 + h_2(\xi_2, \xi_2)\xi_2. \tag{3.33}$$

Proof. Using (3.25) and applying P it follows that

$$P^2A_1X = PA_1PX$$

for any $X \in \chi(M)$.

Using the equality (3.27) and if we put $X = \xi_1$ and $X = \xi_2$ respectively, we obtain

$$P(A_1\xi_1) + A_1\xi_1 - u_1(A_1\xi_1)\xi_1 - u_2(A_1\xi_1)\xi_2 = PA_1P\xi_1.$$

Using equality (3.28), we get

$$(2 - P)A_1\xi_1 + (P - 1)aA_1\xi_1 + (P - 1)bA_1\xi_2 = h_1(\xi_1, \xi_1)\xi_1 + h_1(\xi_1, \xi_2)\xi_2. \tag{3.34}$$

Now,

$$P(A_1\xi_2) + A_1\xi_2 - u_1(A_1\xi_2)\xi_1 - u_2(A_1\xi_2)\xi_2 = PA_1P\xi_2.$$

Using (3.29), we obtain

$$A_1\xi_2 - A_1b\xi_1 - A_1a\xi_2 + A_1\xi_2 - PA_1(\xi_2 - b\xi_1 - a\xi_2) = h_1(\xi_1, \xi_2)\xi_1 + h_1(\xi_2, \xi_2)\xi_2. \tag{3.35}$$

We replace $X \to PX$ in the equality (3.25), so

$$PA_1PX = A_1P^2X.$$

Using equality (3.27) and if we put $X = \xi_1$ and $X = \xi_2$ respectively, we get

$$PA_1P\xi_1 = A_1P\xi_1 + A_1\xi_1 - u_1(\xi_1)A_1\xi_1 - u_2(\xi_1)A_1\xi_2.$$

Using (3.28), we obtain

$$PA_1(\xi_1 - a\xi_1 - b\xi_2) = A_1(\xi_1 - a\xi_1 - b\xi_2) + A_1\xi_1 - u_1(\xi_1)A_1\xi_1 - u_2(\xi_1)A_1\xi_2,$$
$$(P - 2 + \sigma)A_1\xi_1 + (2 - P)aA_1\xi_1 + (2 - P)bA_1\xi_2 = 0 \tag{3.36}$$

and

$$PA_1PA\xi_2 = A_1P\xi_2 + A_1\xi_2 - u_1(\xi_2)A_1\xi_1 - u_2(\xi_2)A_1\xi_2.$$

Using (3.29), we obtain

$$PA_1((1 - a)\xi_2 - b\xi_1) = A_1((1 - a)\xi_2 - b\xi_1) + A_1\xi_2 - bA_1\xi_1 - (a + \sigma)A_1\xi_2,$$
$$(P - 2 + \sigma)A_1\xi_2 + (2 - P)A_1a\xi_2 + (2 - P)bA_1\xi_1 = 0. \tag{3.37}$$

Adding the relations (3.34) and (3.36), we obtain (3.30).

Adding (3.35) and (3.37), we obtain (3.31).

Applying P in the equality (3.26), it follows that

$$P^2 A_2 X = PA_2 PX$$

for any $X \in \chi(M)$ and using in (3.27) and for $X = \xi_1$ and $X = \xi_2$ respectively we obtain

$$(2-P)A_2\xi_1 + (P-1)aA_2\xi_1 + (P-1)bA_2\xi_2 = h_2(\xi_1,\xi_1)\xi_1 + h_2(\xi_1,\xi_2)\xi_2 \tag{3.38}$$

and

$$(2-P)A_2\xi_2 + (P-1)bA_2\xi_1 + (p-1)aA_2\xi_2 = h_2(\xi_1,\xi_2) + h_2(\xi_2,\xi_2)\xi_2.$$

We replace $X \to PX$ in the equality (3.26), so

$$PA_2PX = A_2P^2X \tag{3.39}$$

and using equality (3.27) and if we put $X = \xi_1$ and $X = \xi_2$ we obtain

$$(P-2+\sigma)A_2\xi_1 + (2-P)aA_2\xi_1 + (2-P)bA_2\xi_2 = 0 \tag{3.40}$$

and

$$(P-2+\sigma)A_2\xi_2 + (2-P)aA_2\xi_2 + (2-P)bA_2\xi_1 = 0. \tag{3.41}$$

Adding (3.38) and (3.40), we obtain (3.32).
Adding the relation (3.39) and (3.41), we obtain (3.33). □

Theorem 3.18. *Let M be a submanifold of a golden Riemannian manifold $\overline{M}$ and structure J is parallel to Levi-Civita connection $\overline{\nabla}$ defined on M (i.e $\overline{\nabla}J = 0$). If ξ_α ($\alpha = 1,2,3,,,,r$) are linearly independent, $T_r(P) = constant$ and M is totally umbilical, then M is totally geodesic.*

Proof. Since

$$\nabla_X(a_{\alpha\beta}) = -u_\alpha(A_\beta X) - u_\beta(A_\alpha X) + \sum_\gamma [l_{\alpha\gamma}(X)a_{\gamma\beta} + l_{\beta\gamma}(X)a_{\alpha\gamma}].$$

Putting $\alpha = \beta$, we have

$$\nabla_X(a_{\alpha\alpha}) = -2u_\alpha(A_\alpha X) + \sum_\gamma [l_{\alpha\gamma}(X)a_{\gamma\alpha} + l_{\alpha\gamma}(X)a_{\alpha\gamma}].$$

Since $a_{\alpha\beta}$ is symmetric and $l_{\alpha\beta}$ is skew-symmetric in α,β, then $\sum_{\alpha\gamma} a_{\alpha\gamma} l_{\alpha\gamma}(X) = 0$.
Since, $T_r(P) = constant$, we have $\sum_\alpha a_{\alpha\alpha} = constant$.
Hence,

$$\sum_\alpha u_\alpha(A_{\alpha X}) = 0$$

$$\sum_\alpha g(X, A_\alpha\xi_\alpha) = 0$$

$$\sum_\alpha A_\alpha\xi_\alpha = 0.$$

Since, ξ_α is linearly independent, then

$$A_\alpha = 0.$$

Hence M are totally geodesic. □

Theorem 3.19. *Let M be a submanifold of a golden Riemannian manifold $\overline{M}$ and J is parallel to Levi-Civita connection $\overline{\nabla}$ (i.e $\overline{\nabla}J = 0$). If ξ_α ($\alpha = 1,2,..,r$) are linearly independent, $\sum_j(\nabla_{e_j}P)e_j = 0$ and $T_r(P) = constant$, then M is minimal.*

Proof. Since,

$$(\nabla_X P)(Y) = \sum_\alpha [g(A_\alpha X, Y)\xi_\alpha + u_\alpha(Y)A_\alpha X].$$

Putting $X = Y = e_j$, we obtain

$$\sum_j (\nabla_{e_j}P)(e_j) = \sum_\alpha [A_\alpha \sum_j u_\alpha(e_j)e_j + \sum_j h_\alpha(e_j,e_j)\xi_\alpha].$$

Using (3.8), we obtain

$$\sum_j (\nabla_{e_j} P)(e_j) = \sum_\alpha [A_\alpha \xi_\alpha + \sum_j h_\alpha(e_j, e_j)\xi_\alpha].$$

Since,

$$T_r(P) = constant,$$

then from Theorem 3.18, we have

$$\sum_\alpha h_\alpha(X, \xi_\alpha) = 0.$$

Therefore,

$$\sum_\alpha g(A_\alpha X, \xi_\alpha) = 0$$

then

$$\sum_\alpha A_\alpha \xi_\alpha = 0.$$

Thus,

$$\sum_\alpha \sum_j h_\alpha(e_j, e_j) 0 \xi_\alpha = 0.$$

Since, ξ_α are linearly independent, then

$$h_\alpha(e_j, e_j) = 0.$$

Hence, M is minimal. □

Lemma 3.20. *Let M be a submanifold of a golden Riemannian manifold $\overline{M}$. If ξ_α ($\alpha = 1, 2, ..., r$) are linearly independent, then we have*

$$T_r(P) = -T_r(a_{\alpha\beta}),$$

where ($r = n$).

Lemma 3.21. *Let M be a submanifold of a golden Riemannian manifold $\overline{M}$. If ξ_α ($\alpha = 1, 2, ..., r$) are linearly independent and $\nabla_X P = 0$, then $T_r(a_{\alpha\beta}) = constant$.*

Proof. Let $\{e_1, e_2, ..., e_n\}$ be an orthogonal basis of T_P and extended e_j ($j = 1, 2, ..n$) to local vector field E_j which are covariantly constant at $p \in M$.

Then at $p \in M$,

$$\nabla_X T_r(P) = \nabla_X \sum_j g(Pe_j, e_j)$$

$$\nabla_X T_r(P) = \{\sum_j g(\nabla_X(PE_j, E_j))\}_P$$

$$\nabla_X T_r(P) = \sum_j [g((\nabla_X P)E_j + P\nabla_X E_j, E_j) + g(PE_j, \nabla_X E_j)]$$

$$\nabla_X T_r(P) = \sum_j ((\nabla_X P)E_j, E_j) + \sum_j g(\nabla_X E_j, PE_j) + \sum_j g(\nabla_X E_j, PE_j)$$

$$\nabla_X T_r(P) = 0.$$

Then,

$$T_r(P) = constant.$$

From Lemma 3.13, we have

$$T_r(a_{\alpha\beta}) = constant.$$

□

Example 3.22. We consider that ambient space is a $(2a + b)-$ dimensional Euclidean space E^{2a+b} $(a, b \in N)$. Let $J : E^{2a+b} \to E^{2a+b}$ be an $(1, 1)$ tensor field defined by

$$J(x^1, .., x^a, y^1, .., y^b, z^1, .., z^b) = (\phi x^1, .., \phi x^a, \phi y^1, .., \phi y^b, .., (1-\phi)z^1, .., (1-\phi)z^b)$$

for every point $(x^1, .., x^a, y^1, .., y^a, z^1, .., z^b) \in E^{2a+b}$, where $\phi = \frac{1+\sqrt{5}}{2}$ and $1-\phi = \frac{1-\sqrt{5}}{2}$ are roots of the equation $x^2 = x+1$. On the other hand, for $(x^1, .., x^a, y^1, .., y^a, z^1, .., z^b) \in E^{2a+b}$, we have

$$J^2(x^1, .., x^a, y^1, .., y^a, z^1, .., z^b) = (\phi^2 x^1, \phi^2 x^2, .., \phi^2 x^a, \phi^2 y^1, .., \phi^2 y^a, (1-\phi)^2 z^1, .., (1-\phi)^2 z^b)$$

$$J^2(x^1, .., x^a, y^1, .., y^a, z^1, .., z^b)$$

$$= (x^1, .., x^a, y^1, .., y^a, z^1, .., z^b) + (\phi x^1, .., \phi x^a, \phi y^1, .., \phi y^a, (1-\phi)z^1, .., (1-\phi)z^b)$$

$$J^2 = J + I.$$

For $(x^1, .., x^a, y^1, .., y^a, z^1, .., z^b), (p^1, .., p^a, q^1, .., q^a, r^1, .., r^b) \in E^{2a+b}$, we have

$$\langle J(x^1, .., x^a, y^1, .., y^a, z^1, .., z^b), (p^1, .., p^a, q^1, .., q^a, r^1, .., r^b)\rangle$$

$$= \langle((x^1, .., x^a, y^1, .., y^a, z^1, .., z^b), J(p^1, .., p^a, q^1, .., q^a, r^1, .., r^b)\rangle$$

for every $(x^1, .., x^a, y^1, .., y^a, z^1, .., z^b), (p^1, .., p^a, q^1, .., q^a, r^1, .., r^b) \in E^{2a+b}$.

So, the product $\langle\rangle$ on E^{2a+b} is $J-$compatible.

Therefore, J is a golden structure defined on $(E^{2a+b}, \langle\rangle)$ and $(E^{2a+b}, \langle\rangle, J)$ is a golden Riemannian manifold.

In the following issue, we identify iX with X (where $X \in \chi(E^{2a+b})$). It is obvious that $E^{2a+b} = E^a \times E^a \times E^b$ and in each of spaces E^a, E^a and E^b respectively, we can get a hypersphere

$$S^{a-1}(r_1) = \{(x^1, ..., x^a\}, \sum_{i=1}^{a}(x^i)^2 = r_1^2\},$$

$$S^{a-1}(r_2) = \{(y^1, ..., y^a), \sum_{i=1}^{a}(y^i)^2 = r_2^2\},$$

$$S^{b-1}(r_3) = \{(z^1, ..., z^b\}, \sum_{i=1}^{a}(z^j)^2 = r_3^2$$

respectively, where $r_1^2 + r_2^2 + r_3^2 = R^2$.

We construct the product manifold $S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3)$. Every point of $S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3)$ has the coordinate $(x^1, .., x^a, y^1, .., y^a, z^1, .., z^b) = (x^i, y^i, z^j)$ $(i \in \{1, .., a\}, j\epsilon\{1, .., b\})$ such that:

$$\sum_{i=1}^{a}(x^i)^2 + \sum_{i=1}^{a}(y^i)^2 + \sum_{j=1}^{b}(z^j)^2 = R^2.$$

Thus, $S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3)$ is a submanifold of codimension 3 in E^{2a+b} and $S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3)$ is a submanifold of codimension 2 in $S^{2a+b-1}(R)$ and $S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3)$ is a hypersurface in $S^{2a+b-1}(R)$. Therefore, we have

$$S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3) \hookrightarrow S^{2a+b-2}(r) \hookrightarrow S^{2a+b-1}(R) \hookrightarrow E^{2a+b}$$

The tangent space in a point $(x^1, .., x^a, y^1, .., y^a, z^1, .., z^b) = (x^i, y^i, z^j)$ at the product of spheres $S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3)$ is

$$T_(x^1, .., x^a, 0, .., 0, 0, .., 0)S^{a-1}(r_1) \bigoplus T_(0, .., 0, y^1, .., y^a, 0, .., 0)S^{a-1}(r_2)$$

$$\bigoplus T_(0, .., 0, 0, .., 0, z^1, .., z^b)S^{b-1}(r_3).$$

A vector $(X^1, .., X^a)$ from $T_{(x^1,...,x^a)}E^a$ is tangent to $S^{a-1}(r_1)$ if and only if we have

$$\sum_{i=1}^{a} x^i X^i = 0$$

and it can be identified by $(X^1, .., X^a, 0, .., 0, 0, .., 0)$ from E^{2a+b}.

A vector $(Y^1, .., Y^a)$ from $T_{(y^1,..,y^a)}E^a$ is tangent to $S^{a-1}(r_2)$ if and only if we have

$$\sum_{i=1}^{a} y^i Y^i = 0$$

and it can be identified by $(0, .., Y^1, .., Y^a, 0, .., 0)$ from E^{2a+b}.
A vector $(Z^1, .., Z^b)$ from $T_{(z^1,..,z^b)}E^b$ is tangent to $S^{b-1}(r_3)$ if and only if we have

$$\sum_{i=1}^{b} z^i Z^i = 0$$

and it can be identified by $(0, .., 0, 0, .., 0, Z^1, .., Z^b)$ from E^{2a+b}. Consequently, for every point $(x^i, y^i, z^j) \in S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3)$, we have
$(X^1, .., X^a, Y^1, .., Y^a, Z^1, .., Z^b) = (X^i, Y^i, Z^j) \in T_(x^1, .., x^a, y^1, .., y^a, z^1, .., z^b)(S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3))$.

If the above relations are satisfied, we remark that (X^i, Y^i, Z^j) is a tangent vector field at S^{2a+b-1} and from this it follows that

$$T_{(x^i,y^i,z^j)}(S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3)) \subset T_{(x^i,y^i,z^j)}S^{2a+b}(r)$$

for every point $(x^i, y^i, z^j) \in S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3)$. We consider a local orthonormal basis (N_1, N_2, N_3) of $T^{\perp}_{(x^i,y^i,z^j)}(S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3))$ in every point $(x^i, y^i, z^j) \epsilon S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3)$ given by

$$N_1 = \frac{1}{R}(x^i, y^i, z^j),$$

$$N_2 = \frac{1}{R}(x^i, y^i, -z^j),$$

$$N_3 = \frac{1}{r_3}(\frac{r_2}{r_1}x^i, \frac{-r_1}{r_2}y^i, 0).$$

From decomposition of $J(N_\alpha)$ $(\alpha \in \{1, 2, 3\})$ in tangential and normal components at $S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3)$, we obtain

$$J(N_\alpha) = \xi_\alpha + a_{\alpha 1}N_1 + a_{\alpha 2}N_2 + a_{\alpha 3}N_3,$$

where $\alpha \epsilon \{1, 2, 3\}$.

(i) From $a_{\alpha\beta} = \langle J(N_\alpha), N_\beta\rangle$ $(\alpha, \beta \epsilon \{1, 2, 3\})$, we obtain

$$a_{11} = a_{22} = \frac{1}{R^2}(\phi r_1^2 + \phi r_2^2 + (1-\phi)r_3^2),$$

$$a_{12} = a_{21} = \frac{1}{R^2}(\phi r_1^2 + \phi r_2^2 - (1-\phi)r_3^2),$$

$$a_{13} = a_{23} = 0 = a_{31} = a_{32},$$

$$a_{33} = \frac{\phi r_2^2 + \phi r_1^2}{r_3^2}.$$

Thus, the matrix A = $(a_{\alpha\beta})_3$ is given by

$$\begin{pmatrix} \frac{1}{R^2}(\phi r_1^2 + \phi r_2^2 + (1-\phi)r_3^2) & \frac{1}{R^2}(\phi r_1^2 + \phi r_2^2 - (1-\phi)r_3^2) & 0 \\ \frac{1}{R^2}(\phi r_1^2 + \phi r_2^2 - (1-\phi)r_3^2) & \frac{1}{R^2}(\phi r_1^2 + \phi r_2^2 + (1-\phi)r_3^2) & 0 \\ 0 & 0 & \frac{\phi r_2^2+\phi r_1^2}{r_3^2} \end{pmatrix}. \tag{3.42}$$

(ii)

$$\xi_1 = \frac{(R-2r_3)}{R^3}(\phi x^i, \phi y^i, (1-\phi)z^j), \tag{3.43}$$

$$\xi_2 = \frac{(R-2r_3)}{R^3}(\phi x^i, \phi y^i, -(1-\phi)z^j), \tag{3.44}$$

$$\xi_3 = (\frac{r_2\phi(1-\phi)}{r_1r_3}x^i, \frac{-r_1\phi(1-\phi)}{r_2r_3}y^i, 0). \tag{3.45}$$

(iii) From $u_\alpha(X) = u(X^i, Y^i, Z^j) = \langle (X^i, Y^i, Z^j), \xi_\alpha \rangle$, we obtain

$$u_1 = \frac{1}{R}(\phi X^i x^i + \phi Y^i y^i + (1-\phi) Z^j z^j), \quad (3.46)$$

$$u_2 = \frac{1}{R}(\phi X^i x^i + \phi Y^i y^i - (1-\phi) Z^j z^j), \quad (3.47)$$

$$u_3(X) = \frac{1}{r_3}(\frac{r_2}{r_1}\phi X^i x^i - \frac{r_1}{r_2}\phi Y^i y^i + Z^j). \quad (3.48)$$

(iv)

$$P(X) = (\phi X^i - [\frac{2\phi}{R^2}(X^i x^i + Y^i y^i) - \frac{r_2}{r_1 r_3^2}(\frac{r_2}{r_1}\phi X^i x^i - \frac{r_1}{r_2}\phi Y^i y^i + z^j)]x^i, \phi Y^i$$
$$-[\frac{2\phi}{R^2}(X^i x^i + Y^i y^i) + \frac{r_1}{r_2 r_3}(\frac{r_2}{r_1}\phi X^i x^i - \frac{r_1}{r_2}\phi Y^i y^i + z^j)]y^i, (1-\phi)Z^j - [\frac{2(1-\phi)}{R^2}Z^j z^j]z^j). \quad (3.49)$$

Thus, we have $J(T_{(x^i,y^j)}(S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3))) \subseteq (T_{(x^i,y^j)}(S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3))$ and we obtain $(P, \xi_\alpha, u_\alpha, (a_{\alpha\beta}))$ induced structure on $(S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3))$ by the golden Riemannian structure $(J, \langle\rangle)$ on E^{2a+b}, which is effectively determined by the relations (3.41), (3.43), (3.44), (3.45), (3.46), (3.47), (3.48) and (3.49).

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

References

[1] Ahmad M., Jun, J. B., Haseeb A., *Submanifolds of an almost r-paracontact Riemannian manifold endowed with a semi-symmetric semi-metric connection*, Tensor, N. S., **70**(2008), 311–321. 1

[2] Ahmad M., Jun, J. B., Haseeb A., *Submanifolds of an almost r-paracontact Riemannian manifold endowed with a quarter symmetric non-metric connection*, J. Chungcheong Math. Soc., **24(1)**(2011), 91–104. 1

[3] Ahmad M., Al-Hossain A.Y., Ojha, J.P., *On submanifolds of an almost r-paracontact Riemannian manifold endowed with a quarter symmetric metric connection*, Theoretical Mathematics and Application, **4(4)**(2014), 1–17. 1

[4] Bahadir, O., Uddin, S., *Slant submanifolds of golden Riemannian manifolds*, arXiv:1804.11126v1. 1

[5] Blaga, A.M., Hretcanu, C. E., *Invariant, anti-invariant and slant submanifolds of a metallic Riemannian manifold*, arXiv: 1803.01415. 1

[6] Crasmareanu, M., Hretcanu, C.E., *Golden differential geometry*, Chaos Solitons Fractals, **38(5)**(2008), 1229–1238. 1, 2.1, 2.2, 2.3

[7] Crasmareanu, M., Hretcanu, C.E., *On some invariant submanifolds in a Riemannian manifold with golden structure*, Analele Stintifice Universitatis "Al. I. Cuza" DIN IASI (S.N.) Matematica, Tomul LIII (2007), 199–211. 1, 3.1

[8] Das L. S., Nivas R., Ali S., Ahmad M., *Study of submanifolds in a manifold with quarter symmetric semi-metric connection*, Tensor N. S., **65**(2004), 250–260. 1

[9] Etayo, F., Santamaria, R., Upadhyay, A., *On the geometry of almost golden Riemannian manifolds*, Mediterr. J. Math. **187(14)**(2017), 991–1007. 1

[10] Gezer, A., Cengiz, N., Salimov, A., *On integrability of golden Riemannian structures*, Turk.J.Math., **37**(2013), 366–703. 1

[11] Gherici, B., *Induced structure on golden Riemannian manifolds*, Beitr Algebra Geom., **18(8)**(2018), 366–392. 1

[12] Goldberg, S.I., Yano, K., *Polynomial structures on manifolds*, Kodai Maths.Sem. Rep., **22(2)**(1970), 199–218. 1

[13] Hretcanu, C.E., *Submanifolds in Riemannian manifold with golden structure*, Workshop on Finsler geometry and its applications - Hungary (2007). 1

[14] Hretcanu, C.E., *Induced structure on submanifolds in almost product Riemannian manifolds*, arXiv:math/0608533. 1

[15] Hretcanu, C.E., Crasmareanu, M., *Metallic structure on Riemannian manifolds*, Revista De La Union Mathematica Argentina, **54(2)**(2013), 15–27. 1

[16] Hretcanu, C.E., Blaga, A.M., *Submanifolds in metallic Riemannian manifolds*, arXiv:1803.02184. 1

[17] Ozkan, M., *Prolongations of golden structures to tangent bundles*, Differential Geometry - Dynamical Systems, **16**(2014), 227–238. 1

[18] Ozgur, C., Ahmad, M., Haseeb, A., *CR-submanifolds of LP-Sasakian manifolds with semi-symmetric metric connection*, Hacettepe J. Math. And Stat., **39(4)**(2010), 489–496. 1

[19] Poyraz, N.O., Erol, Y., *Lightlike hypersurfaces of a golden semi-Riemannian manifold*, Mediterr. J. Math., (2017) 14:204, DOI 10.1007/s00009-017-0999-2. 1