Turk. J. Math. Comput. Sci.
11(1)(2019) 29–39
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Bipolar Fuzzy (T, S)-Norm Hyper KU-Ideals (Sub Algebras)

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Received: 10-05-2018 • Accepted: 26-05-2019

Abstract. In this paper, the concepts of (σ, τ)- bipolar fuzzy (T, S)- norm of the notions (strong, weak, s-weak) hyper KU-ideals in hyper KU-algebras are introduced and some properties are discussed.

2010 AMS Classification: 06F35, 03G25, 20N20.

Keywords: KU-algebra, hyper KU-algebra, (σ, τ)- Fuzzy (T, S)- norm hyper KU-ideal.

1. Introduction

In 1956, Zadeh [18] introduced the notion of fuzzy sets. At present this concept has been applied to many mathematical branches. There are several kinds of fuzzy sets extensions in the fuzzy set theory, for example, intuitionistic fuzzy sets, interval valued fuzzy sets, vague sets etc.(see [1–3, 5–7, 12]). Prabpayak and Leerawat [15, 16] introduced a new algebraic structure which is called KU-algebras. They studied ideals and congruencies in KU-algebras. Also, they introduced the concept of homomorphism of KU-algebra and investigated some related properties. Moreover, they derived some straightforward consequences of the relations between quotient KU-algebras and isomorphism. Mostafa et al. [12, 17] introduced the notion of fuzzy KU-ideals of KU-algebras and then they investigated several basic properties which are related to fuzzy KU- ideals .The hyper structure theory (called also multi-algebras) is introduced in 1934 by Marty [11] at the 8th congress of Scandinvian Mathematiciens. Around the 40's, several authors worked on hyper groups, especially in France and in the United States, but also in Italy, Russia and Japan. Hyper structures have many applications to several sectors of both pure and applied sciences. Jun and Xin [3, 6] considered the fuzzification of the notion of a (weak, strong, reflexive) hyper BCK-ideal, and investigated the relations among them. Mostafa et al. [13] applied the hyper structures to KU-algebras and introduced the concept of a hyper KU-algebra which is a generalization of a KU-algebra, and investigated some related properties. They also introduced the notion of a hyper KU-ideal, a weak hyper KU-ideal and gave relations between hyper KU-ideals and weak hyper KU-ideals.
Mostafa et al. [14], stated and proved more several theorems of hyper KU-algebras and studied fuzzy set theory to the hyper KU-sub algebras (ideals). Lee [8] introduced an extension of fuzzy sets named bipolar-valued fuzzy sets. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval [0,1] to [-1,1]. The authors in [4, 8], introduced bipolar-valued fuzzy set on different algebraic structures. Li et al. [9, 10], generalized the operators " $\wedge$ " and " $\vee$ " to T-norm and S-norm and defined the intuitionistic fuzzy groups

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of (T, S)-norms as a generalization of the notion of fuzzy set. Now, in this note, the σ, τ-bipolar fuzzy (T, S)-norms set theory to the (s-weak - strong) hyper KU-ideals in hyper KU-algebras are applied and discussed.

2. Preliminaries

Let H be a nonempty set and $P^*(H) = P(H)\backslash\{\phi\}$ the family of the nonempty subsets of H. A multi valued operation (said also hyper operation) " $\circ$ " on H is a function, which associates with every pair $(x, y) \in H \times H - H^2$ a non empty subset of H denoted $x \circ y$. An algebraic hyper structure or simply a hyper structure is a non empty set H endowed with one or more hyper operations.

Definition 2.1. [13] Let H be a nonempty set and " $\circ$ " a hyper operation on H, such that $\circ : H \times H \to P^*(H)$. Then H is called a hyper KU-algebra if it contains a constant "0" and satisfies the following axioms: for all $x, y, z \in H$
$(HKU_1)[(z \circ y) \circ (z \circ x)] \ll y \circ x$
$(HKU_2) x \circ 0 = \{0\}$
$(HKU_3) 0 \circ x = \{x\}$
$(HkU_4) if\ x \ll y, y \ll x\ implies\ x = y.$
where $x \ll y$ is defined by $0 \in y \circ x$ and for every $A, B \subseteq H, A \ll B$ is defined by $\forall\ \in A, \exists b \in B$ such that $a \ll b$. In such case, we call " $\ll$ " the hyper order in H.
We shall use the $x \circ y$ instead of $x \circ \{y\}, \{x\} \circ y$ or $\{x\} \circ \{y\}$. Note if $A, B \subseteq H$, then by $A \circ B$ we mean the subset $\bigcup_{a \in A, b \in B} a \circ b$ of H.

Example 2.2. (A) Let $H = \{0, 1, 2, \}$ be a set. Define hyper operation $\circ$ on H as follows

$\circ$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{0\}$	$\{0, 1\}$	$\{1, 2\}$
2	$\{0\}$	$\{0, 1\}$	$\{0, 1, 2\}$

Then $(H, \circ, 0)$ is a hyper KU-algebra.
In what follows, H denotes a hyper KU-algebra unless otherwise specified.

Lemma 2.3. *[13] For all $x, y \in H$ and $A \subseteq H$*
(i) $A \circ (y \circ x) = y \circ (A \circ x)$
(ii) $(0 \circ x) \circ x = \{0\}$

Proposition 2.4. *[14] In any hyper KU-algebra H, $0 \circ x = \{x\} \forall x \in H$.*

Theorem 2.5. *[13, 14] For all $x, y, z \in H$ and $A, B.C \subseteq H$*
(i) $x \circ y \ll z \Rightarrow z \circ y \ll x$
(ii) $x \circ y \ll y$
(iii) $x \ll 0 \circ x$
(iv) $A \ll B, B \ll C \Rightarrow A \ll C$
(v) $x \circ A \ll A$
(vi) $A \circ z \ll z \Leftrightarrow z \circ x \ll A$
(vii) $A \ll B \Rightarrow C \circ A \ll C \circ B$ and $B \circ C \ll A \circ C$
(viii) $A \ll 0 \circ A$
(ix) $x \in 0 \circ x$
(x) $x \in 0 \circ 0 \Leftrightarrow x = 0$
(xi) $x \circ x = \{x\} \Leftrightarrow x = 0$

Lemma 2.6. *[13] In hyper KU-algebra $(H, \circ, 0)$, we have*

$$z \circ (y \circ x) = y \circ (z \circ x)\ for\ all\ x, y, z \in H$$

Definition 2.7. [13] Let S be a non-empty subset of a hyper KU-algebra H. Then S is said to be a hyper sub-algebra of H if $S_2 : x \circ y \subseteq S, \forall x, y \in S$

Proposition 2.8. *[14] Let S be a non-empty subset of a hyper KU-algebra $(H, \circ, 0)$. If $y \circ x \subseteq S$ for all x, yS, then $0 \in S$.*

Theorem 2.9. *[14] Let S be a non-empty subset of a hyper KU-algebra $(H, \circ, 0)$. Then S is a hyper subalgebra of H if and only if $y \circ x \subseteq S$ for all $x, y \in S$.*

Definition 2.10. [13] Let I be a non-empty subset of a hyper KU-algebra H and $0 \in I$. Then
(1) I is said to be a weak hyper KU-ideal of H if $x \circ (y \circ z) \subseteq I$ and $x \in I$ imply $y \circ z \in I$, for all $x, y, z \in H$
(2) I is said to be hyper KU-ideal of H if $x \circ (y \circ z) \ll I$ and $x \in I$ imply $y \circ z \in I$, for all $x, y, z \in H$
(3) I is said a strong hyper KU-ideal of H if $x \circ (y \circ z) \cap I \neq \Phi$ and $x \in I$ imply $y \circ z \in I$, for all $x, y, z \in H$.
(4) I is said to be reflexive if $x \circ x \subseteq I$ for all $x \in H$.

Definition 2.11. [13] Let A be a non-empty subset of a hyper KU-algebra H. Then A is said to be a hyper ideal of H if
(HI_1) $0 \in A$,
(HI_2) $y \circ x \ll A$ and $y \in A$ imply $x \in A$ for all $x, y \in H$.

Definition 2.12. [13] A non-empty set A of a hyper KU-algebra H is called a distributive hyper ideal if it satisfies (HI_1) and
$(HI_3$ $(x \circ y) \circ (z \circ (z \circ x)) \ll A$ and $y \in A$ imply $x \in A$.

Definition 2.13. [13] Let I be a non-empty subset of a hyper KU-algebra H and $0 \in I$. Then,
(1) I is called a weak hyper ideal of H if $y \circ x \subseteq I$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$.
(2) I is called a strong hyper ideal of H if $(y \circ x) \cap I \neq \phi$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$.

Lemma 2.14. *[14] Let A be a subset of a hyper KU-algebra H. If I is a hyper ideal of H such that $A \ll I$ then $A \subseteq I$.*

Lemma 2.15. *In hyper KU-algebra $(H, \circ, 0)$, we have*
(i) Any strong hyper KU-ideal of H is a hyper ideal of H.
(ii) Any weak hyper KU-ideal of H is a weak ideal of H.

Definition 2.16. [8] A bipolar valued fuzzy subset B in a nonempty set X is an object having the form $\phi = (H, \mu^P_\Phi, \mu^N_\Phi)$ where $\mu^N : X \to [-1, 0]$ and $\mu^P : X \to [0, 1]$ are mappings. The positive membership degree $\mu^P(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set $\phi = (H, \mu^P_\Phi, \mu^N_\Phi)$, and the negative membership degree $\mu^N(x)$ denotes the satisfaction degree of x to some implicit counter-property of a bipolar-valued fuzzy set $\phi = (H, \mu^P_\Phi, \mu^N_\Phi)$. For simplicity, we shall use the symbol $\phi = (\mu^P_\Phi, \mu^N_\Phi)$ for bipolar fuzzy set $\phi = (H, \mu^P_\Phi, \mu^N_\Phi)$, and use the notion of bipolar fuzzy sets instead of the notion of bipolar-valued fuzzy sets.

Definition 2.17. [9, 10] A triangular norm (t-norm) is a function $T : [0, 1] \times [0, 1] \to [0, 1]$ that satisfies following conditions:
(T_1) boundary condition: $T(x, 1) = x$,
(T_2) commutativity condition: $T(x, y) = T(y, x)$,
(T_3) associativity condition: $T(x, T(y, z)) = T(T(x, y)z)$,
(T_4) monotonicity: $T(x, y) \leq T(x, z)$, whenever $y \leq z$ for all $x, y, z \in [0, 1]$.
A simple example of such defined t-norm is a function $T(\alpha, \beta) = \min\{\alpha, \beta\}$.
In the general case $T(\alpha, \beta) \leq \min\{\alpha, \beta\}$ and $T(\alpha, 0) = 0$ for all $\alpha, \beta \in [0, 1]$.
A simple example of such definition T-norm is a function

$$T(\alpha, \beta) = \min(\alpha, \beta) \forall\, \alpha, \beta \in [0, 1]$$

Definition 2.18. [9, 10] A triangular conorm (t-conorm S) is a mapping $S : [0, 1] \times [0, 1] \to [0, 1]$ that satisfies following conditions:
(S1) $S(x, 0) = x$,
(S2) $S(x, y) = S(y, x)$,
(S3) $S(x, S(y, z)) = S(S(x, y)z)$,
(S4) $S(x, y) \leq S(x, z)$, whenever $y \leq z$ for all $x, y, z \in [0, 1]$.
A simple example of such definition s-norm S is a function $\max(\alpha, \beta) = S(\alpha, \beta)$.

Proposition 2.19. *[9, 10]*
(1) Every t-norm T has a useful property: $T(\alpha,\beta) \leq \min(\alpha,\beta)\forall\, \alpha,\beta) \in [0,1]$.
(2) Every s-norm has a useful property: $\max(\alpha,\beta) \leq S(\alpha,\beta)\forall\, \alpha,\beta) \in [0,1]$.

Remark 2.20. For a t-norm (or s-norm), denote

$$\Delta P = \{\alpha \in [0,1] | T(\alpha,\alpha) = \alpha \ or \ S(\alpha,\alpha) = \alpha\}$$

Remark 2.21. If μ is fuzzy set, we denotes $\mu(x) \wedge \mu(y) = \min\{\mu(x), \mu(y)\}$ and $\mu(x) \vee \mu(y) = \max\{\mu(x), \mu(y)\} \forall x, y, z \in X$.

3. (σ, τ)-Bipolar Fuzzy Hyper KU-Subalgebras (ideals)

Now some fuzzy logic concepts are reviewed. A fuzzy set μ in a set H is a function $\mu : H \to [0,1]$. A fuzzy set μ in a set H is said to satisfy the inf (resp. sup) property if for any subset T of H there exists $x_0 \in T$ such that

$$\mu(x_0) = \inf_{x\in T} \mu(x)(\text{resp.}\mu(x_0) = \sup_{x\in T} \mu(x))$$

Definition 3.1. A fuzzy set $\phi = (H, \mu^P_\Phi, \mu^N_\Phi)$ in H is said to be (σ, τ)-bipolar fuzzy (T, S)-norms hyper KU-subalgebra of H if it satisfies the following inequalities:
(1) $\inf\limits_{z\in x\circ y} \mu^N_\Phi(z) \vee \sigma \geq T\left\{\mu^P_\Phi(x), \mu^P_\Phi(y)\right\} \wedge \tau$.
(2) $\sup\limits_{w\in x\circ y} \mu^N_\Phi(w) \wedge \sigma \leq S\left\{\mu^N_\Phi(x), \mu^N_\Phi(y)\right\} \wedge \tau \,\forall x, y \in H, 0 \leq \sigma < \tau \leq 1$.

Remark 3.2. In definition 3.1., if we take $\sigma = 0, \tau = 1$, we get bipolar fuzzy (T, S)-norms hyper KU-subalgebra

Proposition 3.3. *Let $\phi = (H, \mu^P_\Phi, \mu^N_\Phi)$ be a (σ, τ)-bipolar fuzzy (T, S)-norms hyper KU-sub-algebra of H. Then for all $\forall x \in H, 0 \leq \sigma < \tau \leq 1$*
$\mu^P_\Phi(0) \vee \sigma \geq \mu^P_\Phi(x) \wedge \tau$ *and* $\mu^N_\Phi(0) \wedge \sigma \leq \mu^N_\Phi(x) \vee \tau$

Proof. Using Proposition 2.5 (xi), we see that $0 \in x \circ x$ for all $x \in H$. Hence

$$\inf_{0\in x\circ x} \mu^P_\Phi(0) \vee \sigma \geq T\{\mu^P_\Phi(x), \mu^P_\Phi(x)\} \wedge \tau = \mu^P_\Phi(x) \wedge \tau$$

and

$$\sup_{0\in x\circ x} \mu^N_\Phi(0) \wedge \sigma \leq S\{\mu^N_\Phi(x), \mu^N_\Phi(x)\} \vee \tau = \mu^N_\Phi(x) \vee \tau \text{ for all } x \in H, 0 \leq \sigma < \tau \leq 1$$

□

Example 3.4. Let $H = \{0, 1, 2, 3\}$ be a set. The hyper operations $\circ$ on H are defined as follows.

$\circ_2$	0	1	2	3
0	{0}	{1}	{2}	{3}
1	{0}	{0}	{2,1}	{3}
2	{0}	{0}	{0,1,2}	{0,3}
3	{0}	{0}	{1,2,3}	{0,3}

Then $(H, \circ, 0)$ is a hyper KU-algebra. Define $\mu^N : X \to [-1, 0]$ and $\mu^P : X \to [0, 1]$ by

	0	1	2	3
μ^N	-0.7	-0.7	0.6	0.4
μ^P	0.6	0.5	0.3	0.3

Let $T : [0, 1] \times [0, 1] \to [0, 1]$ be a function defined by $T(\alpha,\beta) = \max\{\alpha + \beta - 1, 0\}$ for all $\alpha,\beta \in [0, 1]$ and $S : [0, 1] \times [0, 1] \to [0, 1]$ be a function defined by $S(\alpha,\beta) = \min\{(\alpha + \beta), \}, \sigma = 0.1, \tau = 0.4$. It is easily verified that by routine calculations, we know that $\Phi = (H, \mu^N, \mu^P)$ is $\sigma, \tau)$-bipolar fuzzy (T, S)-norms hyper sub-algebra of H.

Definition 3.5. For a "hyper KU-algebra" H, a "a bipolar fuzzy set" $\phi = (H, \mu_\Phi^P, \mu_\Phi^N)$ in H is called :
• (σ, τ)-Bipolar fuzzy (T, S)-norms hyper ideal of H, if

$$F_1 : x \ll y \text{ implies } \mu_\Phi^P(x) \vee \sigma \geq \mu_\Phi^P(y) \wedge \tau, \mu_\Phi^N(x) \wedge \sigma \leq \mu_\Phi^N(y) \vee \tau \text{ and}$$

$$F_2 : \mu_\Phi^P(z) \vee \sigma \geq T\left\{\inf_{u \in (y \circ z)} \mu_\Phi^P(u), \mu_\Phi^P(y)\right\} \wedge \tau$$

$$F_3 : \mu_\Phi^N(w) \wedge \sigma \leq S\left\{\sup_{w \in (y \circ z)} \mu_\Phi^N(w), \mu_\Phi^N(y)\right\} \vee \tau, 0 \leq \sigma < \tau \leq 1$$

• (σ, τ)-Bipolar fuzzy (T, S)-norms weak hyper ideal of H, if, for any $y, z \in H$

$$\mu_\Phi^P(0) \vee \sigma \geq \mu_\Phi^P(z) \wedge \tau \geq T\left\{\inf_{u \in (y \circ z)} \mu_\Phi^P(u), \mu_\Phi^P(y)\right\} \wedge \tau$$

and

$$\mu_\Phi^N(0) \wedge \sigma \leq \mu_\Phi^N(w) \vee \tau \leq S\left\{\sup_{w \in (y \circ z)} \mu_\Phi^N(w), \mu_\Phi^N(y)\right\} \vee \tau, 0 \leq \sigma < \tau \leq 1$$

• (σ, τ)-Bipolar (T, S): fuzzy strong hyper ideal of H if, for any $y; z \in H$

$$\inf_{u \in (y \circ z)} \mu_\Phi^P(u) \vee \sigma \geq \mu_\Phi^P(z) \wedge \tau \geq T\left\{\sup_{u \in (y \circ z)} \mu_\Phi^P(u), \mu_\Phi^P(y)\right\} \wedge \tau$$

and

$$\sup_{w \in (y \circ z)} \mu_\Phi^N(w) \wedge \sigma \leq \mu_\Phi^N(z) \vee \tau \leq S\left\{\inf_{w \in (y \circ z)} \mu_\Phi^N(w), \mu_\Phi^N(y)\right\} \vee \tau$$

Definition 3.6. For a "hyper KU-algebra" H, a "bipolar fuzzy set" $\phi = (H, \mu_\Phi^P, \mu_\Phi^N)$ in H is called:
(I) (σ, τ)-Bipolar fuzzy (T, S)-norms hyper KU-ideal of H, if

$$x \ll y \text{ implies } \mu_\Phi^P(x) \vee \sigma \geq \mu_\Phi^P(y) \wedge \tau, \mu_\Phi^N(x) \wedge \sigma \leq \mu_\Phi^N(y) \vee \tau, 0 \leq \sigma < \tau \leq 1$$

$$\mu_\Phi^P(x \circ z) \vee \sigma \geq T\left\{\inf_{u \in (x \circ (y \circ z))} \mu_\Phi^P(u), \mu_\Phi^P(y)\right\} \wedge \tau, 0 \leq \sigma < \tau \leq 1$$

and

$$\mu_\Phi^N(x \circ z) \wedge \sigma \leq S\left\{\sup_{w \in (x \circ (y \circ z))} \mu_\Phi^N(w), \mu_\Phi^N(y)\right\} \vee \tau, 0 \leq \sigma < \tau \leq 1$$

(II) (σ, τ)-Bipolar fuzzy (T, S)-norms weak hyper KU-ideal of H, if any $x; y; z \in H, 0 \leq \sigma < \tau \leq 1$

$$\mu_\Phi^P(0) \vee \sigma \geq \mu_\Phi^P(x \circ z) \vee \sigma \geq T\left\{\inf_{u \in (x \circ (y \circ z))} \mu_\Phi^P(u), \mu_\Phi^P(y)\right\} \wedge \tau$$

and

$$\mu_\Phi^N(0) \wedge \sigma \leq \mu_\Phi^N(x \circ z) \wedge \sigma \leq S\left\{\sup_{w \in (x \circ (y \circ z))} \mu_\Phi^N(w), \mu_\Phi^N(y)\right\} \vee \tau$$

(III) (σ, τ)-Bipolar fuzzy (T, S)-norms strong hyper KU-ideal of H if, for any $x; y; z \in H, 0 \leq \sigma < \tau \leq 1$

$$\inf_{u \in x \circ (y \circ z)} \mu_\Phi^P(u) \vee \sigma \geq \mu_\Phi^P(x \circ z) \vee \sigma \geq T\left\{\sup_{u \in x \circ (y \circ z)} \mu_\Phi^P(u), \mu_\Phi^P(y)\right\} \wedge \tau$$

and

$$\sup_{w \in x \circ (y \circ z)} \mu_\Phi^N(w) \wedge \sigma \leq \mu_\Phi^N(x \circ z) \wedge \sigma \leq S\left\{\inf_{w \in x \circ (y \circ z)} \mu_\Phi^N(w), \mu_\Phi^N(y)\right\} \vee \tau$$

Example 3.7. (1) Consider the hyper KU-algebra in Example 2.2. Define bipolar fuzzy set

	0	1	2
μ^N	-0.7	-0.7	-0.6
μ^P	1	0.5	0

Let $T:[0,1]\times[0,1]\to[0,1]$ be a function defined by $T(\alpha,\beta)=\max\{\alpha+\beta-1,0\}$ for all $\alpha,\beta\in[0,1]$ and $S:[0,1]\times[0,1]\to[0,1]$ be a function defined by $S(\alpha,\beta)=\min\{(\alpha+\beta),1\}$, $\sigma=0.1,\tau=0.4$. Then we can see that $\phi=(H,\mu_\Phi^P,\mu_\Phi^N)$ is (σ,τ)-bipolar fuzzy (T,S)-norms hyper KU-ideal of H and it is (σ,τ)-bipolar fuzzy (T,S)-norms weak hyper KU-ideal of H.

Example 3.8. Consider the hyper KU-algebra H

$\circ$	0	1	2
0	{0}	{1}	{2}
1	{0}	{0}	{2}
2	{0}	{1}	{0,2}

Define bipolar fuzzy set $\phi=(H,\mu_\Phi^P,\mu_\Phi^N)$ in H by

	0	1	2
μ^N	-0.8	-0.6	-0.2
μ^P	0.9	0.5	0.3

Let $T:[0,1]\times[0,1]\to[0,1]$ be a function defined by $T(\alpha,\beta)=\max\{\alpha+\beta-1,0\}$ for all $\alpha,\beta\in[0,1]$ and $S:[0,1]\times[0,1]\to[0,1]$ be a function defined by $S(\alpha,\beta)=\min\{(\alpha+\beta),1\}$, $\sigma=0.2,\tau=0.3$. It is easily that $\phi=(H,\mu_\Phi^P,\mu_\Phi^N)$ is (σ,τ)-bipolar fuzzy (T,S)-norms strong hyper KU-ideal of H

Theorem 3.9. *Any (σ,τ)-bipolar fuzzy (T,S)-norm (weak, strong) hyper KU-ideal is a bipolar (σ,τ)-fuzzy fuzzy (T,S)(weak, strong) hyper ideal.*

Proof. Let $\phi=(H,\mu_\Phi^P,\mu_\Phi^N)$ be a (σ,τ)-bipolar fuzzy (T,S)-norms weak hyper KU-ideal of H, we get for any $x;y;z\in H$

$$\mu_\Phi^P(0)\vee\sigma\geq\mu_\Phi^P(x\circ z)\vee\sigma\geq T\left\{\inf_{u\in(x\circ(y\circ z))}\mu_\Phi^P(u),\mu_\Phi^P(y)\right\}\wedge\tau \text{ ...... (a)}$$

$$\mu_\Phi^N(0)\wedge\sigma\leq\mu_\Phi^N(x\circ z)\wedge\sigma\leq S\left\{\sup_{w\in(x\circ(y\circ z))}\mu_\Phi^N(w),\mu_\Phi^N(y)\right\}\vee\tau \text{ ...... (b)}$$

Put $x=0$ in (a) and (b), we get

$$\mu_\Phi^P(0)\vee\sigma\geq\mu_\Phi^P(0\circ z)\vee\sigma\geq T\left\{\inf_{u\in 0\circ(y\circ z)}\mu_\Phi^P(u),\mu_\Phi^P(y)\right\}\wedge\tau\Rightarrow$$

$$\mu_\Phi^P(0)\vee\sigma\geq\mu_\Phi^P(z)\vee\sigma\geq T\left\{\inf_{u\in(y\circ z)}\mu_\Phi^P(u),\mu_\Phi^P(y)\right\}\wedge\tau$$

and

$$\mu_\Phi^N(0)\wedge\sigma\leq\mu_\Phi^N(0\circ z)\wedge\sigma\leq S\left\{\sup_{w\in 0\circ(y\circ z)}\mu_\Phi^N(w),\mu_\Phi^N(y)\right\}\vee\tau\Rightarrow$$

$$\mu_\Phi^N(0)\wedge\sigma\leq\mu_\Phi^N(z)\wedge\sigma\leq S\left\{\sup_{w\in(y\circ z)}\mu_\Phi^N(w),\mu_\Phi^N(y)\right\}\vee\tau$$

Similarly we can prove that, every $\sigma,\tau)$ bipolar fuzzy (T,S)-norm strong hyper KU-ideal of H is $\sigma,\tau)$ bipolar fuzzy (T,S)-norm strong hyper ideal of H. Ending the proof. □

Definition 3.10. A bipolar fuzzy set $\phi=(H,\mu_\Phi^P,\mu_\Phi^N)$ in H is called (σ,τ)-bipolar fuzzy (T,S)-norm s-weak hyper KU-ideal of H if
(i) $\mu_\Phi^P(0)\vee\sigma\geq\mu_\Phi^P(x)\wedge\tau$, $\mu_\Phi^N(0)\wedge\sigma\leq\mu_\Phi^N(x)\vee\tau$, $\forall\, x\in H$
(ii) for every $x,y,z\in H$ there exists $a,b\in x\circ(y\circ z)$ such that

$$\mu_\Phi^P(x\circ z)\vee\sigma\geq T\left\{\mu_\Phi^P(a),\mu_\Phi^P(y)\right\}\wedge\tau \text{ and } \mu_\Phi^N(x\circ z)\wedge\sigma\leq S\left\{\mu_\Phi^N(b),\mu_\Phi^N(y)\right\}\vee\tau$$

Theorem 3.11. *Every (σ,τ)-bipolar fuzzy (T,S)-norm s-weak hyper KU-ideal of H is (σ,τ)-bipolar fuzzy (T,S)-norm weak hyper KU-ideal of H.*

Proof. Let $\phi = (H, \mu^P_\Phi, \mu^N_\Phi)$ be a (σ, τ)-bipolar fuzzy (T, S)-norms s-weak hyper KU-ideal of H, and let $x; y; z \in H, 0 \leq \sigma < \tau \leq 1$, then there exist $a, b \in x \circ (y \circ z)$ such that

$$\mu^P_\Phi(x \circ z) \vee \sigma \geq T\left\{\mu^P_\Phi(a), \mu^P_\Phi(y)\right\} \wedge \tau \text{ and } \mu^N_\Phi(x \circ z) \wedge \sigma \leq S\left\{\mu^N_\Phi(b), \mu^N_\Phi(y)\right\} \vee \tau$$

Since

$$\mu^P_\Phi(0) \vee \sigma \geq \inf_{c \in (y \circ z)} \mu^P_\Phi(c) \wedge \tau \text{ and } \mu^N_\Phi(0) \wedge \sigma \leq \sup_{d \in (y \circ z)} \mu^N_\Phi(d) \vee \tau,$$

it follows that

$$\mu^P_\Phi(x \circ z) \vee \sigma \geq T\left\{\inf_{c \in x \circ (y \circ z)} \mu^P_\Phi(c), \mu^P_\Phi(y)\right\} \wedge \tau$$

and

$$\mu^N_\Phi(x \circ z) \wedge \sigma \leq S\left\{\sup_{d \in x \circ (y \circ z)} \mu^N_\Phi(d), \mu^P_\Phi(y)\right\} \vee \tau$$

Hence $\phi = (H, \mu^P_\Phi, \mu^N_\Phi)$ is a (σ, τ)-bipolar fuzzy (T, S)-norms weak hyper KU-ideal of H. □

Proposition 3.12. *If $\phi = (H, \mu^P_\Phi, \mu^N_\Phi)$ is (σ, τ)-bipolar fuzzy (T, S)-norms weak hyper KU-ideal of H. satisfying the inf-sup property, then $\phi = (H, \mu^P_\Phi, \mu^N_\Phi)$ is a (σ, τ)-bipolar fuzzy (T, S)-norms s-weak hyper KU-ideal of H.*

Proof. Since $\phi = (H, \mu^P_\Phi, \mu^N_\Phi)$ satisfies the inf-sup property, there exists $a_0.b_0 \in x \circ (y \circ z)$, such that $\mu^P_\Phi(a_0) = \inf_{a \in x \circ (y \circ z)} \mu^P_\Phi(a)$ and $\mu^N_\Phi(b_0) = \sup_{b \in x \circ (y \circ z)} \mu^N_\Phi(b)$. i.e

$$\mu^P_\Phi(a) \vee \sigma \geq \inf_{a \in x \circ (y \circ z)} \mu^P_\Phi(a) \wedge \tau \textit{ and } \mu^N_\Phi(b) \wedge \sigma \leq \sup_{b \in x \circ (y \circ z)} \mu^N_\Phi(b) \wedge \tau$$

It follows that

$$\mu^P_\Phi(x \circ z) \vee \sigma \geq T\left\{\inf_{a \in x \circ (y \circ z)} \mu^P_\Phi(a), \mu^P_\Phi(y)\right\} \wedge \tau \geq T\left\{\mu^P_\Phi(a), \mu^P_\Phi(y)\right\} \wedge \tau$$

and

$$\mu^N_\Phi(x \circ z) \wedge \sigma \leq S\left\{\sup_{b \in x \circ (y \circ z)} \mu^N_\Phi(b), \mu^N_\Phi(y)\right\} \vee \tau \leq S\left\{\mu^N_\Phi(b), \mu^N_\Phi(y)\right\} \vee \tau$$

For every $a, b \in x \circ (y \circ z)$. Hence $\phi = (H, \mu^P_\Phi, \mu^N_\Phi)$ is (σ, τ)-bipolar fuzzy (T, S)-norm s-weak hyper KU-ideal of H. Ending the proof. □

Proposition 3.13. *Let $\phi = (H, \mu^P_\Phi, \mu^N_\Phi)$ be (σ, τ)-bipolar fuzzy (T, S)-norm strong hyper KU-ideal of H and let $x; y; z \in H, 0 \leq \sigma < \tau \leq 1$. Then*
(i) $\mu^P_\Phi(0) \vee \sigma \geq \mu^P_\Phi(x) \wedge \tau, \mu^N_\Phi(0) \wedge \sigma \leq \mu^N_\Phi(x) \vee \tau, \forall x \in H$
(ii) $x \ll y \Rightarrow \mu^P_\Phi(x) \vee \sigma \geq \mu^P_\Phi(y) \wedge \tau$ *and* $\mu^N_\Phi(x) \wedge \sigma \leq \mu^N_\Phi(y) \vee \tau$.
(iii) $\mu^P_\Phi(x \circ z) \vee \sigma \geq T\left\{\mu^P_\Phi(a), \mu^P_\Phi(y)\right\} \wedge \tau, \forall a \in x \circ (y \circ z)$,
$\mu^N_\Phi(x \circ z) \wedge \sigma \leq S\left\{\mu^N_\Phi(b), \mu^N_\Phi(y)\right\} \vee \tau, \forall b \in x \circ (y \circ z)$

Proof. (i) Since $0 \in x \circ x \, \forall x \in H$, we have

$$\mu^P_\Phi(0) \vee \sigma \geq \inf_{a \in x \circ x} \mu^P_\Phi(a) \wedge \tau \geq \mu^P_\Phi(x) \wedge \tau, \mu^N_\Phi(0) \wedge \sigma \leq \sup_{a \in x \circ x} \mu^N_\Phi(a) \vee \tau \leq \mu^N_\Phi(x) \vee \tau.$$

Which proves (i)
(ii) Let $x; y \in H$ be such that $x \ll y$. Then $0 \in y \circ x \, \forall x, y \in H$ and so

$$\sup_{b \in (y \circ x)} \mu^P_\Phi(b) \vee \sigma \geq \mu^P_\Phi(0) \wedge \tau, \ \inf_{w \in (y \circ x)} \mu^N_\Phi(w) \wedge \sigma \leq \mu^N_\Phi(0) \vee \tau$$

It follows from (i) that

$$\mu^P_\Phi(0 \circ x) \vee \sigma = \mu^P_\Phi(x) \vee \sigma \geq T\left\{\sup_{a \in y \circ x} \mu^P_\Phi(a), \mu^P_\Phi(y)\right\} \wedge \tau \geq T\left\{\mu^P_\Phi(0), \mu^P_\Phi(y)\right\} \wedge \tau = \mu^P_\Phi(y) \wedge \tau$$

and

$$\mu^N_\Phi(0 \circ x) \wedge \sigma = \mu^P_\Phi(x) \wedge \sigma \leq S\left\{\inf_{a \in y \circ x} \mu^P_\Phi(a), \mu^P_\Phi(y)\right\} \vee \tau \leq S\left\{\mu^P_\Phi(0), \mu^P_\Phi(y)\right\} \vee \tau = \mu^P_\Phi(y) \vee \tau$$

(iii)

$$\mu_\Phi^P(x\circ z)\vee\sigma \geq T\left\{\sup_{a\in x\circ(y\circ z)}\mu_\Phi^P(a),\mu_\Phi^P(y)\right\}\wedge\tau \geq T\left\{\mu_\Phi^P(a),\mu_\Phi^P(y)\right\}\wedge\tau, \forall a\in x\circ(y\circ z)$$

and

$$\mu_\Phi^N(x\circ z)\wedge\sigma \leq S\left\{\inf_{b\in x\circ(y\circ z)}\mu_\Phi^N(b),\mu_\Phi^N(y)\right\}\vee\tau \leq S\left\{\mu_\Phi^N(b),\mu_\Phi^N(y)\right\}\vee\tau, \forall b\in x\circ(y\circ z)$$

we conclude that (iii) is true. Ending the proof.
Note that, in a finite hyper KU-algebra, every bipolar fuzzy set satisfies inf-sup property. Hence the concept of (σ,τ)-bipolar fuzzy (T,S)-norm weak hyper KU-ideals and (σ,τ)-bipolar fuzzy (T,S)-norm s-weak hyper KU-ideals coincide in a finite hyper KU-algebra. □

Proposition 3.14. *Let $\phi=(H,\mu_\Phi^P,\mu_\Phi^N)$ be a (σ,τ)-bipolar fuzzy (T,S)-norm hyper KU-ideal of H, then:*

$$\mu_\Phi^P(0)\vee\sigma\geq\mu_\Phi^P(x)\wedge\tau, \mu_\Phi^N(0)\wedge\sigma\leq\mu_\Phi^N(x)\vee\tau,$$

if $\phi=(H,\mu_\Phi^P,\mu_\Phi^N)$ satisfies the inf-sup property, then

$$\mu_\Phi^P(x\circ z)\vee\sigma\geq T\left\{\mu_\Phi^P(a)\mu_\Phi^P(y)\right\}\wedge\tau \text{ and } \mu_\Phi^N(x\circ z)\wedge\sigma\leq S\left\{\mu_\Phi^N(b)\mu_\Phi^N(y)\right\}\vee\tau,$$

for every $a,b\in x\circ(y\circ z)$.

Proof. Since $0\ll x\,\forall x\in H$, it follows from Definition 3.5. (I) that $\mu_\Phi^P(0)\vee\sigma\geq\mu_\Phi^P(x)\wedge\tau$ and $\mu_\Phi^N(0)\wedge\sigma\leq\mu_\Phi^N(x)\vee\tau$. Since $\phi=(H,\mu_\Phi^P,\mu_\Phi^N)$ satisfies the inf-sup property, there exists $a_0,b_0\in x\circ(y\circ z)$, such that

$$\mu_\Phi^P(a_0)=\inf_{a\in x\circ(y\circ z)}\mu_\Phi^P(a) \text{ and } \mu_\Phi^N(b_0)=\sup_{b\in x\circ(y\circ z)}\mu_\Phi^N(b)$$

Hence

$$\mu_\Phi^P(x\circ z)\vee\sigma\geq T\left\{\inf_{a\in x\circ(y\circ z)}\mu_\Phi^P(a),\mu_\Phi^P(y)\right\}\wedge\tau\geq T\left\{\mu_\Phi^P(a_0),\mu_\Phi^P(y)\right\}\wedge\tau$$

$$\mu_\Phi^N(x\circ z)\wedge\sigma\leq S\left\{\sup_{b\in x\circ(y\circ z)}\mu_\Phi^N(b),\mu_\Phi^N(y)\right\}\vee\tau\leq S\left\{\mu_\Phi^N(b_0),\mu_\Phi^N(y)\right\}\vee\tau.$$ □

Corollary 3.15. *(1) Every (σ,τ) bipolar fuzzy (T,S)-norm hyper KU-ideal is a bipolar fuzzy (T,S)-norm weak hyper KU-ideal.*
(2) If $\phi=(H,\mu_\Phi^P,\mu_\Phi^N)$ (σ,τ)-bipolar fuzzy (T,S)-norm hyper KU-ideal satisfies the inf-sup property, then $\phi=(H,\mu_\Phi^P,\mu_\Phi^N)$ is (σ,τ)-bipolar fuzzy (T,S)-norm s-weak hyper KU-ideal of H.

Theorem 3.16. *Let $\phi=(H,\mu_\Phi^P,\mu_\Phi^N)$ be (σ,τ)-bipolar fuzzy (T,S)-norm set, then $\phi=(H,\mu_\Phi^P,\mu_\Phi^N)$ is (σ,τ)-bipolar fuzzy (T,S)-norm weak hyper KU-ideal of H if and only if the positive level set $\Phi_{\alpha\wedge\tau}^P=\left\{x\in X|\mu_\Phi^P\geq\alpha\wedge\tau\right\}$ and negative level set $\Phi_{\beta\vee\tau}^N=\left\{x\in X|\mu_\Phi^N\leq\beta\vee\tau\right\}$ for every $(\alpha,\beta)\in[0,1]\times[-1,0]$, are weak hyper KU-ideal of H.*

Proof. Assume that $\phi=(H,\mu_\Phi^P,\mu_\Phi^N)$ is (σ,τ)-bipolar fuzzy (T,S)-norm weak hyper KU-ideal of H and $\Phi_{\alpha\wedge\tau}^P\neq\Phi\neq\Phi_{\beta\vee\tau}^N$ for every $(\alpha,\beta)\in[0,1]\times[-1,0]$. It clear from

$$\mu_\Phi^P(0)\vee\sigma\geq\mu_\Phi^P(x\circ z)\wedge\tau\geq T\left\{\inf_{u\in x\circ(y\circ z)}\mu_\Phi^P(u),\mu_\Phi^P(y)\right\}\wedge\tau \text{ ....(a)}$$

$$\mu_\Phi^N(0)\wedge\sigma\leq\mu_\Phi^N(x\circ z)\vee\tau\leq S\left\{\sup_{w\in x\circ(y\circ z)}\mu_\Phi^N(w),\mu_\Phi^N(y)\right\}\vee\tau \text{ ....(b)}$$

that $0\in\Phi_{\alpha\wedge\tau}\bigcap\Phi_{\beta\vee\tau}$. Let $x;y;z\in H$ be such that $x\circ(y\circ z)\subseteq\Phi_{\alpha\wedge\tau}^P$ and $y\in\Phi_{\alpha\wedge\tau}^P$. Then for any $a\in x\circ(y\circ z), a\in\Phi_{\alpha\wedge\tau}^P$. It follows that $\mu_\Phi^P(a)\vee\sigma\geq\alpha\wedge\tau$ so that $\inf_{a\in x\circ(y\circ z)}\mu_\Phi^P(a)\wedge\tau\geq\alpha\wedge\tau$, thus and so $x\circ z\subseteq\Phi_{\alpha\wedge\tau}^P$,

$\mu_\Phi^P(x\circ z)\vee\sigma\geq T\left\{\inf_{a\in x\circ(y\circ z)}\mu_\Phi^P(a),\mu_\Phi^P(y)\right\}\wedge\tau\geq\alpha\wedge\tau$ therefore $\Phi_{\alpha\wedge\tau}^P$ is weak hyper KU-ideal of H.
Now let $x;y;z\in H$ be such that $x\circ(y\circ z)\subseteq\Phi_{\beta\vee\tau}^N$ and $y\in\Phi_{\beta\vee\tau}^N$. Then for any $b\in x\circ(y\circ z), b\in\Phi_{\beta\vee\tau}^N$. It follows that $\mu_\Phi^N(b)\leq\beta\vee\tau$. so that $\sup_{b\in x\circ(y\circ z)}\mu_\Phi^N(b)\leq\beta\vee\tau$.

Using $\mu^N_\Phi(x \circ z) \wedge \sigma \leq S\left\{\sup_{w\in x\circ(y\circ z)} \mu^N_\Phi(w), \mu^N_\Phi(y)\right\} \vee \tau \leq \beta \vee \tau$, which implies that $x \circ z \subseteq \Phi^N_{\beta\vee\tau}$. Consequently $\Phi^N_{\beta\vee\tau}$ is weak hyper KU-ideal of H. □

Theorem 3.17. *Let $\phi = (H, \mu^P_\Phi, \mu^N_\Phi)$ be bipolar fuzzy (T,S)-norm set, then $\phi = (H, \mu^P_\Phi, \mu^N_\Phi)$ is (σ,τ)-bipolar fuzzy (T,S)-norm hyper KU-ideal of H if and only if the positive level set $\Phi^P_{\alpha\wedge\tau} = \left\{x \in X | \mu^P_\Phi \geq \alpha \wedge \tau\right\}$ and negative level set $\Phi^N_{\beta\vee\tau} = \left\{x \in X | \mu^N_\Phi \leq \beta \vee \tau\right\}$ for every $(\alpha,\beta) \in [0,1] \times l-1,0]$, are hyper KU-ideal of H.*

Proof. Assume that $\phi = (H, \mu^P_\Phi, \mu^N_\Phi)$ is (σ,τ)-bipolar fuzzy (T,S)-norm hyper KU-ideal of H and $\Phi^P_{\alpha\wedge\tau} \neq \Phi \neq \Phi^N_{\beta\vee\tau}$ for every $(\alpha,\beta) \in [0,1] \times [-1,0]$. It clear that $0 \in \Phi^P_{\alpha\wedge\tau} \bigcap \Phi^N_{\beta\vee\tau}$. Let $x; y; z \in H$ be such that $x \circ (y \circ z) \subseteq \Phi^P_{\alpha\wedge\tau}$ and $y \in \Phi^P_{\alpha\wedge\tau}$. Then for any $a \in x \circ (y \circ z), a \in \Phi^P_{\alpha\wedge\tau}$. It follows that $\mu^P_\Phi(a) \geq \alpha \wedge \tau$ so that $\inf_{a\in x\circ(y\circ z)} \mu^P_\Phi(a) \geq \alpha \wedge \tau$, thus

$$\mu^P_\Phi(x \circ z) \vee \sigma \geq T\left\{\inf_{a\in x\circ(y\circ z)} \mu^P_\Phi(a), \mu^P_\Phi(y)\right\} \wedge \tau \geq \alpha \wedge \tau$$

and so $x \circ z \subseteq \Phi^P_{\alpha\wedge\tau}$, there for $\Phi^P_{\alpha\wedge\tau}$ is hyper KU-ideal of H.
Now let $x; y; z \in H$ be such that $x \circ (y \circ z) \subseteq \Phi^N_{\beta\vee\tau}$ and $y \in \Phi^N_{\beta\vee\tau}$. Then for any $b \in x \circ (y \circ z), b \in \Phi^N_{\beta\vee\tau}$. It follows that $\mu^N_\Phi(b) \leq \beta \vee \tau$, so that $\sup_{b\in x\circ(y\circ z)} \mu^N_\Phi(b) \leq \beta \vee \tau$. Using

$$\mu^N_\Phi(x \circ z) \wedge \sigma \leq S\left\{\sup_{w\in x\circ(y\circ z)} \mu^N_\Phi(w), \mu^N_\Phi(y)\right\} \vee \tau \leq \beta \vee \tau,$$

which implies that $x \circ z \subseteq \Phi^N_{\beta\vee\tau}$. Consequently is hyper KU-ideal of H.
Conversely, suppose that the nonempty positive and negative level sets $\Phi^P_{\alpha\wedge\tau}, \Phi^N_{\beta\vee\tau}$ are is hyper KU-ideals of H for every $\alpha,\beta) \in [0,1] \times [-1,0]$. Let $\mu^P_\Phi(x) = \alpha \wedge \tau, \mu^N_\Phi(x)\beta \vee \tau$ for $x \in X$, then by $0 \in \Phi^P_{\alpha\wedge\tau} \in \Phi^N_{\beta\vee\tau}$. It follows that $\mu^P_\Phi(0) \geq \alpha \wedge \tau, \mu^N_\Phi(0) \leq \beta \vee \tau$ and so $\mu^P_\Phi(0) \vee \sigma \geq \mu^P_\Phi(x) \wedge \tau$ and $\mu^N_\Phi(0) \wedge \sigma \leq \mu^N_\Phi(x) \vee \tau$. Now let

$$T\left\{\inf_{a\in x\circ(y\circ z)} \mu^P_\Phi(a), \mu^P_\Phi(y)\right\} \wedge \tau \geq \alpha \wedge \tau \text{ and } S\left\{\sup_{w\in x\circ(y\circ z)} \mu^N_\Phi(w), \mu^N_\Phi(y)\right\} \vee \tau \leq \beta \vee \tau.$$ □

Corollary 3.18. *Every (σ,τ)-bipolar fuzzy (T,S)-norm strong hyper KU-ideal is both a (σ,τ)-bipolar fuzzy (T,S)-norm s-weak hyper KU-ideal (a (σ,τ) bipolar fuzzy (T,S)-norm weak hyper ideal) and (σ,τ)-bipolar fuzzy (T,S)-norm hyper KU-ideal.*

Proof. Straight forward. □

Proposition 3.19. *let $\phi = (H, \mu^P_\Phi, \mu^N_\Phi)$ be (σ,τ)-bipolar fuzzy (T,S)-norm hyper KU-ideal of H and let $x; y; z \in H$. Then*
(i) $\mu^P_\Phi(0) \vee \sigma \geq \mu^P_\Phi(x) \wedge \tau, \mu^N_\Phi(0) \wedge \sigma \leq \mu^N_\Phi(x) \vee \tau$
(ii) if $\phi = (H, \mu^P_\Phi, \mu^N_\Phi)$ satisfies the inf-sup property, then

$$\mu^P_\Phi(x \circ z) \vee \sigma \geq T\left\{\mu^P_\Phi(a), \mu^P_\Phi(y)\right\} \wedge \tau \; for\ some\ \; a \in x \circ (y \circ z)$$

and

$$\mu^N_\Phi(x \circ z) \wedge \sigma \leq S\left\{\mu^N_\Phi(w), \mu^N_\Phi(y)\right\} \vee \tau \; for\ some\ \; w \in x \circ (y \circ z)$$

Proof. Straight forward. □

Corollary 3.20. *(i) Every (σ,τ)-bipolar fuzzy (T,S)-norm hyper KU-ideal of H is (σ,τ)-bipolar fuzzy (T,S)-norm weak hyper KU-ideal of H.*
(ii) If $\phi = (H, \mu^P_\Phi, \mu^N_\Phi)$ is (σ,τ)-bipolar fuzzy (T,S)-norm hyper KU-ideal of H satisfying inf-sup property, then $\phi = (H, \mu^P_\Phi, \mu^N_\Phi)$ is (σ,τ)-bipolar fuzzy (T,S)-norm s-weak Hyper KU-ideal of H.

Proof. Straightforward. □

The following example shows that the converse of Corollary 3.17 and 3.19 (i). may not be true.

Example 3.21. (1) Consider the hyper KU-algebra H

$\circ$	0	1	2
0	{0}	{1}	{2}
1	{0}	{0,1}	{1,2}
2	{0}	{0,1}	{0,1,2}

Define bipolar fuzzy set μ in H by

	0	1	2
μ^N	-0.7	-0.7	-0.6
μ^P	1	0.5	0

Let $T : [0,1] \times [0,1] \to [0,1]$ be a function defined by $T(\alpha,\beta) = \max\{\alpha + \beta - 1, 0\}$ for all $\alpha,\beta \in [0,1]$ and $S : [0,1] \times [0,1] \to [0,1]$ be a function defined by $S(\alpha,\beta) = \min\{(\alpha + \beta), 1\}$, $\sigma = 0.2, \tau = 0.4$. Then we can see that $\phi = (H, \mu^P_\Phi, \mu^N_\Phi)$ is $(\sigma, \tau$-bipolar fuzzy (T, S)-norm hyper KU-ideal of H. and hence it is also $(\sigma, \tau$ bipolar fuzzy (T, S)-norm weak hyper KU-ideal of H. But $\phi = (H, \mu^P_\Phi, \mu^N_\Phi)$ is not $(\sigma, \tau$ bipolar fuzzy (T, S)-norm strong hyper KU-ideal of H since

$$T\left\{\sup_{a\in 0\circ(1\circ 2)} \mu^P_\Phi(a), \mu^P_\Phi(y)\right\} \wedge 0.4 \geq T\left\{\mu^P_\Phi(1), \mu^P_\Phi(1)\right\} \wedge 0.4 = \frac{1}{2} \wedge 0.4 = 0.4 \geq 0 \vee 0.2 = 0.2 \geq= \mu^P_\Phi(2) \vee 0.2. \forall a \in 0 \circ (1 \circ 2)$$

(2) Consider the hyper KU-algebra H in Example 3.14. Define bipolar fuzzy set $\phi = (H, \mu^P_\Phi, \mu^N_\Phi)$ in H by

	0	1	2
μ^N_Φ	-0.7	-0.7	-0.6
μ^P_Φ	1	0	0.5

Let $T : [0,1] \times [0,1] \to [0,1]$ be a function defined by $T(\alpha,\beta) = \max\{\alpha + \beta - 1, 0\}$ for all $\alpha,\beta \in [0,1]$ and $S : [0,1] \times [0,1] \to [0,1]$ be a function defined by $S(\alpha,\beta) = \min\{(\alpha + \beta), 1\}$, $\sigma = 0.2$, $\tau = 0.4$. Then $\phi = (H, \mu^P_\Phi, \mu^N_\Phi)$ is (σ, τ)-bipolar fuzzy (T, S)-norm weak hyper KU-ideal of H but it is not a (σ, τ)-bipolar fuzzy (T, S)-norm hyper KU-ideal of H since $1 \ll 2$ but $\mu^P_\Phi(1) \vee 0.2 \ngeq \mu^P_\Phi(2) \wedge 0.4$.

Theorem 3.22. *If $\phi = (H, \mu^P_\Phi, \mu^N_\Phi)$ is bipolar fuzzy (T, S)-norm strong hyper KU-ideal of H, then the set $\mu_{t,s} = \left\{x \in H, \mu^P_\Phi(x) \geq t, \mu^N_\Phi(x) \leq s\right\}$ is a strong hyper KU-ideal of H, when $\mu_{t,s} \neq \Phi$, for $t \in [0,1]$, $s \in [-1,0]$.*

Proof. Let $\phi = (H, \mu^P_\Phi, \mu^N_\Phi)$ be a fuzzy (T, S)-norm strong hyper KU-ideal of H and $\mu_{t,s} \neq \Phi$ for $t \in [0,1]$. $s \in [-1,0]$. Then there $a, b \in \mu_{t,s}$ and so $\mu^P_\Phi(a) \geq t, \mu^N_\Phi(b) \leq s$. By Proposition 3.12 (i), $\mu^P_\Phi 0 \geq \mu^P_\Phi(a) \geq t, \mu^N_\Phi(0) \leq \mu^N_\Phi(b) \leq s$ and so $0 \in \mu_{t,s}$. Let $x, y, z \in H$ such that $x \circ (y \circ z) \cap \mu_{t,s} \neq \Phi$ and $y \in \mu_{t,s}$, Then there exists $a_0, b_0 \in x \circ (y \circ z) \cap \mu_{t,s}$ and hence $\mu^P_\Phi(a_0) \geq t, \mu^N_\Phi(b_0) \leq s$. By definition 3.5 (iii), we have

$$\mu^P_\Phi(x \circ z) \geq T\left\{\sup_{a\in x\circ(y\circ z)} \mu^P_\Phi(a), \mu^P_\Phi(y)\right\} \geq T\left\{\mu^P_\Phi(a_0), \mu(y)\right\} \geq T\{t, t\} = t$$

and

$$\mu^N_\Phi(x \circ z) \leq S\left\{\inf_{a\in x\circ(y\circ z)} \mu^N_\Phi(a), \mu^N_\Phi(y)\right\} \leq S\left\{\mu^N_\Phi(b_0), \mu^N_\Phi(y)\right\} \leq S\{s, s\} = s$$

So $(x \circ z) \in \mu_{t,s}$. It follows that $\mu_{t,s}$ is a strong hyper KU-ideal of H □

Theorem 3.23. *Let $\phi = (H, \mu^P_\Phi, \mu^N_\Phi)$ is bipolar fuzzy (T, S)-norm in H satisfying the inf-sup property,. If the set $\mu_{t,s} = \left\{x \in H, \mu^P_\Phi(x) \geq t, \mu^N_\Phi(x) \leq s\right\} \neq \Phi$ is a strong hyper KU-ideal of H for all $t \in [0,1]$. $s \in [-1,0]$, then $\phi = (H, \mu^P_\Phi, \mu^N_\Phi)$ is bipolar fuzzy (T, S)-norm strong hyper KU-ideal of H.*

Proof. Assume that $\mu_{t,s} \neq \Phi$ is a strong hyper KU-ideal of H for all $t \in [0,1]$. $s \in [-1,0]$. Then there is $x \in \mu_{t,s}$ such that $x \circ x \ll x \in \mu_{t,s}$. Using Proposition 2.8, we have $x \circ x \subseteq \mu_{t,s}$. Thus for $a, b \in x \circ x$, we have $a, b \in \mu_{t,s}$ and hence

$\mu_\Phi^P(a) \geq t, \mu_\Phi^N(b) \leq s$. It follows that $\inf\limits_{a\in x\circ(y\circ z)} \mu_\Phi^P(a) \geq t = \mu_\Phi^P(x)$ and $\sup\limits_{b\in x\circ(y\circ z)} \mu_\Phi^N(b) \leq s = \mu_\Phi^N(x)$. Moreover let $x, y, z \in H$ and $\mu_{\alpha',\beta'}$, where

$$\alpha' = T\left\{\sup_{a\in x\circ(y\circ z)} \mu_\Phi^P(a), \mu_\Phi^P(y)\right\}, \ \beta' = S\left\{\inf_{b\in x\circ(y\circ z)} \mu_\Phi^N(b), \mu_\Phi^N(y)\right\}$$

By hypothesis $\mu_{\alpha',\beta'}$ is a strong hyper KU-ideal of H.
Since $\phi = (H, \mu_\Phi^P, \mu_\Phi^N)$ satisfies the inf-sup property there is $a_0, b_0 \in x\circ(y\circ z)$, such that $\mu_\Phi^P(a_0) = \sup\limits_{a\in x\circ(y\circ z)} \mu_\Phi^P(a), \mu_\Phi^N(b_0) = \inf\limits_{b\in x\circ(y\circ z)} \mu_\Phi^N(b)$. Thus

$$\mu_\Phi^P(a_0) = \sup_{a\in x\circ(y\circ z)} \mu_\Phi^P(a) \geq T\left\{\sup_{a\in x\circ(y\circ z)} \mu_\Phi^P(a), \mu_\Phi^P(y)\right\} = \alpha'$$

and

$$\mu_\Phi^N(b_0) = \inf_{b\in x\circ(y\circ z)} \mu_\Phi^N(b) \leq S\left\{\inf_{b\in x\circ(y\circ z)} \mu_\Phi^N(b), \mu_\Phi^N(y)\right\} = \beta'.$$

This shows that $a_0, b_0 \in \mu_{\alpha',\beta'}, a_0, b_0 \in x\circ(y\circ z)\cap\mu_{\alpha',\beta'}$ and hence $x\circ(y\circ z)\cap\mu_{\alpha',\beta'} \neq \Phi$. Combining $y \in \mu_{\alpha',\beta'}$ and noticing that any bipolar fuzzy (T, S)-norm (weak, strong) hyper KU-ideal is a bipolar fuzzy (T, S)-norm (weak, strong) hyper ideal., we get $x\circ z \in \mu_{\alpha',\beta'}$. Hence

$$\mu_\Phi^P(x\circ z) \geq T\left\{\sup_{a\in x\circ(y\circ z)} \mu_\Phi^P(a), \mu_\Phi^P(y)\right\}, \mu_\Phi^N(x\circ z) \leq S\left\{\inf_{b\in x\circ(y\circ z)} \mu_\Phi^N(b), \mu_\Phi^N(y)\right\}$$

Therefore $\phi = (H, \mu_\Phi^P, \mu_\Phi^N)$ is bipolar fuzzy (T, S)-norm strong hyper KU-ideal of H. □

Acknowledgment

The author is greatly appreciate the referees for their valuable comments and suggestions for improving the paper.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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