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Strong Roman Domination Number of Complementary Prism Graphs

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Abstract. Let $G = (V, E)$ be a simple graph with vertex set $V = V(G)$, edge set $E = E(G)$ and from maximum degree $\Delta = \Delta(G)$. Also let $f : V \to \{0, 1, ..., \lceil \frac{\Delta}{2} \rceil + 1\}$ be a function that labels the vertices of G. Let $V_i = \{v \in V : f(v) = i\}$ for $i = 0, 1$ and let $V_2 = V - (V_0 \bigcup V_1) = \{w \in V : f(w) \geq 2\}$. A function f is called a strong Roman dominating function (StRDF) for G, if every $v \in V_0$ has a neighbor w, such that $w \in V_2$ and $f(w) \geq 1 + \lceil \frac{1}{2} |N(w) \bigcap V_0| \rceil$. The minimum weight, $\omega(f) = f(V) = \Sigma_{v \in V} f(v)$, over all the strong Roman dominating functions of G, is called the strong Roman domination number of G and we denote it by $\gamma_{StR}(G)$. An StRDF of minimum weight is called a $\gamma_{StR}(G)$-function. Let $\overline{G}$ be the complement of G. The complementary prism $G\overline{G}$ of G is the graph formed from the disjoint union G and $\overline{G}$ by adding the edges of a perfect matching between the corresponding vertices of G and $\overline{G}$. In this paper, we investigate some properties of Roman, double Roman and strong Roman domination number of $G\overline{G}$.

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1. Introduction

Let $G = (V, E)$ be a simple undirected graph with the set of vertices $V = V(G)$ of order $n = |V|$ and the set of edges $E = E(G)$. We refer the reader to [16] for any terminology and notation not here in. We denote minimum degree of a graph G with $\delta(G)$ and maximum degree with $\Delta(G)$. The open neighborhood of a vertex $v \in V$ is the set $N(v) = \{u : uv \in E(G)\}$. The open neighborhood of a set $S \subseteq V$ is the set $N(S) = \bigcup_{v \in S} N(v)$. The closed neighborhood of a set $S \subseteq V$ is the set $N[S] = N(S) \bigcup S$. Let E_v be the set of edges incident with v in G that is, $E_v = \{uv \in E(G) : u \in N(v)\}$. We denote the degree of v by $d_G(v) = |E_v|$. A vertex of degree zero is called an isolated vertex. A set $M \subseteq E(G)$ is called a matching if no two edges of M have a common end vertex. If M is a matching in a graph G with the property that every vertex of G is incident with an edge of M, then M is a perfect matching in G. The vertex chromatic number $\chi(G)$ of G is the minimum integer k such that G is k-colorable.

A complementary prism of G, denoted by $G\overline{G}$, is the graph obtained by taking a copy of G and a copy of its complement $\overline{G}$ and then joining corresponding vertices by an edge. According to the definition of complementary prism of G, it is easy to see that $G\overline{G}$, contains a perfect matching. We note that complementary prisms are a generalization of

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the Petersen graph. For example, the graph $C_5\overline{C_5}$ is the Petersen graph. Also if $G = K_n$ the graph $K_n\overline{K_n}$ is the corona $K_n \circ K_1$, where the corona $H \circ K_1$, of a graph H is the graph obtained from H by attaching a pendent edge to each vertex of H. For notational convenience, we let $V(\overline{G}) = \overline{V}$. Also, note that $V(G\overline{G}) = V \cup \overline{V}$. To simplify our discussion of complementary prisms, we say simply G and $\overline{G}$ to refer to the subgraph copies of G and $\overline{G}$, respectively, in $G\overline{G}$. Also, for a vertex v of G, we let $\overline{v}$ be the corresponding vertex in $\overline{G}$, and for a set $X \subseteq V$, we let $\overline{X}$ be the corresponding set of vertices in $\overline{V}$. Further, for any function f on $G\overline{G}$, we let $\omega(f_V)$ denote the weight of f on G, and $\omega(f_{\overline{V}})$ denote the weight of f on $\overline{G}$. Clearly, $G\overline{G}$ is isomorphic to $\overline{G}G$, so our results stated in terms of G also apply to $\overline{G}$ unless otherwise stated. A complementary prism is a specific case of complementary product of graphs introduced by Haynes et al. [12] in 2009. Haynes et al. ([10–12]) studied some parameters of complementary prism of graphs such as the vertex independence number, the chromatic number, the domination number, total domination number, independent domination number and Roman domination number.

Let $G = (V, E)$ be a graph, $X \subseteq V$ and $B(X)$ be the set of vertices in $V - X$ that have a neighbor in the set X. We define the differential of a set X to be $\partial(X) = |B(X)| - |X|$ [14], and the differential of a graph to be equal to $\partial(G)$ =max $\{\partial(X) : X \subseteq V\}$. A set D satisfying $\partial(D) = \partial(G)$ is called a ∂-set or differential set. One of the variations of the differential of a graph is the B-differential of a graph. We denote this parameter with $\Psi(G)$ and we define $\Psi(G)$ =max $\{|B(X)| : X \subseteq V\}$ [14]. We define the B-differential of a set $X \subseteq V(G)$ to be $\Psi(X) = |B(X)|$ [14]. A set X satisfying $\Psi(X) = \Psi(G)$ is called a $\Psi(G)$-set or B-differential set.

A set $S \subseteq V$ is a dominating set if $N[S] = V$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set $S \subseteq V$ is called a $\gamma(G)$-set if $|S| = \gamma(G)$ [16].

For a graph $G = (V, E)$, let $f : V \to \{0, 1, 2\}$ be a function, and let $f = (V_0, V_1, V_2)$ be the ordered partition of V induced by f, where $V_i = \{v \in V(G) : f(v) = i\}$. A Roman dominating function (or just an RDF) on graph G is a function $f : V \to \{0, 1, 2\}$ such that if $v \in V_0$ for some $v \in V$, then there exists a vertex $w \in N(v)$ such that $f(w) = 2$. The weight of a Roman dominating function is the sum $w_f = \sum_{v\in V(G)} f(v)$, and the minimum weight of w_f for every Roman dominating function f on G is called Roman domination number of G. We denote this number with $\gamma_R(G)$. A Roman dominating function of G with weight $\gamma_R(G)$ is called a γ_R-function of G. For more on Roman domination number see for example [5, 6].

Let $f : V \to \{0, 1, 2, 3\}$ be a function, and let $f = (V_0, V_1, V_2, V_3)$ be the ordered partition of V induced by f, where $V_i = \{v \in V(G) : f(v) = i\}$. A double Roman dominating function (or just a DRDF) on graph G is a function $f : V \to \{0, 1, 2, 3\}$ such that the following conditions are met:

(a) if $f(v) = 0$, then vertex v must have at least two neighbors in V_2 or one neighbor in V_3.

(b) if $f(v) = 1$, then vertex v must have at least one neighbor in $V_2 \bigcup V_3$.

The weight of a double Roman dominating function is the sum $w_f = \sum_{v\in V(G)} f(v)$, and the minimum weight of w_f for every double Roman dominating function f on G is called double Roman domination number of G. We denote this number with $\gamma_{dR}(G)$. A double Roman dominating function of G with weight $\gamma_{dR}(G)$ is called a γ_{dR}-function of G [4].

Also let $f : V \to \{0, 1, ..., \lceil\frac{\Delta}{2}\rceil + 1\}$ be a function that labels the vertices of G. Let $V_i = \{v \in V : f(v) = i\}$ for $i = 0, 1$ and let $V_2 = V - (V_0 \bigcup V_1) = \{w \in V : f(w) \geq 2\}$. A function f is called a strong Roman dominating function (StRDF) for G, if every $v \in V_0$ has a neighbor w, such that $w \in V_2$ and $f(w) \geq 1 + \lceil\frac{1}{2}|N(w) \bigcap V_0|\rceil$. The minimum weight, $\omega(f) = f(V) = \Sigma_{v\in V} f(v)$, over all the strong Roman dominating functions of G, is called the strong Roman domination number of G and we denote it by $\gamma_{StR}(G)$. An StRDF of minimum weight is called a $\gamma_{StR}(G)$-function [2].

The following results are useful for the proofs of our main contributions in this paper.

Theorem A [14]. For any graph G of order n, $\Psi(G) = n - \gamma(G)$.

Theorem B [3]. If G is a graph of order n, then $\gamma_R(G) = n - \partial(G)$.

Theorem C [2]. If G a graph of order n, then $\gamma_{StR}(G) \geq \lceil \frac{n+1}{2} \rceil$.

Theorem D [6]. If G is a connected graph of order n, then $\gamma_R(G) \leq \frac{4n}{5}$.

Theorem E [9]. If a graph G has no isolated vertices, then $\gamma(G) \leq \frac{n}{2}$.

Theorem F [15]. Let G be a graph without isolated vertices. Then $\gamma_{dR}(G) \leq 2n - \Psi(G) - \partial(G)$.

Theorem G [13]. Let G be a graph. Then $G\overline{G}$ is even order and connected.

Theorem H [8]. For any graph G, $\gamma(G\overline{G}) \leq \gamma(G) + \gamma(\overline{G})$.
Theorem I [6]. For any graph G, $\gamma(G) + \gamma(\overline{G}) \leq n + 1$.

Theorem J [7]. Let G be a graph. Then the following hold.
(1) $\gamma(\overline{G}) \leq \delta(G) + 1$.
(2) $\gamma(\overline{G}) \leq \chi(G)$.
Theorem K [2]. Let G be a graph of order n. Then $\gamma_{StR}(G) \leq n - \lfloor \frac{\Delta}{2} \rfloor$.

2. γ_R and γ_{dR} of complementary prism of a graph

In this section we investigate Roman domination number of $(G\overline{G})$ and double Roman domination number of $(G\overline{G})$.

Theorem 2.1. *For any graph G, $\Psi(G) \leq \partial(G\overline{G}) \leq \partial(G) + \partial(\overline{G})$.*

Proof. Let $X \subseteq V(G)$ be a $\Psi(G)$-set on graph G. We consider the set $Y = X$ as a subset of $V(G\overline{G})$, that is $Y \subseteq V(G\overline{G})$. Thus, by the definition differential of graphs, $\partial(G\overline{G}) \geq \partial_{G\overline{G}}(Y) = |B_{G\overline{G}}(Y)| - |Y| = |B_{G\overline{G}}(X)| - |X|$. Now by the definition of $G\overline{G}$, we have $|B_{G\overline{G}}(X)| = |B_G(X)| + |\overline{X}|$. Hence, $\partial(G\overline{G}) \geq \partial_{G\overline{G}}(X) = |\overline{X}| + |B_G(X)| - |X| = |B_G(X)| = \Psi(X) = \Psi(G)$. Now to prove the second part of inequality. Suppose that a set $Y = X \cup Z \subseteq V(G\overline{G})$ is a $\partial(G\overline{G})$-set on graph $G\overline{G}$ such that $X \subseteq V(G)$ and $Z \subseteq V(\overline{G})$. We have $\partial(X) \leq \partial(G)$, $\partial(Z) \leq \partial(\overline{G})$ and $|B(X \cup Z)| \leq |B(X)| + |B(Z)|$. Also we have $\partial(G\overline{G}) = \partial(Y) = \partial(X \cup Z) = |B(X \cup Z)| - |X \cup Z|$. Since $X \cap Z = \emptyset$, we conclude $\partial(G\overline{G}) = \partial(Y) = |B(X \cup Z)| - |X \cup Z| = |B(X \cup Z)| - |X| - |Z| \leq |B(X)| + |B(Z)| - |X| - |Z| = \partial(X) + \partial(Z) \leq \partial(G) + \partial(\overline{G})$. □

Theorem 2.2. *For any graph G of order n, $\gamma_R(G\overline{G}) \leq n + \gamma(G)$.*

Proof. By Theorems A, B and Theorem 2.1 we have $\gamma_R(G\overline{G}) = 2n - \partial(G\overline{G}) \leq 2n - \Psi(G) = 2n - (n - \gamma(G)) = n + \gamma(G)$. □

As an immediate result, we will improve the bound of γ_R in Theorem D for complementary prism of a graph $G\overline{G}$.

Corollary 2.3. *If G is a graph with no isolated vertices, then $\gamma_R(G\overline{G}) \leq \frac{3n}{2}$.*

Proof. By Theorem 2.1 and Theorem E, the result holds. □

We now establish the relation between double Roman domination number of complementary prism of a graph G and domination number of G.

Theorem 2.4. *For any graph G of order n, $\gamma_{dR}(G\overline{G}) \leq 2n + 1 + \gamma(G)$.*

Proof. By Theorem G, the graph $G\overline{G}$ is connected. Thus, it has no isolated vertex. Now according to the Theorem F, we have $\gamma_{dR}(G\overline{G}) \leq 4n - \Psi(G\overline{G}) - \partial(G\overline{G})$. But by Theorems A, 2.1, we conclude

$$\begin{aligned}\gamma_{dR}(G\overline{G}) &\leq 4n - (2n - \gamma(G\overline{G})) - \Psi(G) = 4n - (2n - \gamma(G\overline{G})) - (n - \gamma(G))\\ &= n + \gamma(G\overline{G}) + \gamma(G).\end{aligned}$$

On the other hand, by Theorems H, I we have $\gamma_{dR}(G\overline{G}) \leq n + \gamma(G) + \gamma(\overline{G}) + \gamma(G) \leq n + n + 1 + \gamma(G) = 2n + 1 + \gamma(G)$. □

Corollary 2.5. *Let G be a graph of order n and without isolated vertex. Then $\gamma_{dR}(G\overline{G}) \leq 2n + \gamma(\overline{G})$.*

Proof. By Theorem E, we have $\gamma(G) \leq \frac{n}{2}$. Thus, by previous Theorem, we have $\gamma_{dR}(G\overline{G}) \leq n + 2\gamma(G) + \gamma(\overline{G}) \leq n + 2\frac{n}{2} + \gamma(\overline{G}) = 2n + \gamma(\overline{G})$. □

Corollary 2.6. *Let G be a graph of order n and without isolated vertex. Then*

- $\gamma_{dR}(G\overline{G}) \leq 2n + \delta(G) + 1$.
- $\gamma_{dR}(G\overline{G}) \leq 2n + \chi(G)$.

Proof. By Theorem 2.4 and Theorem J, the result holds. □

3. γ_{StR} OF COMPLEMENTARY PRISM OF A GRAPH

In this section we establish upper bound of strong Roman domination number of complementary prism of a graph. We compare the strong Roman domination number of complementary prism of a graph and the strong Roman domination number of the graph. First we study some special graphs.

Theorem 3.1. *Let P_n be a path with vertices $v_1, v_2, \ldots, v_n$ and $\overline{P_n}$ with vertices $\overline{v_1}, \overline{v_2}, \ldots, \overline{v_n}$. Them $\gamma_{StR}(P_n\overline{P_n}) = n + \lceil \frac{n}{3} \rceil + 1$.*

Proof. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{StR}(P_n)$-function on P_n. If $n \equiv 0$ (mod 3), then f can be chosen in such a way that $V_1 = \emptyset$, $V_2 = \{v_i : i = 3t + 2,\ 0 \leq t \leq \frac{n-3}{3}\}$ and $V_0 = V - V_2$. We define a function $g = (V_0', V_1', V_2')$ on $P_n\overline{P_n}$ by

$$g(v) = \begin{cases} 1, & \text{if } v \in \overline{V_2}; \\ f(v) & v \in V_0 \cup V_1 \cup V_2; \\ \frac{n-\frac{n}{3}}{2} + 1 & \text{for } v = \overline{v_1} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, g is a StRDF on $P_n\overline{P_n}$. Hence,

$$\gamma_{StR}(P_n\overline{P_n}) \leq \omega(g) = \frac{n}{3} + \frac{2n}{3} + \frac{n - \frac{n}{3}}{2} + 1 = n + \frac{n}{3} + 1$$

Conversely, it is well known that the least value that must be assigned to the vertices of $\overline{P_n}$ and P_n in $P_n\overline{P_n}$ by any StRDF are $1 + \lceil \frac{1}{2}(\frac{2n}{3} + 1) \rceil + \frac{n}{3}$ and $\frac{2n}{3}$ respectively or $1 + \lceil \frac{1}{2}(n-1) \rceil$ and n respectively. Therefore $\gamma_{StR}(P_n\overline{P_n}) \geq n + \frac{n}{3} + 1$ and thus $\gamma_{StR}(P_n\overline{P_n}) = n + \frac{n}{3} + 1$.

If $n \equiv 1$ (mod 3), then f can be chosen in such a way that $V_1 = \{v_n\}$, $V_2 = \{v_i : i = 3t + 2,\ 0 \leq t \leq \frac{n-4}{3}\}$ and $V_0 = V - V_2 \cup V_1$. If $n \equiv 2$ (mod 3), then f can be chosen in such a way that $V_1 = \emptyset$, $V_2 = \{v_i : i = 3t + 2,\ 0 \leq t \leq \frac{n-2}{3}\}$ and $V_0 = V - V_2$.
Thus we define a function $h = (V_0'', V_1'', V_2'')$ on $P_n\overline{P_n}$ by

$$h(v) = \begin{cases} 1, & v \in \overline{V_2}; \\ f(v) & v \in V_0 \cup V_1 \cup V_2; \\ \lceil \frac{n-\lceil \frac{n}{3} \rceil + 1}{2} \rceil + 1 & \text{for } v = \overline{v_1} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, h is a StRDF on $P_n\overline{P_n}$. Hence,

$$\gamma_{StR}(P_n\overline{P_n}) \leq \omega(h) = \lceil \frac{n}{3} \rceil - 1 + \lceil \frac{2n}{3} \rceil + \lceil \frac{n - \lceil \frac{n}{3} \rceil + 1}{2} \rceil + 1 = n + \lceil \frac{n}{3} \rceil + 1.$$

Conversely, it can be proved that like the first part. Thus the proof is completed. □

It can be easily verified $\gamma_{StR}(C_3\overline{C_3}) = 5$. In the follow we investigate the $\gamma_{StR}(C_n\overline{C_n})$ for $n \geq 4$.

Theorem 3.2. *For any cycle C_n,* $\gamma_{StR}(C_n\overline{C_n}) = \begin{cases} n + \lceil \frac{n}{3} \rceil + 2, & \textit{if } n \equiv 0 \ (mod\ 3)\,, \\ n + \lceil \frac{n}{3} \rceil + 1, & \textit{otherwise.} \end{cases}$

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_{StR}-function on C_n and $n \equiv 0\ (mod\ 3)$, $(n \geq 4)$. Then f can be chosen in such a way that $V_1 = \emptyset$, $V_2 = \{v_i : i = 3t + 2,\ 0 \leq t \leq \frac{n-3}{3}\}$ and $V_0 = V - V_2$. We define a function g on $C_n\overline{C_n}$ by

$$g(v) = \begin{cases} 1, & v \in \overline{V_2},\ \text{and } v = \overline{v_n}, \\ f(v), & v \in V_2, \\ \lceil \frac{n-2-\lceil \frac{n}{3} \rceil}{2} \rceil + 1 & v = \overline{v_1}; \\ 0, & \text{otherwise.} \end{cases}$$

g is a StRDF on $C_n\overline{C_n}$. Thus $\gamma_{StR}(C_n\overline{C_n}) \leq \omega(g) = \frac{n}{3} + 1 + \frac{2n}{3} + \lceil \frac{n-2-(\frac{n}{3}-1)}{2} \rceil + 1 = n + \lceil \frac{n}{3} \rceil + 2$.
Conversely, it is well known that the least value that must be assigned to the vertices of $\overline{C_n}$ and C_n in $C_n\overline{C_n}$ by any StRDF are $1 + \lceil \frac{1}{2}(n-2) - (\frac{n}{3} - 1) \rceil + \frac{n}{3} + 1$ and $\frac{2n}{3}$ respectively or $1 + \lceil \frac{1}{2}(n-2) \rceil$ and $n + 1$ respectively. Therefore $\gamma_{StR}(C_n\overline{C_n}) \geq n + \frac{n}{3} + 2$ and thus $\gamma_{StR}(C_n\overline{C_n}) = n + \frac{n}{3} + 2 = n + \lceil \frac{n}{3} \rceil + 2$.
Now similar to the proof of the first part and using Theorem 3.1, one can prove $\gamma_{StR}(C_n\overline{C_n}) = n + \lceil \frac{n}{3} \rceil + 1$ if $n \equiv 1\ (mod\ 3)$ or $n \equiv 2\ (mod\ 3)$. □

Theorem 3.3. *For any complete graph K_n, $\gamma_{StR}(K_n\overline{K_n}) = n + \lceil \frac{n}{2} \rceil$*

Proof. Let v be a vertex in G. Clearly, $degv = n - 1$. Now we define a function f on $G\overline{G}$ by $f(v) = \lceil \frac{n}{2} \rceil + 1$, $f(\overline{v}) = 0$ for $\overline{v} \in \overline{V}$, $f(x) = 1$ for any $x \in \overline{V} - \{\overline{v}\}$ and 0 otherwise. Thus f is a StRDF of $G\overline{G}$. So, we conclude $\gamma_{StR}(G\overline{G}) \leq \omega(f) = n - 1 + \lceil \frac{n}{2} \rceil + 1 = n + \lceil \frac{n}{2} \rceil$. Now to see that for any StRDF on $G\overline{G}$ has at least weight $\geq n + \lceil \frac{n}{2} \rceil$. Let f be a StRDF of $G\overline{G}$. For any vertex $\overline{v} \in \overline{V}$, we have either $f(\overline{v}) = 0$ or $f(\overline{v}) = 1$. If $f(\overline{v}) = 0$ and k vertices of K_n has value 0, and $n - (k + 1)$ vertices has value 1, then $n - 1$ vertices of $\overline{K_n}$ assigned by value 1. Therefore $\omega(f) = n - 1 + n - (k + 1) + \lceil \frac{k+1}{2} \rceil + 1 = 2n - k - 1 + \lceil \frac{k+1}{2} \rceil \geq n + \lceil \frac{n}{2} \rceil$. Thus $\gamma_{StR}(K_n\overline{K_n}) = n + \lceil \frac{n}{2} \rceil$. If for any $v \in \overline{V}$ $f(v) = 1$, then it is simply verified that $\omega(f) \geq n + \lceil \frac{n}{2} \rceil$. Anyway $\gamma_{StR}(K_n\overline{K_n}) = n + \lceil \frac{n}{2} \rceil$. □

A. Alhashim and others observed in the article [1] that for any graph G, $\gamma_R(G\overline{G}) \leq \gamma_R(G) + \gamma_R(\overline{G})$. But in general, this proposition for the parameter γ_{StR} is not correct. For example, this inequality is not true for the graph $G = K_3 + K_3$, because we have $\overline{G} = K_{3,3}$, $\gamma_{StR}(G\overline{G}) = 10$, $\gamma_{StR}(G) = 4$ and $\gamma_{StR}(\overline{G}) = 5$. In the next Theorem, we prove the correct form of this inequality for the parameter γ_{StR}.

Theorem 3.4. *Let G be a simple graph of order n. Then we have*

$$\gamma_{StR}(G\overline{G}) - \gamma_{StR}(G) \leq n.$$

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_{StR}-function on G. We define an function g on $G\overline{G}$ by $g = (V_0', V_1', V_2')$ such that

$$V_0' = \{w \in V(\overline{G}) : w \in N_{G\overline{G}}(V_2)\} \cup V_0$$

$$V_1' = \{w \in V(\overline{G}) : w \in N_{G\overline{G}}(V_1 \cup V_0)\} \cup V_1$$

$$V_2' = V_2,$$

where $V_2 = \{v \in G |\ f(v) \geq 2\} = U_2 \cup U_3 \cup \ldots \cup U_{\lceil \frac{\Delta(G)}{2} \rceil + 1}$ and $U_i = \{v \in V(G) |\ f(v) = i\}$. Clearly, g is an StRDF on $G\overline{G}$. Since $V_2' = \{v \in V(G\overline{G}) |\ g(v) \geq 2\}$, hence $V_2' = V_2'' \cup V_3'' \cup \ldots \cup V''_{\lceil \frac{\Delta}{2} \rceil + 1}$ where $V_i'' = \{v \in V(G\overline{G}) |\ g(v) = i\}$ *and* $\Delta = \Delta(G\overline{G})$. Thus we have $\gamma_{StR}(G\overline{G}) \leq \omega(g) = |V_1'| + 2|V_2''| + 3|V_3''| + \ldots + (\lceil \frac{\Delta}{2} \rceil + 1)|V''_{\lceil \frac{\Delta}{2} \rceil + 1}|$. By definition of StRDF g, we have

$$|V_1'| = |V_0| + |V_1|$$

and

$$2|V_2''| + 3|V_3''| + \ldots + (\lceil \frac{\Delta}{2} \rceil + 1)|V''_{\lceil \frac{\Delta}{2} \rceil + 1}| \leq$$
$$|U_2| + \ldots + |U_{\lceil \frac{\Delta(G)}{2} \rceil + 1}| + \gamma_{StR}(G).$$

Thus, we conclude

$$\gamma_{StR}(G\overline{G}) \leq |V_0| + |V_1| + |U_2| + ... + |U_{\lceil \frac{\Delta(G)}{2} \rceil+1}| + \gamma_{StR}(G) \leq n + \gamma_{StR}(G).$$

□

Corollary 3.5. *Let G be a simple graph of order n. If every vertex of G has odd degree, then*

$$\gamma_{StR}(G\overline{G}) - \gamma_{StR}(G) \leq n - 1.$$

Proof. Since each vertex of G is odd degree, by using the notations of the proof of Theorem 3.4, we have $V_2'' = U_2, \ldots, V''_{\lceil \frac{\Delta}{2} \rceil+1} = U_{\lceil \frac{\Delta(G)}{2} \rceil+1}$ and for any vertex v in $V_2' = V_2$, $g(v) = f(v)$. Thus

$$\begin{aligned}\gamma_{StR}(G\overline{G}) &\leq \omega(g) = |V_1'| + 2|V_2''| + 3|V_3''| + ... + (\lceil \frac{\Delta}{2} \rceil + 1)|V''_{\lceil \frac{\Delta}{2} \rceil+1}| \\ &= |V_0| + |V_1| + 2|U_2| + 3|U_3| + ... + (\lceil \frac{\Delta(G)}{2} \rceil + 1)|U_{\lceil \frac{\Delta(G)}{2} \rceil+1}| \\ &= |V_0| + \gamma_{StR}(G) \leq n - 1 + \gamma_{StR}(G).\end{aligned}$$

□

Theorem 3.6. *For any graph G with maximum degree $\Delta = \Delta(G)$, $2 \leq \gamma_{StR}(G\overline{G}) \leq 2n - \Delta + \lceil \frac{\Delta+1}{2} \rceil - 1$ and the bounds are sharp.*

Proof. Let v be a vertex of G with $degv = \Delta$. We establish a function f on $G\overline{G}$ by $f(v) = \lceil \frac{\Delta+1}{2} \rceil + 1$, $f(x) = 0$ for any $x \in N_{G\overline{G}}(v)$ and $f(y) = 1$ otherwise. The function f is a StRDF on $G\overline{G}$ and $\omega(f) = \lceil \frac{\Delta+1}{2} \rceil + 1 + 2n - (\Delta + 2)$. Thus, $\gamma_{StR}(G\overline{G}) \leq \omega(f) = \lceil \frac{\Delta+1}{2} \rceil + 2n - \Delta - 1$. Since the lower bound is trivial, the result is proved. For upper bound sharpness, let $G = K_n$, and using of Theorem 3.3, for lower bound sharpness, consider $G = K_1$. Thus the proof is completed. □

Using Theorem C, we establish a lower bound for strong domination number of $G\overline{G}$ in terms of order of G.

Corollary 3.7. *Let G be a graph of order n. Then $\gamma_{StR}(G\overline{G}) \geq n + 1$.*

Now we determine the complementary prisms $G\overline{G}$ having small strong Roman domination numbers, namely, the graphs G with $\gamma_{StR}(G\overline{G}) \in \{2, 3, 4, 5\}$.

Theorem 3.8. *Let G be a graph. Then,*

- 1. $\gamma_{StR}(G\overline{G}) = 2$ *if and only if* $G = K_1$.
- 2. $\gamma_{StR}(G\overline{G}) = 3$ *if and only if* $G = K_2$ *or* $\overline{G} = K_2$.
- 3. *For any graph G,* $\gamma_{StR}(G\overline{G}) \neq 4$.
- 4. $\gamma_{StR}(G\overline{G}) = 5$ *if and only if* $G = P_3$ *or* $G = K_3$.

Proof. (1) If $G = K_1$, then $G\overline{G} = K_2$ and $\gamma_{StR}(K_2) = 2$.
Conversely, assume that $\gamma_{StR}(G\overline{G}) = 2$. Then by Corollary 3.7 we have $2 \geq n + 1$. Thus $n = 1$. Hence we must have $G = K_1$.

(2) If $G = K_2$, then $G\overline{G}$ is isomorphic to the path P_4 and $\gamma_{StR}(P_4) = 3$.
Conversely, assume that $\gamma_{StR}(G\overline{G}) = 3$. Then by Corollary 3.7 we have $3 \geq n + 1$. Thus $n \leq 2$. But with regard to the first part of the Theorem we must have $n = 2$. Hence, we conclude $G = K_2$ or $\overline{G} = K_2$.

(3) Let G be a graph of order n such that $\gamma_{StR}(G\overline{G}) = 4$. Then by 3.7 we have $4 \geq n + 1$. Thus $n \leq 3$. But according to the two preceding parts of the Theorem we must have $n = 3$. Hence, we must have $G = K_3$ or $G = P_3$. Now with simple calculation we have $\gamma_{StR}(G\overline{G}) = 5$. And so it is a contradiction.

(4) If $G = K_3$ or $G = P_3$, then $\gamma_{StR}(G\overline{G}) = 5$.
Conversely, assume that $\gamma_{StR}(G\overline{G}) = 5$. Then by Corollary 3.7 we have $5 \geq n + 1$. Thus $n \leq 4$. But according to the

preceding parts of the Theorem we must have $n = 3$ or $n = 4$. If $n = 4$, then there are 12 graphs G of the order of 4, which with simple calculations for $\gamma_{StR}(G\overline{G})$ we conclude $\gamma_{StR}(G\overline{G}) = 6$. Then we must have $n = 3$. Hence, $G = K_3$ or $G = P_3$. □

Another lower bound for strong Roman domination number of complementary prism of a graph G in terms of order maximum degree and minimum degree is established as follows.

Theorem 3.9. *Let G be a graph of order n. Then*

$$\gamma_{StR}(G\overline{G}) \geq max\{\gamma_{StR}(G) + \lceil\frac{n-\delta}{2}\rceil, \gamma_{StR}(G) + \lceil\frac{\Delta-1}{2}\rceil + 1, \gamma_{StR}(\overline{G}) + \lceil\frac{n-\delta}{2}\rceil, \gamma_{StR}(\overline{G}) + \lceil\frac{\Delta-1}{2}\rceil + 1\}.$$

This bound is sharp.

Proof. Without loss of generality, let $max\{\gamma_{StR}(G), \gamma_{StR}(\overline{G})\} = \gamma_{StR}(G)$. Thus by Theorems C and K, we have $\gamma_{StR}(G\overline{G}) \geq \lceil\frac{2n+1}{2}\rceil = n + 1$ and $\gamma_{StR}(G) \leq n - \lfloor\frac{\Delta}{2}\rfloor$. Hence, we conclude

$$\gamma_{StR}(G\overline{G}) - \gamma_{StR}(G) \geq n + 1 - (n - \lfloor\frac{\Delta}{2}\rfloor) = \lfloor\frac{\Delta}{2}\rfloor + 1 = \lceil\frac{\Delta-1}{2}\rceil + 1,$$

On the other hand, for the graph $\overline{G}$, since $\overline{\Delta} = n - \delta - 1$ by Theorem K we have $\gamma_{StR}(\overline{G}) \leq n - \lfloor\frac{n-\delta-1}{2}\rfloor$. But $\lfloor\frac{n-\delta-1}{2}\rfloor + 1 = \lceil\frac{n-\delta-1-1}{2}\rceil + 1 = \lceil\frac{n-\delta}{2}\rceil$. So, we have $\gamma_{StR}(\overline{G}) \leq n - \lceil\frac{n-\delta}{2}\rceil + 1$. Hence,

$$\gamma_{StR}(G\overline{G}) - \gamma_{StR}(\overline{G}) \geq n + 1 - (n - \lceil\frac{n-\delta}{2}\rceil + 1) = \lceil\frac{n-\delta}{2}\rceil.$$

Thus by the assumption $\gamma_{StR}(G) \geq \gamma_{StR}(\overline{G})$ we conclude

$$\gamma_{StR}(G\overline{G}) \geq \gamma_{StR}(G) + \lceil\frac{\Delta-1}{2}\rceil + 1 \geq \gamma_{StR}(\overline{G}) + \lceil\frac{\Delta-1}{2}\rceil + 1,$$

Therefore,

$$\gamma_{StR}(G\overline{G}) \geq max\{\gamma_{StR}(\overline{G}) + \lceil\frac{n-\delta}{2}\rceil, \gamma_{StR}(\overline{G}) + \lceil\frac{\Delta-1}{2}\rceil + 1\}.$$

Similarly, by changing the role of $\overline{G}$ with G, we have

$$\gamma_{StR}(\overline{G}G) = \gamma_{StR}(G\overline{G}) \geq max\{\gamma_{StR}(G) + \lceil\frac{n-\delta}{2}\rceil, \gamma_{StR}(G) + \lceil\frac{\Delta-1}{2}\rceil + 1\}.$$

Thus, the result is established, that is,

$$\gamma_{StR}(G\overline{G}) \geq max\{\gamma_{StR}(G) + \lceil\frac{n-\delta}{2}\rceil, \gamma_{StR}(G) + \lceil\frac{\Delta-1}{2}\rceil + 1, \gamma_{StR}(\overline{G}) + \lceil\frac{n-\delta}{2}\rceil, \gamma_{StR}(\overline{G}) + \lceil\frac{\Delta-1}{2}\rceil + 1\}.$$

For sharpness, let $G = K_n$. □

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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