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Characterizations of Curves According to Frenet Frame in Euclidean 3-Space

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Abstract. In this paper, we investigate the conditions of being an harmonic curve and research differential equations characterizing any differentiable curve in Euclidean 3-space. By means of the Laplacian image of the mean curvature vector field of a curve, it is stated which type of harmonic the curve is. Then we write the theorems related to the characterization of the curves and proved these theorems. When the differentiable curve, used throughout this paper, is specifically replaced to the unit speed curve then it is seen that the results coincide with the study [4]. In addition we elucidate the characterizations of helix as an example.

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1. Introduction and Preliminaries

To establish a relationship between the curvatures and characterization of a curve in Euclidean space and non-Euclidean spaces and to expound it from the language of geometry has been the focus of interest for many researchers. Thanks to meticulous studies, it has been revealed that curves can be classified, [2]. After this classification, a considerable number of articles are written, [1, 4–6, 8]. In a three dimensional Euclidean space, characterizations of a unit speed curve is stated clearly, [1, 4]. In this paper, we explore the necessary and sufficient conditions of being biharmonic curve and 1-type harmonic curve. Also we research the differential equations characterizing the differentiable curve in Euclidean 3-space, according to both connection and normal connection. For any differentiable curve α with the Euclidean coordinate mappings $(\alpha_1, \alpha_2, \alpha_3)$, the real valued function $\alpha'(t)$ that is, $\alpha'(t) = (\alpha_1', \alpha_2', \alpha_3')\mid_{t=0}$ is called a *speed vector* of the curve α, [3]. If any differentiable curve α, $\| \alpha' \| = \vartheta$, is given in E^3 then the relationship between the Frenet vector fields and its curvatures is stated as [7]

$$T = \frac{\alpha'}{\| \alpha' \|}, \qquad N = B \times T, \quad B = \frac{\alpha' \times \alpha''}{\| \alpha' \times \alpha'' \|}, \qquad \kappa = \frac{\| \alpha' \times \alpha'' \|}{\| \alpha' \|^3}, \qquad \tau = \frac{\langle \alpha' \times \alpha'', \alpha''' \rangle}{\| \alpha' \times \alpha'' \|^2}.$$

Theorem 1.1. *Frenet vector fields can be expressed by means of covariant derivative of these vectors and this relation is known as Frenet formulas* [7]*,*

$$D_T T = \vartheta \kappa N, \qquad D_T N = \vartheta(-\kappa T + \tau B), \qquad D_T B = -\vartheta \tau N. \tag{1.1}$$

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Definition 1.2. Let any differentiable curve α and a continuous function f, $f \in C(E^3, \mathbb{R})$, are given then the operator D defined as

$$D : T_{E^3}(\alpha(t)) \times C(E^3, \mathbb{R}) \to \mathbb{R}, \qquad D(\alpha'(t), f) = D_{\alpha'(t)} f = \alpha'(t)(f)$$

and it is called a *Levi-Civita connection*. Here the value of $\alpha'(t)(f) \in \mathbb{R}$ is called as *covariant derivative* of the function f along the curve α, [3].

Theorem 1.3. *Let two vector fields X and W defined on E^3 and another two vector fields Y , Z from C^2-class defined on E^3 be given. Then the following propositions are true,* [7]

$$\begin{aligned} D_X(Y+Z) &= D_X Y + D_X Z, \\ D_{X+W}(Y) &= D_X Y + D_W Y, \\ D_{f(P)X}(Y) &= f(P) D_X Y\,,\ f : E^3 \to \mathbb{R}\,,\ P \in E^3, \\ D_X(fY) &= X(f)Y + f D_X Y\,,\ f \in C(E^3, \mathbb{R}). \end{aligned} \tag{1.2}$$

Definition 1.4. Let α be the unit speed curve, then the mean curvature vector field $\mathbb{H}$ along the curve α is defined as [4, 5]

$$\mathbb{H} = D_{\alpha'}\alpha' = D_T T = \kappa N. \tag{1.3}$$

Definition 1.5. Let α be the unit speed curve with the mean curvature vector field $\mathbb{H}$, then the operator Δ defined as

$$\Delta : \chi(\alpha(I))^{\perp} \to \chi(\alpha(I))\,, \qquad \Delta\mathbb{H} = -D_T^2 \mathbb{H} \tag{1.4}$$

on α is called a Laplace operator [1, 4] .

Definition 1.6. Let us denote the normal bundle of the curve α by $\chi^{\perp}(\alpha(s))$. Then the normal connection $D^{\perp}$ is defined as

$$D_T^{\perp} : \chi^{\perp}(\alpha(I)) \to \chi^{\perp}(\alpha(I))\,, \qquad D_T^{\perp} X = D_T X - \langle D_T X, T\rangle T \tag{1.5}$$

and the normal Laplace operator $\Delta^{\perp}$ is given by the following mapping [4, 6]

$$\Delta_T^{\perp} : \chi^{\perp}(\alpha(I)) \to \chi^{\perp}(\alpha(I))\,, \qquad \Delta^{\perp} X = -D_T^{\perp} D_T^{\perp} X\,, \quad \forall X \in \chi^{\perp}(\alpha(I)). \tag{1.6}$$

Corollary 1.7. *If α is a unit speed curve with the mean curvature vector field $\mathbb{H}$, then we have the following propositions*
a) If $\Delta\mathbb{H} = 0$ then α is a biharmonic curve,
b) If $\Delta\mathbb{H} = \lambda\mathbb{H}$ then α is a 1-type harmonic curve,
c) If $\Delta^{\perp}\mathbb{H} = 0$ then α is a weak biharmonic curve,
d) If $\Delta^{\perp}\mathbb{H} = \lambda\mathbb{H}$ then α is a 1-type harmonic curve $\lambda \in \mathbb{R}$, [4, 6].

2. Discussions and Result

Theorem 2.1. *Let α be a differentiable curve in $\mathbb{E}^3$, then the following propositions are true*

1) *α is a biharmonic curve if and only if*

$$3(\vartheta\kappa)'\vartheta\kappa = 0, \quad (\vartheta\kappa)^3 + \vartheta\kappa(\vartheta\tau)^2 - (\vartheta\kappa)'' = 0\,, \quad -2(\vartheta\kappa)'\vartheta\tau - \vartheta\kappa(\vartheta\tau)' = 0.$$

2) *α is a 1-type harmonic curve if and only if*

$$3(\vartheta\kappa)'\vartheta\kappa = 0, \quad (\vartheta\kappa)^3 + \vartheta\kappa(\vartheta\tau)^2 - (\vartheta\kappa)'' = \lambda\vartheta\kappa, \quad -2(\vartheta\kappa)'\vartheta\tau - \vartheta\kappa(\vartheta\tau)' = 0, \ \vartheta = \| \alpha'(s) \|, \ \lambda \in \mathbb{R}.$$

Proof. From (1.3) we can write $\mathbb{H} = \vartheta\kappa N$ and from (1.4) we have

$$\begin{aligned}\Delta\mathbb{H} &= -D_T^2(\vartheta\kappa N) = -D_T(D_T(\vartheta\kappa N)) \\ \Delta\mathbb{H} &= 3(\vartheta\kappa)'(\vartheta\kappa)T + ((\vartheta\kappa)^3 + (\vartheta\kappa)(\vartheta\tau)^2 - (\vartheta\kappa)'')N + (-2(\vartheta\kappa)')\vartheta\tau - \vartheta\kappa(\vartheta\tau)')B.\end{aligned}$$

If we take the condition that $\Delta\mathbb{H} = 0$ into account, then 1. proposition holds and if we take the condition that $\Delta\mathbb{H} = \lambda\mathbb{H}$ into account, then 2. proposition holds. This completes the proof. □

Theorem 2.2. *Let α be a differentiable curve in $\mathbb{E}^3$, then the following propositions are true according to normal Levi- Civita connection $D^\perp$.*

1) *α is a 1-type harmonic curve if and only if* $\vartheta^3\tau^2\kappa - (\vartheta\kappa)'' = \lambda\kappa\,, \quad (\vartheta\kappa)'\vartheta\tau + (\vartheta^2\kappa\tau)' = 0.$

2) *α is a weak biharmonic curve if and only if* $\vartheta^3\tau^2\kappa - (\vartheta\kappa)'' = 0\,, \quad (\vartheta\kappa)'\vartheta\tau + (\vartheta^2\kappa\tau)' = 0.$

Proof. From (1.6) we have the image mapping of the mean curvature under normal Laplace operator as

$$D_T\mathbb{H} = (\vartheta\kappa)'N - (\vartheta\kappa)^2T + \vartheta^2\kappa\tau B.$$

Hence we get, $D_T^\perp\mathbb{H} = (\vartheta\kappa)'N + \vartheta^2\kappa\tau B.$ By making use of (1.6),

$$\begin{aligned}D_T(D_T^\perp\mathbb{H}) &= D_T((\vartheta\kappa)'N + \vartheta^2\kappa\tau B) = (\vartheta\kappa)''N + (\vartheta\kappa)'D_TN + (\vartheta^2\kappa\tau)'B + \vartheta^2\kappa\tau D_TB \\ D_T^\perp D_T^\perp\mathbb{H} &= ((\vartheta\kappa)'' - \vartheta^3(\tau)^2\kappa)N + ((\vartheta\kappa)'\vartheta\tau + (\vartheta^2\kappa\tau)')B\end{aligned}$$

and it follows

$$\Delta^\perp\mathbb{H} = (\vartheta^3\tau^2\kappa - (\vartheta\kappa)'')N - ((\vartheta\kappa)'\vartheta\tau + (\vartheta^2\kappa\tau)')B.$$

If we take the condition that $\Delta^\perp\mathbb{H} = \lambda\mathbb{H}$ into account, then 1. proposition holds and if we take the condition that $\Delta^\perp\mathbb{H} = 0$ into account, then 2. proposition holds. This yields the required result and completes the proof. □

Theorem 2.3. *Let a differentiable curve α be given. Then the differential equation characterizing the curve α according to unit tangent vector T is given as*

$$D_T^3T + \lambda_2D_T^2T + \lambda_1D_TT + \lambda_0T = 0$$

where

$$\lambda_0 = \vartheta^2\kappa\tau(\frac{\kappa}{\tau})'\,, \quad \lambda_1 = \vartheta^2(\kappa^2+\tau^2) - \frac{\vartheta''}{\vartheta} - \frac{\kappa''}{\kappa} + (\frac{\vartheta'}{\vartheta} + \frac{\kappa'}{\kappa})(3\frac{\vartheta'}{\vartheta} + \frac{\tau'}{\tau}) + 2(\frac{\kappa'}{\kappa})^2 \;\; and \;\; \lambda_2 = -(3\frac{\vartheta'}{\vartheta} + 2\frac{\kappa'}{\kappa} + \frac{\tau'}{\tau})\,.$$

Proof. From (1.1) we can write

$$D_TT = \vartheta\kappa N \implies N = \frac{1}{\vartheta\kappa}D_TT \quad \text{and} \quad D_TN = \vartheta(-\kappa T + \tau B) \implies B = \frac{1}{\vartheta\tau}D_TN + \frac{\kappa}{\tau}.$$

Now let us put $(1/\vartheta\kappa)D_TT$ instead of N, thence we get

$$B = \frac{1}{\vartheta\tau}D_T(\frac{1}{\vartheta\kappa}D_TT) + \frac{\kappa}{\tau}T.$$

It follows that

$$B = \frac{1}{\vartheta\tau}(\frac{1}{\vartheta\kappa})'D_TT + \frac{1}{\vartheta^2\tau\kappa}D_T^2T + \frac{\kappa}{\tau}T.$$

Taking the tangential derivative of both sides of above equality gives us,

$$D_T B = D_T\Big(\frac{1}{\vartheta\tau}\big(\frac{1}{\vartheta\kappa}\big)' D_T T\Big) + D_T\Big(\frac{1}{\vartheta^2\kappa\tau} D_T^2 T\Big) + D_T\Big(\frac{\kappa}{\tau} T\Big). \tag{2.1}$$

We may take into account of $N = (1/\vartheta\kappa) D_T T$ and find out $D_T B = (-\tau/\kappa) D_T T$. If we put the equivalent of $D_T B$ into (2.1) we work out,

$$-\frac{\tau}{\kappa} D_T T = D_T\Big(\frac{1}{\vartheta\tau}\big(\frac{1}{\vartheta\kappa}\big)' D_T T\Big) + D_T\Big(\frac{1}{\vartheta^2\kappa\tau} D_T^2 T\Big) + D_T\Big(\frac{\kappa}{\tau} T\Big).$$

After performing necessary operations we get,

$$\frac{1}{\vartheta^2\kappa\tau} D_T^3 T + \Big(\frac{1}{\vartheta\tau}\big(\frac{1}{\vartheta\kappa}\big)' + \big(\frac{1}{\vartheta^2\kappa\tau}\big)'\Big) D_T^2 T + \Big(\frac{\kappa^2+\tau^2}{\kappa\tau} + \big(\frac{1}{\vartheta\tau}\big(\frac{1}{\vartheta\kappa}\big)'\big)'\Big) D_T T + \big(\frac{\kappa}{\tau}\big)' T = 0. \tag{2.2}$$

Now it remains only to compute the coefficients of the expressions $D_T^2 T$ and $D_T T$ respectively , that is,

$$\frac{1}{\vartheta\tau}\big(\frac{1}{\vartheta\kappa}\big)' + \big(\frac{1}{\vartheta^2\kappa\tau}\big)' = -\frac{\vartheta'\kappa + \vartheta\kappa'}{\vartheta^3\tau\kappa^2} - \frac{2\vartheta\vartheta'\kappa\tau + \vartheta^2\kappa'\tau + \vartheta^2\kappa\tau'}{\vartheta^4\kappa^2\tau^2}$$

and

$$\begin{aligned}\frac{\kappa}{\tau} + \frac{\tau}{\kappa} + \big(\frac{1}{\vartheta\tau}\big(\frac{1}{\vartheta\kappa}\big)'\big)' &= \frac{\kappa^2+\tau^2}{\kappa\tau} - \frac{\vartheta''\kappa + 2\vartheta'\kappa' + \vartheta\kappa''}{\vartheta^3\tau\kappa^2} + \\ &\quad \frac{(3\vartheta^2\vartheta'\tau\kappa^2 + \vartheta^3\tau'\kappa^2 + 2\vartheta^3\tau\kappa\kappa')(\vartheta'\kappa + \vartheta\kappa')}{\vartheta^6\tau^2\kappa^4}.\end{aligned}$$

Finally we can put these values into (2.2) and then multiplying both sides of (2.2) by $\vartheta^2\kappa\tau$ we obtain,

$$\begin{aligned} D_T^3 T - \big(3\frac{\vartheta'}{\vartheta} + 2\frac{\kappa'}{\kappa} + \frac{\tau'}{\tau}\big) D_T^2 T &+ \Big(\vartheta^2(\kappa^2+\tau^2) - \frac{\vartheta''}{\vartheta} - \frac{\kappa''}{\kappa} \\ &+ \big(\frac{\vartheta'}{\vartheta} + \frac{\kappa'}{\kappa}\big)\big(3\frac{\vartheta'}{\vartheta} + \frac{\tau'}{\tau}\big) + 2\big(\frac{\kappa'}{\kappa}\big)^2\Big) D_T T + \vartheta^2\kappa\tau\big(\frac{\kappa}{\tau}\big)' T = 0\end{aligned}$$

and this completes the proof. □

Theorem 2.4. *Let a differentiable curve α be given. Then the differential equations characterizing the curve α according to principal normal vector N and binormal vector B are given as*

$$1)\quad D_T^{\perp} D_T^{\perp} N - \frac{(\vartheta\tau)'}{\vartheta\tau} D_T^{\perp} N + (\vartheta\tau)^2 N = 0,$$

$$2)\quad D_T^{\perp} D_T^{\perp} B - \frac{(\vartheta\tau)'}{\vartheta\tau} D_T^{\perp} B + (\vartheta\tau)^2 B = 0.$$

Proof. 1) From (1.6) we have $D_T^{\perp} N = \vartheta\tau B$ and then $B = \frac{1}{\vartheta\tau} D_T^{\perp} N$. By the similar way

we have $D_T^{\perp} B = -\vartheta\tau N$ and then we can write $N = \frac{-1}{\vartheta\tau} D_T^{\perp} B$. It follows that

$$D_T^{\perp} N = \vartheta\tau B \quad\Longrightarrow\quad D_T^{\perp}(D_T^{\perp} N) = D_T^{\perp}(\vartheta\tau B)$$

and this gives us

$$D_T^{\perp} D_T^{\perp} N - \frac{(\vartheta\tau)'}{\vartheta\tau} D_T^{\perp} N + (\vartheta\tau)^2 N = 0.$$

2) By applying the (1.6) we claim that

$$D_T^{\perp} B = -\vartheta\tau N \quad\Longrightarrow\quad D_T^{\perp}(D_T^{\perp} B) = D_T^{\perp}(-\vartheta\tau N)$$

it follows

$$D_T^{\perp}D_T^{\perp}B - \frac{(\vartheta\tau)'}{\vartheta\tau}D_T^{\perp}B + (\vartheta\tau)^2B = 0.$$

Thus we obtain the desired results which complete the proof. □

Example 2.5. Let a differentiable curve $\alpha(s) = (a\cos s, a\sin s, bs)$ be given, $a, b \in \mathbb{R}^+$. Then it is obvious that $\vartheta = \alpha'(s) = \sqrt{a^2+b^2}$. From (1.1) Frenet apparatus of $\alpha(s)$ can be evaluated as

$$T = \frac{-a\sin s, a\cos s, b}{\sqrt{a^2+b^2}}, \quad N = (-\cos s, -\sin s, 0), \quad B = \frac{b\sin s, -b\cos s, a}{\sqrt{a^2+b^2}}, \quad \kappa = \frac{a}{a^2+b^2}, \quad \tau = \frac{b}{a^2+b^2}$$

According to these outcomes we can decide which kind of harmonic curve an helix is. From (1.1), we have

$$\mathbb{H} = \vartheta\kappa N = a/\sqrt{a^2+b^2}(-\cos s, -\sin s, 0)$$

and by the (2.1) we get

$$\begin{aligned}\Delta\mathbb{H} &= 3(\vartheta\kappa)'(\vartheta\kappa)T + ((\vartheta\kappa)^3 + (\vartheta\kappa)(\vartheta\tau)^2 - (\vartheta\kappa)'')N + (-2(\vartheta\kappa)')\vartheta\tau - \vartheta\kappa(\vartheta\tau)')B \\ &= \frac{a}{\sqrt{a^2+b^2}}(-\cos s, -\sin s, 0)\end{aligned}$$

therefore we realize that helices are of 1-type harmonic curves according to connection. By referring (2.2), we obtain

$$\begin{aligned}\Delta_T^{\perp}\mathbb{H} &= (\vartheta^3\tau^2\kappa - (\vartheta\kappa)'')N - ((\vartheta\kappa)'\vartheta\tau + (\vartheta^2\kappa\tau)')B \\ &= \frac{ab^2}{(a^2+b^2)^{\frac{3}{2}}}N\end{aligned}$$

and thus an helix is not a 1-type of harmonic curve according to normal connection. Let us look at the differential equations characterizing the helices. From (2.3), it is straightforward computation that

$$D_T^3T + D_TT = 0.$$

Eventually from (2.4), we get the equations

$$1)\quad D_T^{\perp}D_T^{\perp}N + (\frac{b^2}{a^2+b^2})N = 0, \qquad 2)\quad D_T^{\perp}D_T^{\perp}B + (\frac{b^2}{a^2+b^2})B = 0.$$

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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