



POSITION VECTORS OF CURVES WITH RESPECT TO DARBOUX FRAME IN THE GALILEAN SPACE G^3

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ABSTRACT. In this paper, we investigate the position vector of a curve on the surface in the Galilean 3-space G^3 . Firstly, the position vector of a curve with respect to the Darboux frame is determined. Secondly, we obtain the standard representation of the position vector of the curve with respect to Darboux frame in terms of the geodesic, normal curvature and geodesic torsion. As a result of this, we define the position vectors of geodesic, asymptotic and normal line along with some special curves with respect to Darboux frame. Finally, we elaborate on some examples and provide their graphs.

1. INTRODUCTION

The fundamental theorem of curves state that curves are determined by curvatures [1]. Thus, curvature functions provide us with some special and important information about curves. For example, a circular helix is a geometric curve with curvature $\kappa \equiv \text{const.} \neq 0$, torsion $\tau \equiv \text{const.} \neq 0$ [2]. Straight lines and circles are curves that $\kappa \equiv 0$ and $\kappa \equiv \text{cons.}, \tau \equiv 0$, respectively. Also, these curves are degenerate helices. Helices appear in many different branch of science such as engineering, biology, chemistry, CAD, etc.

In addition, curvature functions give us information about not only curves but also surfaces on which curves lie. The geodesic curvature κ_g , normal curvature κ_n and geodesic torsion τ_g characterize geodesic, asymptotic curve, and line of curvature, respectively. The curves emerges from the solution of some important physical problems. They are also important in the theory of curves and surfaces. For example, geodesics arise from the problem of finding 'shortest curves' joining two points of a surface M . It was first considered by Johann Bernoulli (1697). Obviously this is a typically problem of calculus of variations. Also, a curve C on a surface M is called a *geodesic curve* or *geodesic* if its geodesic curvature κ_g vanishes identically [1]. In what follows, we state three different definition lines in planes.

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We want to emphasize that geodesics can be seen as extension of this idea to curves in surfaces.

A "line" on a surface can be seen as extension of the familiar properties of lines in the plane: For example, lines are (1) the curves of shortest length joining two points (Archimedes). (2) The curves of plane curvature identically zero (Huygens, Leibniz, Newton). (3) The curves whose tangent and its derivative are linearly dependent [2].

As stated in [3], the problem of the determination of the position vector of a space curve with respect to the Frenet frame is still open in the Euclidean space. Generally, it is hard to solve this problem. However, it is solved for some special curves such as plane curves, helix and slant helix [4, 5, 6]. On the other hand, in the Galilean space G^3 , the foregoing problem is solved for all curves [3].

The main aim of this study is to solve the above problem for all curves on a surface in G^3 with respect to the Darboux Frame. Firstly, we determine the position vector of a curve on a surface in G^3 in terms of geodesic, normal curvature and geodesic torsion with respect to the Darboux and standard frame. Secondly, we shall give position vectors of some special curves such as geodesic, asymptotic curve, line of curvature on a surface in G^3 .

Also, we will relate foregoing curves with helix, Salkowski curve and anti-Salkowski curve (see (30)). That is, we shall give special cases of these curves such as: geodesics that are circular helix, generalized helix or Salkowski, etc. Furthermore, we provide graphs of some special curves.

Last but not least, we want to emphasize that the results of this study can be extended to families of surfaces that have common geodesic curve.

2. PRELIMINARIES

As it is well known, Galilean geometry is associated with the Galilean principle of relativity. The Galilean space G^3 is one of the Cayley-Klein spaces equipped with the projective metric of signature $(0, 0, +, +)$ [7]. The absolute figure of the Galilean space is the ordered triple $\{w, f, I\}$, where w is an ideal (absolute) plane, f is a line (absolute line) in w , and I is a fixed elliptic involution of points of f .

In non-homogeneous coordinates the group of motion of G^3 (i.e. the group of isometries of G^3) has the form define :

$$\begin{aligned}\bar{x} &= a_1 + x, \\ \bar{y} &= a_2 + a_3x + y \cos \varphi + z \sin \varphi, \\ \bar{z} &= a_4 + a_5x - y \sin \varphi + z \cos \varphi,\end{aligned}\tag{1}$$

where a_1, a_2, a_3, a_4, a_5 , and φ are real numbers [8]. If the first component of a vector is not zero, then the vector is called as non-isotropic, otherwise it is called isotropic vector [8].

The scalar product of two vectors $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ in G^3 is defined by

$$\mathbf{v} \cdot_G \mathbf{w} = \begin{cases} v_1 w_1, & \text{if } v_1 \neq 0 \text{ or } w_1 \neq 0 \\ v_2 w_2 + v_3 w_3, & \text{if } v_1 = 0 \text{ and } w_1 = 0. \end{cases}$$

If $\mathbf{v} \cdot_G \mathbf{w} = 0$, then \mathbf{v} and \mathbf{w} are perpendicular. In particular, every isotropic vector is perpendicular to every non-isotropic vector. The norm of \mathbf{v} is defined by

$$\|\mathbf{v}\|_G = \sqrt{|\mathbf{v} \cdot_G \mathbf{v}|}.$$

Let $I \subset \mathbb{R}$ and let $\alpha : I \rightarrow G^3$ be a curve parameterized by arc length (we abbreviate as p.b.a.l) with curvature $\kappa > 0$ and torsion τ . If α is a curve p.b.a.l. that is,

$$\alpha(x) = (x, y(x), z(x)),$$

then the Frenet frame fields are given by

$$\begin{aligned} T(x) &= \alpha'(x), \\ N(x) &= \frac{\alpha''(x)}{\|\alpha''(x)\|_G} \end{aligned} \tag{2}$$

$$\begin{aligned} B(x) &= T(x) \times_G B(x) \\ &= \frac{1}{\kappa(x)} (0, -z''(x), y''(x)), \end{aligned} \tag{3}$$

where $\kappa(x)$ and $\tau(x)$ are defined by

$$\kappa(x) = \|\alpha''(x)\|_G, \quad \tau(x) = \frac{\det(\alpha'(x), \alpha''(x), \alpha'''(x))}{\kappa^2(x)}. \tag{4}$$

Also, where \times_G is the Galilean cross product defined by

$$\mathbf{v} \times_G \mathbf{w} = \begin{vmatrix} 0 & \mathbf{e}_2 & \mathbf{e}_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \tag{5}$$

for $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ [9].

The vectors T, N and B are called the tangent, the principal normal and the binormal vector field, respectively [9]. Therefore, the Frenet-Serret formulae can be written as

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}' = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{6}$$

Frame fields constitute a very useful tool for studying curves and surfaces. However, the Frenet frame T, N, B of α is not useful to describe the geometry of surface M . Since N and B in general will be neither tangent nor perpendicular to M . Therefore, we require another frame of α for study the relation between the geometry of α and M . There is such a frame field that is called Darboux frame field of α with respect to M . The Darboux frame field consists of the triple of vector fields T, Q, n . The first and last vector fields of this frame T and n are a unit tangent

vector field of α and unit normal vector field of M at the point $\alpha(x)$ of α . Let $Q = n \times_G T$ be the tangential-normal.

Theorem 1. *Let $\alpha : I \subset \mathbb{R} \rightarrow M \subset G^3$ be a unit-speed curve, and let T, Q, n be the Darboux frame field of α with respect to M . Then*

$$\begin{bmatrix} T \\ Q \\ n \end{bmatrix}' = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ 0 & 0 & \tau_g \\ 0 & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} T \\ Q \\ n \end{bmatrix}. \quad (7)$$

where κ_g and κ_n give the tangential and normal component of the curvature vector, and these functions are called the geodesic and the normal curvature, respectively [10].

Proof. We have

$$\begin{aligned} T' &= (T' \cdot_G Q)Q + (T' \cdot_G n)n \\ &= (\alpha'' \cdot_G Q)Q + (\alpha'' \cdot_G n)n \\ &= \kappa_g Q + \kappa_n n. \end{aligned} \quad (8)$$

The other formulae are proved in a similar fashion. \square

Also, the equation (8) implies the important relations

$$\kappa^2(x) = \kappa_g^2(x) + \kappa_n^2(x), \quad \tau(x) = -\tau_g(x) + \frac{\kappa'_g(x)\kappa_n(x) - \kappa_g(x)\kappa'_n(x)}{\kappa_g^2(x) + \kappa_n^2(x)} \quad (9)$$

where $\kappa^2(x)$ and $\tau(x)$ are the square curvature and the torsion of α , respectively. We refer to [8, 9, 11, 12] for detailed treatment of Galilean and pseudo-Galilean geometry.

3. POSITION VECTORS OF A CURVE IN GALILEAN SPACE

In this section, we will get an arbitrary curve on a surface in G^3 . We will analyze position vector of the curve with respect to the Darboux and standard frame in G^3 .

Theorem 2. *The position vector $\beta(x)$ of an arbitrary curve on a surface with respect to the Darboux frame in the Galilean space G^3 is given by:*

$$\begin{aligned} \beta(x) = (x + c_1)\mathbf{T} + & \left\{ -\frac{(x + c_1)\kappa_n(x)}{\tau_g(x)} + \left(c_2 - \int f(x)\tau_g(x) \sin [t(x)] dx \right) \sin [t(x)] \right. \\ & \left. - \left(c_3 + \int f(x)\tau_g(x) \cos [t(x)] dx \right) \cos [t(x)] \right\} \mathbf{Q} \\ & + \left\{ \left(c_2 - \int f(x)\tau_g(x) \sin [t(x)] dx \right) \cos [t(x)] \right. \\ & \left. + \left(c_3 + \int f(x)\tau_g(x) \cos [t(x)] dx \right) \sin [t(x)] \right\} \mathbf{n} \end{aligned} \tag{10}$$

where $f(x) = \frac{\lambda_1(x)\kappa_g(x)}{\tau_g(x)} - \left(\frac{\lambda_1(x)\kappa_n(x)}{\tau_g(x)} \right)' \frac{1}{\tau_g(x)}$ and $t(x) = \int \tau_g(x) dx$.

Proof. Let $\beta(x)$ be an arbitrary curve on a surface in the G^3 , then, we may express its position vectors with respect to the Darboux frame as follows:

$$\beta(x) = \lambda_1(x)\mathbf{T} + \lambda_2(x)\mathbf{Q} + \lambda_3(x)\mathbf{n} \tag{11}$$

where $\lambda_1(x)$, $\lambda_2(x)$ and $\lambda_3(x)$ are differentiable functions of $x \in I \subset \mathbb{R}$. By differentiating (11) and using (7), we get

$$\begin{aligned} \lambda_1'(x) - 1 &= 0 \\ \lambda_1(x)\kappa_g(x) + \lambda_2'(x) - \lambda_3(x)\tau_g(x) &= 0 \\ \lambda_1(x)\kappa_n(x) + \lambda_2(x)\tau_g(x) + \lambda_3'(x) &= 0 \end{aligned} \tag{12}$$

The first equation of (12) leads to

$$\lambda_1(x) = x + c_1 \tag{13}$$

where c_1 is an arbitrary real constant. To solve (12) for λ_i , we use the following change of variable $t = \int \tau_g(x)dx$ so that

$$\begin{aligned} \lambda_1(t) &= (\lambda_1 \circ x)(t), \\ \tau_g(t) &= (\tau_g \circ x)(t), \\ \lambda_2(t) &= -\frac{\lambda_1(t)\kappa_n(t)}{\tau_g(t)} - \dot{\lambda}_3(t). \end{aligned} \tag{14}$$

Here, " $\dot{\cdot}$ " stands for derivative with respect to t .

Substituting (14) into (12) we get the following equation

$$\ddot{\lambda}_3(t) + \lambda_3(t) = \frac{\lambda_1(t)\kappa_g(t)}{\tau_g(t)} - \left(\frac{\lambda_1(t)\kappa_n(t)}{\tau_g(t)} \right)'. \tag{15}$$

The general solution becomes

$$\lambda_3(t) = \left[c_2 - \int f(t) \sin t dt \right] \cos t + \left[c_3 + \int f(t) \cos t dt \right] \sin t \quad (16)$$

where c_2, c_3 are arbitrary real constants and $f(t) = \frac{\lambda_1(t)\kappa_g(t)}{\tau_g(t)} - \left(\frac{\lambda_1(t)\kappa_n(t)}{\tau_g(t)} \right)'$. By differentiating (16) and plug the resulting equation into (14), we obtain

$$\lambda_2(t) = -\frac{\lambda_1(t)\kappa_n(t)}{\tau_g(t)} + \left[c_2 - \int f(t) \sin t dt \right] \sin t - \left[c_3 + \int f(t) \cos t dt \right] \cos t. \quad (17)$$

As a result the equations (16) and (17) becomes

$$\begin{aligned} \lambda_2(x) = & -\frac{(x+c_1)\kappa_n(x)}{\tau_g(x)} + \left(c_2 - \int f(x)\tau_g(x) \sin [t(x)] dx \right) \sin [t(x)] \\ & - \left(c_3 + \int f(x)\tau_g(x) \cos [t(x)] dx \right) \cos [t(x)]. \end{aligned} \quad (18)$$

and

$$\begin{aligned} \lambda_3(x) = & \left(c_2 - \int f(x)\tau_g(x) \sin [t(x)] dx \right) \cos [t(x)] \\ & + \left(c_3 + \int f(x)\tau_g(x) \cos [t(x)] dx \right) \sin [t(x)] \end{aligned} \quad (19)$$

where $f(x) = \frac{\lambda_1(x)\kappa_g(x)}{\tau_g(x)} - \left(\frac{\lambda_1(x)\kappa_n(x)}{\tau_g(x)} \right)' \frac{1}{\tau_g(x)}$ and $t(x) = \int \tau_g(x) dx$.

Substituting equations (13), (18) and (19) to (11) we obtain (10). This completes the proof. \square

Theorem 3. *The position vector $\beta(x)$ of an arbitrary curve on a surface with respect to the standard frame in the Galilean space G^3 is computed from the natural representation form:*

$$\begin{aligned} \beta(x) = & \left(x, \int \left[\int \left(\kappa_g(x)S_{\tau_g} - \kappa_n(x) \int \tau_g(x)S_{\tau_g} dx \right) dx \right] dx, \right. \\ & \left. \int \left[\int \left(\kappa_g(x)C_{\tau_g} - \kappa_n(x) \int \tau_g(x)C_{\tau_g} dx \right) dx \right] dx \right) \end{aligned} \quad (20)$$

where $C_{\tau_g} = \cos [\int \tau_g(x) dx]$ and $S_{\tau_g} = \sin [\int \tau_g(x) dx]$.

Proof. If $\beta(x)$ is a curve on a surface in Galilean space G^3 , then the Frenet equations (7) are hold. It is easy to see that the following differential equation is obtained by using (7).

$$\left(\frac{1}{\tau_g(x)} \mathbf{Q}'(\mathbf{x}) \right)' = -\tau_g(x) \mathbf{Q}(\mathbf{x})$$

The above equation can be written in the form

$$\frac{d^2\mathbf{Q}}{dt^2} + \mathbf{Q} = 0, \tag{21}$$

where t is the new variable that equals to $t = \int \tau_g(x) dx$.

Thus, we can write \mathbf{Q} as follows:

$$\mathbf{Q} = (0, \sin[\theta(t)], \cos[\theta(t)]) \tag{22}$$

If we substitute (22) into (21), and solve componentwise, we get the following two equations

$$\dot{\theta}(t) = \pm 1, \quad \ddot{\theta}(t) = 0$$

which lead to $\theta(t) = \pm t = \pm \int \tau_g(x) dx$. Without loss of generality, we can assume that $\theta(t)$ has a positive sign. Then we get

$$\mathbf{Q}(x) = \left(0, \sin\left[\int \tau_g(x) dx\right], \cos\left[\int \tau_g(x) dx\right] \right). \tag{23}$$

From (7), we obtain

$$\begin{aligned} \mathbf{n}(x) &= - \int \tau_g(x) \mathbf{Q}(x) dx \\ &= - \int \tau_g(x) \left(0, \sin\left[\int \tau_g(x) dx\right], \cos\left[\int \tau_g(x) dx\right] \right) dx + \mathbf{c} \end{aligned}$$

where \mathbf{c} is a constant vector. Since the first component of normal vector is zero, then we can take $\mathbf{c} = (0, 0, 0)$, and then

$$\mathbf{n}(x) = \left(0, \cos\left(\int \tau_g(x) dx\right), -\sin\left(\int \tau_g(x) dx\right) \right) \tag{24}$$

From (23) and (7), we have

$$\begin{aligned} \mathbf{T}'(x) &= \kappa_g(x) \mathbf{Q}(x) + \kappa_n(x) \mathbf{n}(x) \\ &= \kappa_g(x) \left(0, \sin\left[\int \tau_g(x) dx\right], \cos\left[\int \tau_g(x) dx\right] \right) \\ &\quad + \kappa_n(x) \left(0, -\int \tau_g(x) \sin\left[\int \tau_g(x) dx\right] dx, -\int \tau_g(x) \cos\left[\int \tau_g(x) dx\right] dx \right) \\ &= \left(0, \kappa_g(x) \sin\left[\int \tau_g(x) dx\right] - \kappa_n(x) \int \tau_g(x) \sin\left[\int \tau_g(x) dx\right] dx \right. \\ &\quad \left. , \kappa_g(x) \cos\left[\int \tau_g(x) dx\right] - \kappa_n(x) \int \tau_g(x) \cos\left[\int \tau_g(x) dx\right] dx \right). \end{aligned} \tag{25}$$

If we let $C_{\tau_g} = \cos[\int \tau_g(x) dx]$ and $S_{\tau_g} = \sin[\int \tau_g(x) dx]$, then we have

$$\mathbf{T}'(x) = \left(0, \kappa_g(x)S_{\tau_g} - \kappa_n(x) \int \tau_g(x)S_{\tau_g} dx, \kappa_g(x)C_{\tau_g} - \kappa_n(x) \int \tau_g(x)C_{\tau_g} dx \right). \quad (26)$$

Taking the integral of (26) with respect to, we get

$$\mathbf{T}(x) = \left(0, \int \left(\kappa_g(x)S_{\tau_g} - \kappa_n(x) \int \tau_g(x)S_{\tau_g} dx \right) dx, \int \left(\kappa_g(x)C_{\tau_g} - \kappa_n(x) \int \tau_g(x)C_{\tau_g} dx \right) dx \right) + \mathbf{d} \quad (27)$$

where \mathbf{d} is a constant vector. Since the first component of tangent vector is one, we can take $\mathbf{d} = (1, 0, 0)$, and then

$$\mathbf{T}(x) = \left(1, \int \left(\kappa_g(x)S_{\tau_g} - \kappa_n(x) \int \tau_g(x)S_{\tau_g} dx \right) dx, \int \left(\kappa_g(x)C_{\tau_g} - \kappa_n(x) \int \tau_g(x)C_{\tau_g} dx \right) dx \right). \quad (28)$$

Integrating (28) with respect to x , we have

$$\beta(x) = \left(x, \int \left[\int \left(\kappa_g(x)S_{\tau_g} - \kappa_n(x) \int \tau_g(x)S_{\tau_g} dx \right) dx \right] dx, \int \left[\int \left(\kappa_g(x)C_{\tau_g} - \kappa_n(x) \int \tau_g(x)C_{\tau_g} dx \right) dx \right] dx \right) \quad (29)$$

where $C_{\tau_g} = \cos[\int \tau_g(x) dx]$ and $S_{\tau_g} = \sin[\int \tau_g(x) dx]$ which leads to the equation (20) and the proof is complete. \square

4. APPLICATIONS

We begin a study of important special curves lying on surfaces. For example, geodesic, asymptotic and curvature (or principal) line. Let β be regular curve on the oriented surface in G^3 with the curvature κ , the torsion τ , the geodesic curvature κ_g , the normal curvature κ_n and the geodesic torsion τ_g .

Definition 4. We can say that β is

$$\begin{aligned} \text{geodesic curve} &\iff \kappa_g \equiv 0, \\ \text{asymptotic curve} &\iff \kappa_n \equiv 0, \\ \text{line of curvature} &\iff \tau_g \equiv 0. \end{aligned}$$

Also, We can say that β is called:

κ, τ	\Rightarrow	β
$\kappa \equiv 0$	\Rightarrow	a straight line.
$\tau \equiv 0$	\Rightarrow	a plane curve.
$\kappa \equiv \text{cons.} > 0, \tau \equiv \text{cons.} > 0$	\Rightarrow	a circular helix or W-curve.
$\frac{\tau}{\kappa} \equiv \text{cons.}$	\Rightarrow	a generalized helix.
$\kappa \equiv \text{cons.}, \tau \neq \text{cons.}$	\Rightarrow	Salkowski curve [12,13].
$\kappa \neq \text{cons.}, \tau \equiv \text{cons.}$	\Rightarrow	anti-Salkowski curve [13].

4.1. The position vector of a family of geodesic line in the Galilean space G^3 .

Theorem 5. *The position vector $\beta_g(x)$ of a family of geodesic line in Galilean space G^3 is given by*

$$\beta_g(x) = \left(x, - \int \int \kappa_n(x) \int \tau_g(x) S_{\tau_g} dx dx dx, - \int \int \kappa_n(x) \int \tau_g(x) C_{\tau_g} dx dx dx \right). \tag{31}$$

Proof. By using $\kappa_g(x) \equiv 0$ in the equation (20), we obtain the above equation. \square

Corollary 6. *The position vector of a geodesic that is a circular helix is defined by the equation*

$$\beta_{\text{gch}}(\mathbf{x}) = \left(x, -\frac{e}{c^2} \cos(cx + c_1) + e_1x^2 + e_2x + e_3, \frac{e}{c^2} \sin(cx + c_1) + f_1x^2 + f_2x + f_3 \right)$$

where $c, c_1, e, e_1, e_2, e_3, f_1, f_2$ and f_3 are constants.

Proof. By using the definition (4) and the equations (9) in (5), we get the above equation. \square

Corollary 7. *The position vector of a geodesic that is a generalized helix is defined by the equation*

$$\beta_{\text{ggh}}(\mathbf{x}) = \left(x, \int \int \kappa_n(x) \left[\cos \left(d \int \kappa_n(x) dx \right) + d_1 \right] dx dx, - \int \int \kappa_n(x) \left[\sin \left(d \int \kappa_n(x) dx \right) + d_2 \right] dx dx \right)$$

where d, d_1 and d_2 are constants.

Proof. By using the definition (4) and the equations (9), we obtain $\tau_g(x) = d\kappa_n(x)$. By using this equation in (5), we get the above equation. \square

Corollary 8. *The position vector of a geodesic that is a Salkowski curve is defined by the equation*

$$\beta_{\mathbf{g}_s}(\mathbf{x}) = \left(x, m \int \int \left(\cos \left(\int \tau_g(x) dx \right) + m_1 \right) dx dx, \right. \\ \left. -m \int \int \left(\sin \left(\int \tau_g(x) dx \right) + m_2 \right) dx dx \right)$$

where m, m_1 and m_2 are constants.

Proof. By using the definition (4) and the equations (9), we obtain $\kappa_n(x) \equiv d(const.)$ and $\tau_g(x) \neq const.$. By using this equation in (5), we get the above equation. \square

Corollary 9. *The position vector of a geodesic that is a anti-Salkowski curve is defined by the equation*

$$\beta_{\mathbf{g}_{as}}(\mathbf{x}) = \left(x, \int \int \left(\kappa_n(x) \left[\cos(bx + b_1) + b_2 \right] \right) dx dx, \right. \\ \left. - \int \int \left(\kappa_n(x) \left[\sin(bx + b_1) + b_3 \right] \right) dx dx \right)$$

where b, b_1, b_2 and b_3 are constants.

Proof. By using the definition (4) and the equations (9), we obtain $\tau_g(x) \equiv b(const.)$ and $\kappa_n(x) \neq const.$. By using this equation in (5), we get the above equation. \square

4.2. The position vector of a family of asymptotic line in the Galilean space G^3 .

Theorem 10. *The position vector $\beta_a(x)$ of a family of asymptotic line in Galilean space G^3 is given by*

$$\beta_{\mathbf{a}}(x) = \left(x, \int \int \kappa_g(x) \sin \left(\int \tau_g(x) dx \right) dx dx, \right. \\ \left. \int \int \kappa_g(x) \cos \left(\int \tau_g(x) dx \right) dx dx \right) \quad (32)$$

Proof. By using $\kappa_n(x) \equiv 0$ in the equation (20), we obtain the above equation. \square

Corollary 11. *The position vector of a asymptotic that is a circular helix is defined by the equation*

$$\beta_{\mathbf{a}_{ch}}(\mathbf{x}) = \left(x, -\frac{e}{c^2} \sin(cx + c_1) + c_2x + c_3, -\frac{e}{c^2} \cos(cx + c_1) + c_4x + c_5 \right)$$

where $c, c_1, c_2, c_3, c_4, c_5$ and e are constants.

Proof. By using the definition (4) and the equations (9) in (32), we obtain $\kappa_g(x) \equiv e, \tau_g(x) \equiv c$ where e and c are constants. By using this relation in (32), we get the above equation. \square

Corollary 12. *The position vector of a asymptotic that is a generalized helix is defined by the equation*

$$\beta_{\mathbf{agh}}(\mathbf{x}) = \left(x, -\frac{1}{k} \int \cos \left(k \int \kappa_g(x) dx \right) dx + k_1 x + k_2, \right. \\ \left. \frac{1}{k} \int \sin \left(k \int \kappa_g(x) dx \right) dx + k_3 x + k_4 \right)$$

where k, k_1, k_2, k_3 and k_4 are constants.

Proof. By using the definition (4) and the equations (9), we obtain $\tau_g(x) = k\kappa_g(x)$ where k is a constant. By using this relation in (32), we get the above equation. \square

Corollary 13. *The position vector of a asymptotic that is a Salkowski curve is defined by the equation*

$$\beta_{\mathbf{as}}(\mathbf{x}) = \left(x, \int \int \left(e \sin \left(\int \tau_g(x) dx \right) \right) dx dx, \int \int \left(e \cos \left(\int \tau_g(x) dx \right) \right) dx dx \right)$$

where e is a constant.

Proof. By using the definition (4) and the equations (9), we obtain $\kappa_g(x) \equiv e(const.)$ and $\tau_g(x) \neq const.$. By using this equation in (32), we get the above equation. \square

Corollary 14. *The position vector of a asymptotic that is a anti-Salkowski curve is defined by the equation*

$$\beta_{\mathbf{aas}}(\mathbf{x}) = \left(x, \int \int \left(\kappa_g(x) \sin(dx + d_1) \right) dx dx, \int \int \left(\kappa_g(x) \cos(dx + d_1) \right) dx dx \right)$$

where d and d_1 are constants.

Proof. By using the definition (4) and the equations (9), we obtain $\tau_g(x) \equiv d(const.)$ and $\kappa_g(x) \neq const.$. By using this equation in (32), we get the above equation. \square

4.3. The position vector of a family of line of curvature in the Galilean space G^3 .

Theorem 15. *The position vector $\beta_c(x)$ of a family of line of curvature in Galilean space G^3 is given by*

$$\beta_{\mathbf{p}}(x) = \left(x, \int \int \left(c_1 \kappa_g(x) - c_2 \kappa_n(x) \right) dx dx, \int \int \left(c_3 \kappa_g(x) - c_4 \kappa_n(x) \right) dx dx \right) \quad (33)$$

where c_1, c_2, c_3 and c_4 are constants.

Proof. By using $\tau_g(x) \equiv 0$ in the equation (7), we obtain the above equation. \square

Corollary 16. *The position vector of a line of curvature is a circular helix if and only if the below system of differential equations is satisfied.*

$$\begin{aligned} \kappa_g(x) \kappa'_g(x) + \kappa_n(x) \kappa'_n(x) &= 0, \\ \kappa_n(x) \kappa''_g(x) - \kappa_g(x) \kappa''_n(x) &= 0. \end{aligned} \quad (34)$$

Proof. By using the definition (30) and the equations (9), we get the above equation. \square

Special Case:

If $\kappa_g(x)$ and $\kappa_n(x)$ are constant functions, then the equation (34) is satisfied. Therefore, the line of curvature with $\kappa_g(x) \equiv \text{const.}$ and $\kappa_n(x) \equiv \text{const.}$ is a circular helix, and its position vector is defined by

$$\beta_{\mathbf{p}_{\text{ch}}}(x) = \left(x, a_1 x^2 + a_2 x + a_3, b_1 x^2 + b_2 x + b_3 \right) \quad (35)$$

where a_1, a_2, a_3 and b_1, b_2, b_3 are constants.

Corollary 17. *The position vector of a line of curvature is a generalized helix if and only if the below differential equation is satisfied.*

$$\begin{aligned} \kappa''_g \kappa_n^3 + \kappa''_g \kappa_n \kappa_g^2 - \kappa_n^2 \kappa_g \kappa''_n - 3\kappa_n^2 \kappa'_g \kappa'_n - 3\kappa_n \kappa_g \kappa'_g{}^2 + 3\kappa_n \kappa_g \kappa'_n{}^2 \\ - \kappa_g^3 \kappa''_n + 3\kappa_g^2 \kappa'_g \kappa'_n = 0 \end{aligned} \quad (36)$$

Proof. By using the definition (30) and the equations (9), we get the above equation. \square

Corollary 18. *The position vector of a line of curvature is a Salkowski curve if and only if the following equation is satisfied:*

$$\kappa_g \kappa'_g + \kappa_n \kappa'_n = 0$$

Proof. By using the definition (30) and the equations (9), we get the above equation. \square

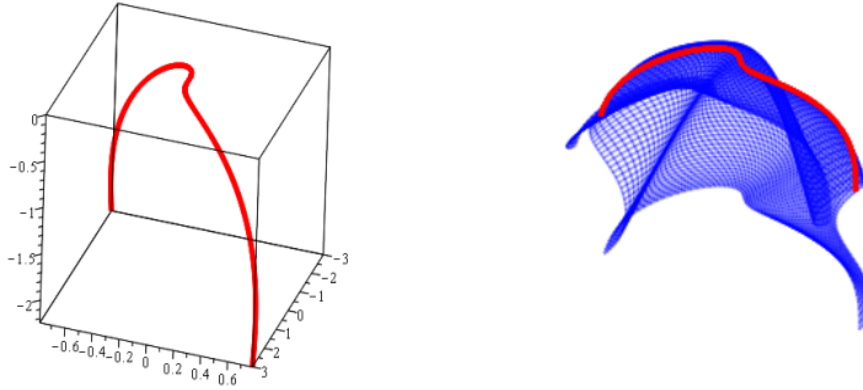


FIGURE 1. The geodesic curve and surface

Corollary 19. *The position vector of a line of curvature is a anti-Salkowski curve if and only if the following equation is satisfied:*

$$\begin{aligned} \kappa_g'' \kappa_g^2 \kappa_n + \kappa_g'' \kappa_n^3 - \kappa_n'' \kappa_g^3 - \kappa_n'' \kappa_g \kappa_n^2 - 2\kappa_g'^2 \kappa_g \kappa_n + 2\kappa_g' \kappa_n' \kappa_g^2 \\ - 2\kappa_g' \kappa_n' \kappa_n^2 + 2\kappa_n'^2 \kappa_g \kappa_n = 0 \end{aligned} \tag{37}$$

Proof. By using the definition (30) and the equations (9), we get the above equation. □

We now consider an example for geodesic curve on surface along with their graphs.

Example 20. *In (5), if we let $\kappa(x) = \sin x$ and $\tau(x) \equiv 1$, we obtain*

$$\alpha(x) = \left(x, \frac{x - \sin(x) \cos(x)}{4}, \frac{\sin(x)^2 - x^2}{4} \right)$$

A surface on which this curve lies can be taken as follows:

$$\phi(u, v) = \left(u + v, \frac{u - \sin(u + v) \cos(u + v)}{4}, \frac{\sin(u + v)^2 - u^2}{4} \right).$$

5. CONCLUSIONS

This study is obtained the position vectors of all curves on a surface in G^3 with respect to the Darboux Frame. Firstly, the position vector of a curve on a surface in G^3 in terms of geodesic, normal curvature and geodesic torsion with respect to the Darboux and standard frame is investigated. As result of these, position vectors

of some special curves such as geodesic, asymptotic curve, line of curvature on a surface is obtained in G^3 .

Consequently, relations of foregoing curves with helix, Salkowski curve and anti-Salkowski curve are given(see (30)). That is, special cases of these curves such as: geodesics that are circular helix, generalized helix or Salkowski, etc is given. Furthermore, the graphs of some special curves is drawn .

In the light of these results, special smarandache curves with respect to Darboux frame in G^3 is studied in [15]. Also, we want to emphasize that the results of this study can be extended to families of surfaces that have common geodesic, asymptotic curve and line of curvature.

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