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ON (λ, A) −STATISTICAL CONVERGENCE OF ORDER α

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Abstract. In the paper [B. de Malafosse and V. Rakočević, Linear Algebra Appl. 420, no. 2-3, (2007), 377–387], authors defined the concept of (λ, A) −statistical convergence. In this paper, the concept of (λ, A) −statistical convergence is generalized to (λ, A) −statistical convergence of order α. Also, we introduce the concept of strong (V, λ, A) −convergence of order α and give some inclusion relations between the concepts of (λ, A) −statistical convergence of order α and strong (V, λ, A) −convergence of order α.

1. Introduction

In 1951, Steinhaus [34] and Fast [22] introduced the concept of statistical convergence and later in 1959, Schoenberg [32] reintroduced independently. Some arguments related to statistical convergence and its applications may be found in ([2], [5], [6], [7], [8], [9], [10],[18],[19], [20], [23], [35], [25], [26], [31], [16], [15], [38], [30], [33], [1], [17], [24]).

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$. The generalized de la Vallée-Poussin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$ for $n = 1, 2, \ldots$. A sequence $x = (x_k)$ is said to be (V, λ) −summable to a number L if $t_n(x) \to L$ as $n \to \infty$. If $\lambda_n = n$, then (V, λ) −summability is reduced to Cesàro summability. By Λ we denote the class of all non-decreasing sequence of positive real numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$.

$\lambda = (\lambda_n)$ sequence spaces were studied in ([11],[12],[21],[27],[28],[13],[14],[29],[36]) and A−statistical convergence for $A = (a_{ik})$ an infinite matrix of complex numbers were studied in ([15],[14],[37],[3],[4]).

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Recently, the concept of $S(\lambda, A)$ −convergence was defined by de Malafosse and Rakočević [14] as below:

Let $A = (a_{km})$ be an infinite matrix of complex numbers and $[AX]_k = A_k(X) = \sum_{m=1}^{\infty} a_{km}x_m$ for $k \geq 0$. A sequence $X = (x_n)_{n\geq 1}$ is said to be (λ, A) −statistically convergent to L (or $S(\lambda, A)$ −convergent to L) if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : |[AX]_k - L| \geq \varepsilon\}| = 0$$

where $I_n = [n - \lambda_n + 1, n]$. In this case we write $x_k \to L(S(\lambda, A))$. The set of all λ−statistically convergent sequences will be denoted by $S(\lambda, A)$. If $\lambda_n = n$, we write $x_k \to L(S(A))$ and in the special case $A = I$, we write $x_k \to L(S(I))$ means that $x_k \to L(S)$.

2. Main Results

In this section, we will give the definition of $S^{\alpha}(\lambda, A)$ −convergence and strong $W_p^{\alpha}(\lambda, A)$ −convergence for $0 < p < \infty$ where $A = (a_{km})$ is an infinite matrix of complex numbers and $0 < \alpha \leq 1$ and give some results related to these concepts.

Definition 1. *Let $\alpha \in (0, 1]$ and $A = (a_{km})$ be an infinite matrix of complex numbers. A sequence $X = (x_k)$ is said to be (λ, A) −statistically convergent of order α to L (or $S^{\alpha}(\lambda, A)$ −convergent to L) if for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{\lambda_n^{\alpha}} |\{k \in I_n : |[AX]_k - L| \geq \varepsilon\}| = 0$$

where $I_n = [n - \lambda_n + 1, n]$ and λ_n^{α} denotes the αth power $(\lambda_n)^{\alpha}$ of λ_n, that is $\lambda^{\alpha} = (\lambda_n^{\alpha}) = (\lambda_1^{\alpha}, \lambda_2^{\alpha}, ..., \lambda_n^{\alpha}, ...)$. In this case we write $S^{\alpha}(\lambda, A) - \lim x_k = L$ or $x_k \to L(S^{\alpha}(\lambda, A))$. The set of all (λ, A)− statistically convergent sequences of order α will be denoted by $S^{\alpha}(\lambda, A)$. For $\lambda_n = n$, we shall write $S^{\alpha}(A)$ instead of $S^{\alpha}(\lambda, A)$ and in the special case $A = I$, $\alpha = 1$ and $\lambda_n = n$ we shall write S instead of $S^{\alpha}(\lambda, A)$.

The (λ, A) −statistical convergence of order α is well defined for $\alpha \in (0, 1]$, but it is not well defined for $\alpha > 1$ in general. $X = (x_m)$ and $A = (a_{km})$ are defined as follows: For $A = (a_{km})$ row matrix and $i = 1, 2, ...$

$$x_m = \begin{cases} 3, & \text{if} \quad m = 3i \\ 0, & \text{if} \quad m \neq 3i. \end{cases}$$

and

$$a_{km} = \begin{cases} 2, & \text{if} \quad m = 3i \\ 0, & \text{if} \quad m \neq 3i. \end{cases}$$

Both for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n^{\alpha}} |\{k \in I_n : |[AX]_k - 6| \geq \varepsilon\}| \leq \lim_{n\to\infty} \frac{[\lambda_n] + 1}{3\lambda_n^{\alpha}} = 0$$

and

$$\lim_{n\to\infty}\frac{1}{\lambda_n^\alpha}\left|\{k\in I_n:|[AX]_k-0|\geq\varepsilon\}\right|\leq\lim_{n\to\infty}\frac{2[\lambda_n]+1}{3\lambda_n^\alpha}=0$$

for $\alpha>1$. *So* $S^\alpha(\lambda,A)-\lim x_k=6$ *and* $S^\alpha(\lambda,A)-\lim x_k=0$, *but this is impossible.*

Theorem 2. *Let* $\alpha\in(0,1]$ *be positive real number. If* $S^\alpha(\lambda,A)-\lim x_k=L_1$ *and* $S^\alpha(\lambda,A)-\lim x_k=L_2$, *then* $L_1=L_2$.

Proof. Since $S^\alpha(\lambda,A)-\lim x_k=L_1$ and $S^\alpha(\lambda,A)-\lim x_k=L_2$, we can write

$$\lim\frac{1}{\lambda_n^\alpha}\left|\{k\in I_n:|[AX]_k-L_1|\geq\varepsilon\}\right|=0$$

and

$$\lim\frac{1}{\lambda_n^\alpha}\left|\{k\in I_n:|[AX]_k-L_2|\geq\varepsilon\}\right|=0.$$

We have

$$\begin{aligned}|L_1-L_2| &= |L_1-L_2+[AX]_k-[AX]_k|\\ &\leq |[AX]_k-L_1|+|[AX]_k-L_2|\end{aligned}$$

for $I_n=[n-\lambda_n+1,n]$. We get

$$\begin{aligned}\frac{1}{\lambda_n^\alpha}\left|\{k\in I_n:|L_1-L_2|\geq\varepsilon\}\right| &\leq \frac{1}{\lambda_n^\alpha}\left|\{k\in I_n:|[AX]_k-L_1|\geq\varepsilon\}\right|\\ &\quad+\frac{1}{\lambda_n^\alpha}\left|\{k\in I_n:|[AX]_k-L_2|\geq\varepsilon\}\right|.\end{aligned}$$

This is possible with $L_1=L_2$. □

Theorem 3. *Let* $\alpha\in(0,1]$ *be positive real number,* $A=(a_{km})$ *be an infinite matrix of complex numbers and* $X=(x_k)$, $Y=(y_k)$ *be sequences of real numbers, then*

(*i*) *If* $S^\alpha(\lambda,A)-\lim x_k=x_0$ *and* $S^\alpha(\lambda,A)-\lim y_k=y_0$, *then* $S^\alpha(\lambda,A)-\lim(x_k+y_k)=(x_0+y_0)$,

(*ii*) *If* $S^\alpha(\lambda,A)-\lim x_k=x_0$ *and* $c\in\mathbb{C}$, *then* $S^\alpha(\lambda,A)-\lim(cx_k)=cx_0$.

Proof. Ommited. □

Definition 4. *Let* $\alpha\in(0,1]$, $0<p<\infty$ *and* $A=(a_{km})$ *be an infinite matrix of complex numbers. We say that the sequence* $X=(x_k)$ *is strong* $(V,\lambda,A)-$*convergent of order* α *to a number* L *(or* $W_p^\alpha(\lambda,A)-$*convergent to* L*) if*

$$\lim_{n\to\infty}\frac{1}{\lambda_n^\alpha}\sum_{k\in I_n}|[AX]_k-L|^p=0.$$

In this case, we write $W_p^\alpha(\lambda,A)-\lim x_k=L$ *or* $x_k\to L(W_p^\alpha(\lambda,A))$.

Theorem 5. *Let* $\alpha\in(0,1]$ *be positive real numbers and* $A=(a_{km})$ *be an infinite matrix of complex numbers, then* $W_p^\alpha(\lambda,A)\subseteq S^\alpha(\lambda,A)$ *and the inclusion is strict.*

Proof. $\varepsilon > 0$ and $x_k \to L(W_p^\alpha(\lambda, A))$. In this case, we have

$$\sum_{k\in I_n} |[AX]_k - L|^p \geq \varepsilon^p \left|\{k \in I_n : |[AX]_k - L| \geq \varepsilon\}\right|$$

and

$$\frac{1}{\lambda_n^\alpha} \left|\{k \in I_n : |[AX]_k - L| \geq \varepsilon\}\right| \leq \frac{1}{\lambda_n^\alpha \varepsilon^p} \sum_{k\in I_n} |[AX]_k - L|^p .$$

So $x_k \to L(S^\alpha(\lambda, A))$.

To show that the inclusion is strict define a sequence $X = (x_m)$ and a row matrix $A = (a_{km})$ such that for $i = 1, 2, ...$

$$x_m = \begin{cases} 4, & \text{if } m = i^2 \\ 0, & \text{if } m \neq i^2 \end{cases}$$

and

$$a_{km} = \begin{cases} 1, & \text{if } m = i^2 \\ 0, & \text{if } m \neq i^2 \end{cases}$$

Let $\lambda_n = n$, $p = 1$ and $L = 0$. For $\frac{1}{2} < \alpha \leq 1$

$$\frac{1}{\lambda_n^\alpha} \left|\{k \in I_n : |[AX]_k - 0| \geq \varepsilon\}\right| \leq \frac{\sqrt{n}}{n^\alpha} \to 0.$$

i.e. $x_k \to 0(S^\alpha(\lambda, A))$. For $0 < \alpha < \frac{1}{2}$

$$\frac{1}{\lambda_n^\alpha} \sum_{k\in I_n} |[AX]_k - L|^p = \frac{1}{n^\alpha} \sum_{k=1}^{n} |[AX]_k| \leq \frac{4\sqrt{n}}{n^\alpha} \to \infty$$

and for $\alpha = \frac{1}{2}$

$$\frac{1}{n^\alpha} \sum_{k=1}^{n} |[AX]_k| \leq \frac{4\sqrt{n}}{n^\alpha} \to 4$$

i.e. $x_k \nrightarrow 0(W_p^\alpha(\lambda, A))$. □

Theorem 6. *Let $\alpha, \beta \in (0, 1]$ be positive real numbers such that $\alpha \leq \beta$, then $S^\alpha(\lambda, A) \subseteq S^\beta(\lambda, A)$.*

Proof. For $\varepsilon > 0$, we can write

$$\frac{1}{\lambda_n^\beta} \left|\{k \in I_n : |[AX]_k - L| \geq \varepsilon\}\right| \leq \frac{1}{\lambda_n^\alpha} \left|\{k \in I_n : |[AX]_k - L| \geq \varepsilon\}\right| .$$

So $S^\alpha(\lambda, A) \subseteq S^\beta(\lambda, A)$ for $0 < \alpha \leq \beta \leq 1$.

To show that the inclusion is strict define a sequence $X = (x_k)$ by

$$x_k = \begin{cases} 3, & \text{if } k \text{ is square} \\ 0, & \text{otherwise} \end{cases} .$$

Let $A = I$. Then $X \in S^\beta(\lambda, A)$ for $\beta \in \left(\frac{1}{2}, 1\right]$, but $X \notin S^\alpha(\lambda, A)$ for $\alpha \in \left(0, \frac{1}{2}\right]$. □

Theorem 7. *Let* $\alpha \in (0,1]$, $S^{\alpha}(\lambda, A) - \lim x_k = x_0$ *and* $S^{\alpha}(\lambda, A) - \lim y_k = y_0$. *If* $|[AX]_k| = |A_k(X)| = \left|\sum_{m=1}^{\infty} a_{km} x_m\right| < M,\ (M > 0)$, *then* $\lim_{n\to\infty} \frac{1}{\lambda_n^{\alpha}} |\{k \in I_n : |([AX]_k[AY]_k) - (x_0 y_0)| \geq \varepsilon\}| = 0$.

Proof. For $\varepsilon > 0$, we can write

$$\begin{aligned}
& \frac{1}{\lambda_n^{\alpha}} |\{k \in I_n : |([AX]_k[AY]_k) - (x_0 y_0)| \geq \varepsilon\}| \\
= & \frac{1}{\lambda_n^{\alpha}} |\{k \in I_n : |[AX]_k[AY]_k - (x_0 y_0) + [AX]_k y_0 - [AX]_k y_0| \geq \varepsilon\}| \\
= & \frac{1}{\lambda_n^{\alpha}} |\{k \in I_n : |[AX]_k ([AY]_k - y_0) + y_0 ([AX]_k - x_0)| \geq \varepsilon\}| \\
\leq & \frac{1}{\lambda_n^{\alpha}} \left|\left\{k \in I_n : |[AX]_k ([AY]_k - y_0)| \geq \frac{\varepsilon}{2}\right\}\right| \\
& + \frac{1}{\lambda_n^{\alpha}} \left|\left\{k \in I_n : |y_0 ([AX]_k - x_0)| \geq \frac{\varepsilon}{2}\right\}\right| \\
= & \frac{1}{\lambda_n^{\alpha}} \left|\left\{k \in I_n : |([AY]_k - y_0)| \geq \frac{\varepsilon}{2[AX]_k} > \frac{\varepsilon}{2M}\right\}\right| \\
& + \frac{1}{\lambda_n^{\alpha}} \left|\left\{k \in I_n : |([AX]_k - x_0)| \geq \frac{\varepsilon}{2|y_0|}\right\}\right|.
\end{aligned}$$

Therefore, $\lim_{n\to\infty} \frac{1}{\lambda_n^{\alpha}} |\{k \in I_n : |([AX]_k[AY]_k) - (x_0 y_0)| \geq \varepsilon\}| = 0$. □

Theorem 8. *Let* $\alpha \in (0,1]$. $S(A) \subseteq S^{\alpha}(\lambda, A)$ *if and only if*

$$\lim_{n\to\infty} \inf \frac{\lambda_n^{\alpha}}{n} > 0. \tag{1}$$

Proof. For a given $\varepsilon > 0$, since

$$\{k \leq n : |[AX]_k - L| \geq \varepsilon\} \supset \{k \in I_n : |[AX]_k - L| \geq \varepsilon\},$$

we can write

$$\begin{aligned}
\frac{1}{n} |\{k \leq n : |[AX]_k - L| \geq \varepsilon\}| & \geq \frac{1}{n} |\{k \in I_n : |[AX]_k - L| \geq \varepsilon\}| \\
& = \frac{\lambda_n^{\alpha}}{n} \frac{1}{\lambda_n^{\alpha}} |\{k \in I_n : |[AX]_k - L| \geq \varepsilon\}|.
\end{aligned}$$

Conversely, suppose that $\lim_{n\to\infty} \inf \frac{\lambda_n^{\alpha}}{n} = 0$. We can choose a subsequence $(n(j))_{j=1}^{\infty}$ such that $\frac{\lambda_{n(j)}^{\alpha}}{n(j)} < \frac{1}{j}$. Define a sequence $X = (x_k)$ by for $j = 1, 2, ...$

$$x_k = \begin{cases} 1, & \text{if } k \in I_{n(j)} \\ 0, & \text{otherwise} \end{cases}.$$

Let $A = I$. Then $X \in S(A)$, but $X \notin S(\lambda, A)$. From Theorem 6, since $S^{\alpha}(\lambda, A) \subseteq S(\lambda, A)$, we have $X \notin S^{\alpha}(\lambda, A)$. Hence (1) is necessary. □

Theorem 9. *Let $\alpha, \beta \in (0,1]$ be positive real numbers such that $\alpha \leq \beta$, then $W_p^{\alpha}(\lambda, A) \subseteq W_p^{\beta}(\lambda, A)$.*

Proof. For $\varepsilon > 0$, we can write

$$\frac{1}{\lambda_n^{\beta}} \sum_{k \in I_n} |[AX]_k - L|^p \leq \frac{1}{\lambda_n^{\alpha}} \sum_{k \in I_n} |[AX]_k - L|^p .$$

So $W_p^{\alpha}(\lambda, A) \subseteq W_p^{\beta}(\lambda, A)$ for $0 < \alpha \leq \beta \leq 1$.

To show that the inclusion is strict define a sequence $X = (x_k)$ by

$$x_k = \begin{cases} 2, & \text{if } k \text{ is square} \\ 0, & \text{otherwise} \end{cases} .$$

Let $A = I$. Then $X \in W_p^{\beta}(\lambda, A)$ for $\beta \in \left(\frac{1}{2}, 1\right]$, but $X \notin W_p^{\alpha}(\lambda, A)$ for $\alpha \in \left(0, \frac{1}{2}\right]$. □

Theorem 10. *Let $\lambda = (\lambda_n), \mu = (\mu_n) \in \Lambda$ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$ and $\alpha, \beta \in (0,1]$ be positive real numbers such that $0 < \alpha \leq \beta \leq 1$.*

(i) If

$$\lim_{n \to \infty} \inf \frac{\lambda_n^{\alpha}}{\mu_n^{\beta}} > 0 \tag{2}$$

then $S^{\beta}(\mu, A) \subseteq S^{\alpha}(\lambda, A)$,

(ii) If

$$\lim_{n \to \infty} \frac{\mu_n}{\lambda_n^{\beta}} = 1 \tag{3}$$

then $S^{\alpha}(\lambda, A) \subseteq S^{\beta}(\mu, A)$, where $I_n = [n - \lambda_n + 1, n]$, $J_n = [n - \mu_n + 1, n]$.

Proof. (i) Suppose that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$ and let (2) be satisfied. For given $\varepsilon > 0$ we have

$$\{k \in J_n : |[AX]_k - L| \geq \varepsilon\} \supseteq \{k \in I_n : |[AX]_k - L| \geq \varepsilon\}$$

and so

$$\frac{1}{\mu_n^{\beta}} |\{k \in J_n : |[AX]_k - L| \geq \varepsilon\}| \geq \frac{\lambda_n^{\alpha}}{\mu_n^{\beta}} \frac{1}{\lambda_n^{\alpha}} |\{k \in I_n : |[AX]_k - L| \geq \varepsilon\}|$$

for all $n \in \mathbb{N}$.

(ii) Let $X = (x_k) \in S^{\alpha}(\lambda, A)$ and (3) be satisfied. We have

$$\lim_{n \to \infty} \frac{1}{\lambda_n^{\alpha}} |\{k \in I_n : |[AX]_k - L| \geq \varepsilon\}| = 0.$$

Since $I_n \subset J_n$, for $\varepsilon > 0$ we may write

$$\begin{aligned}
& \frac{1}{\mu_n^\beta}\left|\{k \in J_n : |[AX]_k - L| \geq \varepsilon\}\right| \\
= & \frac{1}{\mu_n^\beta}\left|\{n-\mu_n+1 < k \leq n-\lambda_n : |[AX]_k - L| \geq \varepsilon\}\right| \\
& + \frac{1}{\mu_n^\beta}\left|\{k \in I_n : |[AX]_k - L| \geq \varepsilon\}\right| \\
\leq & \frac{(\mu_n-\lambda_n)}{\mu_n^\beta} + \frac{1}{\lambda_n^\alpha}\left|\{k \in I_n : |[AX]_k - L| \geq \varepsilon\}\right| \\
\leq & \frac{\mu_n-\lambda_n^\beta}{\lambda_n^\beta} + \frac{1}{\lambda_n^\alpha}\left|\{k \in I_n : |[AX]_k - L| \geq \varepsilon\}\right| \\
\leq & \left(\frac{\mu_n}{\lambda_n^\beta} - 1\right) + \frac{1}{\lambda_n^\alpha}\left|\{k \in I_n : |[AX]_k - L| \geq \varepsilon\}\right|
\end{aligned}$$

for all $n \in \mathbb{N}$. This implies that $S^\alpha(\lambda, A) \subseteq S^\beta(\mu, A)$. □

Corollary 11. *Let* $\lambda = (\lambda_n)$, $\mu = (\mu_n) \in \Lambda$ *such that* $\lambda_n \leq \mu_n$ *for all* $n \in \mathbb{N}$ *and* $0 < \alpha \leq \beta \leq 1$.

If (2) *holds, then*
(*i*) $S^\alpha(\mu, A) \subseteq S^\alpha(\lambda, A)$,
(*ii*) $S(\mu, A) \subseteq S^\alpha(\lambda, A)$,
(*iii*) $S(\mu, A) \subseteq S(\lambda, A)$.

If (3) *holds, then*
(*i*) $S^\alpha(\lambda, A) \subseteq S^\alpha(\mu, A)$,
(*ii*) $S^\alpha(\lambda, A) \subseteq S(\mu, A)$,
(*iii*) $S(\lambda, A) \subseteq S(\mu, A)$.

Theorem 12. *Let* $\lambda = (\lambda_n)$, $\mu = (\mu_n) \in \Lambda$ *such that* $\lambda_n \leq \mu_n$ *for all* $n \in \mathbb{N}$ *and* $0 < \alpha \leq \beta \leq 1$. *Then*
(*i*) *If* (2) *holds, then* $W_p^\beta(\mu, A) \subset W_p^\alpha(\lambda, A)$,
(*ii*) *If* (3) *holds and* $\sup_k |A_k(x)| < \infty$ *then* $W_p^\alpha(\lambda, A) \subset W_p^\beta(\mu, A)$.

Proof. (*i*) Suppose that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$ and let (2) be satisfied. For given $\varepsilon > 0$ we have

$$\frac{1}{\mu_n^\beta}\sum_{k\in J_n} |[AX]_k - L|^p > \frac{\lambda_n^\alpha}{\mu_n^\beta}\frac{1}{\lambda_n^\alpha}\sum_{k\in I_n} |[AX]_k - L|^p .$$

This implies that $W_p^\beta(\mu, A) \subset W_p^\alpha(\lambda, A)$.

(*ii*) Let $X = (x_k) \in W_p^\alpha(\lambda, A)$ and suppose that (3) holds. Then

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha}\sum_{k\in I_n} |[AX]_k - L|^p = 0.$$

Since $\sup_k |A_k(x)| < \infty$ then there exists some $M > 0$ such that $|[AX]_k - L| \leq M$ for all k. Now, since $I_n \subseteq J_n$ and $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$, we may write

$$\begin{aligned}
\frac{1}{\mu_n^\beta} \sum_{k \in J_n} |[AX]_k - L|^p &= \frac{1}{\mu_n^\beta} \sum_{k \in J_n - I_n} |[AX]_k - L|^p + \frac{1}{\mu_n^\beta} \sum_{k \in I_n} |[AX]_k - L|^p \\
&\leq \frac{\mu_n - \lambda_n}{\mu_n^\beta} M^p + \frac{1}{\mu_n^\beta} \sum_{k \in I_n} |[AX]_k - L|^p \\
&\leq \left(\frac{\mu_n}{\lambda_n^\beta} - \frac{\lambda_n^\beta}{\lambda_n^\beta} \right) M^p + \frac{1}{\lambda_n^\beta} \sum_{k \in I_n} |[AX]_k - L|^p \\
&\leq \left(\frac{\mu_n}{\lambda_n^\beta} - 1 \right) M^p + \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} |[AX]_k - L|^p
\end{aligned}$$

for every $n \in \mathbb{N}$. Therefore $W_p^\alpha(\lambda, A) \subset W_p^\beta(\mu, A)$. □

Corollary 13. *Let* $\lambda = (\lambda_n)$, $\mu = (\mu_n) \in \Lambda$ *such that* $\lambda_n \leq \mu_n$ *for all* $n \in \mathbb{N}$ *and* $0 < \alpha \leq \beta \leq 1$.

If (2) *holds, then*
(*i*) $W_p^\alpha(\mu, A) \subset W_p^\alpha(\lambda, A)$,
(*ii*) $W_p(\mu, A) \subset W_p^\alpha(\lambda, A)$,
(*iii*) $W_p(\mu, A) \subset W_p(\lambda, A)$.

If (3) *holds and* $\sup_k |A_k(x)| < \infty$, *then*
(*i*) $W_p^\alpha(\lambda, A) \subset W_p^\alpha(\mu, A)$,
(*ii*) $W_p^\alpha(\lambda, A) \subset W_p(\mu, A)$,
(*iii*) $W_p(\lambda, A) \subset W_p(\mu, A)$.

References

[1] Atlihan, Ö. G., Ünver, M. and Duman, O., Korovkin theorems on weighted spaces: revisited, *Period. Math. Hungar.,* 75(2) (2017) 201–209.
[2] Bhardwaj, V. K. and Dhawan, S., Density by moduli and lacunary statistical convergence, *Abstr. Appl. Anal.,* 2016, Art. ID 9365037, 11 pp.
[3] Bilgin, T., Lacunary strong $A-$convergence with respect to a modulus, *Studia Univ. Babeş-Bolyai Math.,* 46(4) (2001) 39–46.
[4] Bilgin, T., Lacunary strong $A-$convergence with respect to a sequence of modulus functions, *Appl. Math. Comput,.* 151(3) (2004) 595–600.
[5] Caserta, A., Di Maio, G. and Kočinac, L. D. R., Statistical convergence in function spaces, *Abstr. Appl. Anal.,* 2011, Art. ID 420419, 11 pp.
[6] Connor, J. S., The statistical and strong $p-$Cesaro convergence of sequences, *Analysis,* 8 (1988) 47–63.
[7] Çakallı, H., Lacunary statistical convergence in topological groups, *Indian J. Pure Appl. Math.,* 26(2) (1995) 113–119.
[8] Çakallı, H., A study on statistical convergence, *Funct. Anal. Approx. Comput.,* 1(2) (2009) 19–24.

[9] Çınar, M., Karakaş, M. and Et, M., On pointwise and uniform statistical convergence of order α for sequences of functions, *Fixed Point Theory and Applications*, 2013(33) (2013) 11 pp.

[10] Çolak, R., Statistical convergence of order α, *Modern Methods in Analysis and Its Applications, New Delhi, India: Anamaya Pub.*, 2010 (2010) 121–129.

[11] Çolak, R., On $\lambda-$Statistical Convergence, *Conference on Summability and Aplications,* Commerce University, May 12-13 (2011), Istanbul, Turkey.

[12] Çolak, R. and Bektaş, Ç. A., On $\lambda-$Statistical Convergence of Order α, *Acta Math. Sci.,* 31(3) (2011) 953–959.

[13] Çolak, R., Altın, Y. and Et, M., $\lambda-$almost statistical convergence of order α. *An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.),* 60(2) (2014) 437–448.

[14] De Malafosse, B. and Rakočević, V., Matrix transformation and statistical convergence, *Linear Algebra Appl.,* 420(2-3) (2007) 377–387.

[15] Demirci, K., Strong $A-$summability and $A-$statistical convergence, *Indian J. Pure Appl. Math.,* 27(6) (1996) 589–593.

[16] Di Maio, G. and Kočinac, L. D. R., Statistical convergence in topology, *Topology Appl.,* 156 (2008) 28–45.

[17] Doğru, O. and Duman, O., Statistical approximation of Meyer-König and Zeller operators based on $q-$integers, *Publ. Math. Debrecen,* 68(1-2) (2006) 199–214.

[18] Et, M., Çınar, M. and Karakaş, M., On $\lambda-$statistical convergence of order α of sequences of functions, *J. Inequal. Appl.,* 2013(204) (2013) 8 pp.

[19] Et, M., Alotaibi, A. and Mohiuddine, S. A., On (Δ^m, I) statistical convergence of order α, *Scientific World Journal*, 2014 (2014) Article Number: 535419.

[20] Et, M. and Şengül, H., Some Cesaro-type summability spaces of order α and lacunary statistical convergence of order α, *Filomat,* 28(8) (2014) 1593–1602.

[21] Et, M., On pointwise $\lambda-$statistical convergence of order α of sequences of fuzzy mappings, *Filomat,* 28(6) (2014) 1271–1279.

[22] Fast, H., Sur la convergence statistique,*Colloq. Math.,* 2 (1951) 241–244.

[23] Fridy, J., On statistical convergence, *Analysis* 5 (1985), 301–313.

[24] Gadjiev, A. D. and Orhan, C., Some approximation theorems via statistical convergence, *Rocky Mountain J. Math.,* 32(1) (2002) 129–138.

[25] Işık, M. and Akbaş, K. E., On $\lambda-$statistical convergence of order α in probability, *J. Inequal. Spec. Funct.,* 8(4) (2017) 57–64.

[26] Işık, M., Generalized vector-valued sequence spaces defined by modulus functions, *J. Inequal, Appl.,* 2010 (2010) Art. ID 457892, 7 pp.

[27] Işık, M. and Et, M., Almost λ_r-statistical and strongly almost λ_r-convergence of order β of sequences of fuzzy numbers, *J. Funct. Spaces,* 2015 (2015) Art. ID 451050, 6 pp.

[28] Işık, M. and Altın, Y., $f_{(\lambda,\mu)}-$statistical convergence of order $\widetilde{\alpha}$ for double sequences, *J. Inequal. Appl.,* 246 (2017) 8 pp.

[29] Malkowsky, E., The continuous duals of the spaces $c_0(\Lambda)$ and $c(\Lambda)$ for exponentially bounded sequences Λ, *Acta Sci. Math. (Szeged),* 61(1-4) (1995) 241–250.

[30] Sağıroğlu, S. and Ünver, M., Statistical convergence of sequences of sets in hyperspaces, *Hacet. J. Math. Stat.,* 47(4) (2018) 889–896.

[31] Salat, T., On statistically convergent sequences of real numbers, *Math. Slovaca,* 30 (1980) 139–150.

[32] Schoenberg, I. J., The integrability of certain functions and related summability methods, *Amer. Math. Monthly,* 66 (1959) 361–375.

[33] Söylemez, D. and Ünver, M., Korovkin type theorems for Cheney-Sharma operators via summability methods, *Results Math.,* 72(3) (2017) 1601–1612.

[34] Steinhaus, H., Sur la convergence ordinaire et la convergence asymptotique, *Colloq. Math.,* 2 (1951) 73–74.

[35] Şengül, H. and Et, M., On lacunary statistical convergence of order α, *Acta Math. Sci. Ser. B Engl. Ed.,* 34 (2) (2014) 473–482.

[36] Şengül, H. and Et, M., On (λ, I) −statistical convergence of order α of sequences of function, *Proc. Nat. Acad. Sci. India Sect. A,* 88(2) (2018) 181–186.

[37] Şengül, H. and Arıca, Z., Lacunary A−statistical convergence and lacunary strong A−convergence of order α with respect to a modulus, *AIP Conference Proceedings* 2086, 030037 (2019), doi: https://doi.org/10.1063/1.5095122.

[38] Turan, C. and Duman, O., Statistical convergence on timescales and its characterizations, *Advances in applied mathematics and approximation theory, Springer, New York,* (2013), pp. 57–71.

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