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ON α-ALMOST QUASI ARTINIAN MODULES

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Abstract. In this article we introduce and study the concepts of α-almost quasi Artinian and α-quasi Krull modules. Using these concepts we extend some of the basic results of α-almost Artinian and α-Krull modules to α-almost quasi Artinian and α-quasi Krull modules. We observe that if M is an α-quasi Krull module then the quasi Krull dimension of M is either α or $\alpha+1$.

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1. Introduction

The concept of Noetherian dimension of a module M, (the dual of Krull dimension of M, in the sense of Rentschler and Gabriel, see [17,26]) introduced in Lemonnier [27], and Karamzadeh [20], is almost as old as Krull dimension of M, and their existence are equivalent. Later, Chambless [4] studied dual Krull dimension and called it N-dimension. Roberts [28] calls this dual dimension again Krull dimension. The latter dimension is also called dual Krull dimension in some other articles, see for example, [1,2]. The former dimension has received some attention; see [1,2,16,21,23,24]. In this article, all rings are associative with $1 \neq 0$, and all modules are unital right modules. If M is an R-module, then $n\text{-dim}\, M$ and $k\text{-dim}\, M$ will denote the Noetherian dimension and the Krull dimension of M, respectively.

Davoudian, Karamzadeh and Shirali in [14] introduce and study the concepts of α-short modules and α-almost Noetherian modules. We recall that an R-module M is called an α-short module, if for each submodule N of M, either $n\text{-dim}\, N \leq \alpha$ or $n\text{-dim}\, \frac{M}{N} \leq \alpha$ and α is the least ordinal number with this property. We also recall that an R-module M is called α-almost Noetherian, if for each proper submodule N of M, $n\text{-dim}\, N < \alpha$ and α is the least ordinal number with this property, see [14]. Later Davoudian, Halali and Shirali undertook a systematic study of the concepts of α-almost Artinian and α-Krull modules, which are the dual of the concepts of α-almost Noetherian and α-short modules, respectivly, see [12]. We introduce and extensively investigate quasi-Krull dimension and quasi-Noetherian dimension of an R-module M, see [5]. The quasi-Noetherian dimension (resp., quasi-Krull dimension), which is denoted by $qn\text{-}\dim M$ (resp., $qk\text{-}\dim M$) is defined to be the codeviation (resp., deviation) of the poset of all non-finitely generated submodules of M. We recall that an R-module M is called α-quasi critical, where α is an ordinal, if $qk\text{-}\dim M = \alpha$ and $qk\text{-}\dim \frac{M}{N} < \alpha$ for any non-finitely generated submodule N of M. M is said to be quasi-critical if it is α-quasi critical for some α. We also extensively investigate the concepts of α-almost quasi Noetherian and α-quasi short modules, see [6]. Recall that an R-module M is called α-almost quasi Noetherian if for each non-finitely generated submodule N of M, $qn\text{-}\dim N < \alpha$ and α is the least ordinal with this property. We also recall that an R-module M is called α-quasi short if for each non-finitely generated submodule N of M, either $qn\text{-}\dim N < \alpha$ or $qn\text{-}\dim \frac{M}{N} < \alpha$ and α is the least ordinal number with this property. It is convenient, when we are dealing with the latter dimensions, to begin our list of ordinales with -1. In this article we introduce and study the concepts of α-almost quasi Artinian and α-quasi Krull modules. These concepts are the dual of the concepts of α-almost quasi Noetherian and α-quasi short modules, respectively; and at the same time are the extension of the concepts of α-almost Artinian and α-Krull modules, respectively. Let us give a brief outline of this paper. Section 1 is the introduction. In Section 2, we introduce and study the concept of α-almost quasi Artinian and α-quasi Krull modules. Hein [19] introduced almost Artinian modules and studied some of the properties of these modules. Later Davoudian, Halali and Shirali undertook a systematic study of the concept of α-almost Artinian modules. We recall that an R-module M is called α-almost Artinian, if for each non-zero submodule N of M, $k\text{-}\dim \frac{M}{N} < \alpha$ and α is the least ordinal number with this property, see [14]. We shall call an R-module M to be α-almost quasi Artinian if for each non-finitely generated submodule N of M, $qk\text{-}\dim \frac{M}{N} < \alpha$ and α is the least ordinal number with this property. Using this concept we extend some of the basic results of α-almost Artinian modules to α-almost quasi Artinian modules. In particular, we observe that each α-almost quasi Artinian module M has quasi Krull dimension and $qk\text{-}\dim M \leq \alpha$. We also introduce and study the concept of α-quasi Krull modules, which is the dual of α-quasi short modules, see [6]. We recall that an R-module M is called an α-Krull module, if for each submodule N of M, either $k\text{-}\dim N \leq \alpha$ or $k\text{-}\dim \frac{M}{N} \leq \alpha$ and α is the least ordinal number with this property. We shall call an R-module M to be α-quasi Krull if for each non-finitely generated submodule N of M, either $qk\text{-}\dim N \leq \alpha$ or $qk\text{-}\dim \frac{M}{N} \leq \alpha$ and α is the least ordinal number with this property. In the last section we also investigate some properties of α-almost quasi Artinian and α-quasi Krull modules. Finally, we should emphasize here that the results in Section 2 and Section 3 are new and are the dual of the corresponding results in [6] and at the same time are the extensions of the results in [12]. For all concepts and basic properties of rings and modules which are not defined in this paper, we refer the reader to [3,5,7,8,9,10,11,13,17].

2. α-Almost quasi Artinian and α-quasi Krull modules

In this section we introduce and study α-almost quasi Artinian and α-quasi Krull modules. We extend some of the basic results of α-almost Artinian modules to α-almost quasi Artinian modules.

Let us recall that the deviation of an arbitrary partially ordered set $E = (E, \leq)$, (shortly poset), denoted by $dev(E)$ is defined as follows: $dev(E) = -1$ if and only if E is a trivial poset, i.e., E has no two distinct comparable elements. If E is nontrivial but satisfies the descending chain condition on its elements, then $dev(E) = 0$. For a general ordinal α, we define $dev(E) = \alpha$, provided:

(i) $dev(E) \neq \beta < \alpha$;

(ii) for any descending chain $x_1 \geq x_2 \geq \cdots \geq x_n \geq \cdots$ of elements of E there is some $n_0 \in N$ such that for all $n \geq n_0$ the deviation of the poset

$$\frac{x_n}{x_{n+1}} := \{x \in E | x_{n+1} \leq x \leq x_n\}$$

already defined and satisfies

$$dev(\frac{x_n}{x_{n+1}}) < \alpha.$$

If no ordinal α exists such that $dev(E) = \alpha$, we say E does not have deviation. For any R-module M we shall denote by NF(M) the poset of all non-finitely generated submodules of M. The quasi-Krull dimension of the right R-module M, denoted by $qk\text{-dim}\, M$, is defined to be the deviation of the poset $(\mathrm{NF(M)}, \subseteq)$, see [5, Definition 1].

We continue with our definition of α-almost quasi Artinian modules.

Definition 2.1. An R-module M is called α-almost quasi Artinian, if for each non-finitely generated submodule N of M, $qk\text{-dim}\, \frac{M}{N} < \alpha$ and α is the least ordinal number with this property.

We should remind the reader that the above concept is in fact the dual of α-almost quasi Noetherian modules, see [6, Definition 2.1].

Remark 2.2. If M is an α-almost quasi Artinian module, then each submodule and each factor module of M is β-almost quasi Artinian for some $\beta \leq \alpha$.

We recall that an R-module M is called α-quasi critical, if $qk\text{-}\dim M = \alpha$ and for each non-finitely generated submodule N of M we have $qk\text{-}\dim \frac{M}{N} < \alpha$. M is called quasi-critical if it is α-quasi critical for some ordinal number α, see [5, Definition 2]. The next three trivial, but useful facts, which are the dual of the corresponding facts in [6, Lemmas 2.2, 2.3, 2.4] are needed.

Lemma 2.3. *If M is an α-almost quasi Artinian module, then M has quasi Krull dimension and $qk\text{-}\dim M \leq \alpha$. In particular, $qk\text{-}\dim M = \alpha$ if and only if M is α-quasi critical.*

Proof. For each proper non-finitely generated submodule N of M, we have $qk\text{-}\dim \frac{M}{N} < \alpha$. In view of [5, Lemma 4], we get $qk\text{-}\dim M \leq \alpha$. The final part is now evident. □

Lemma 2.4. *If M is a module with $qk\text{-}\dim M = \alpha$, then either M is α-quasi critical, in which case it is α-almost quasi Artinian, or it is $\alpha + 1$-almost quasi Artinian.*

Proof. Let M be an α-quasi critical module, then for each non-finitely generated submodule N of M, we have $qk\text{-}\dim \frac{M}{N} < \alpha$. Hence M is β-almost quasi Artinian, for some ordinal number $\beta \leq \alpha$. If $\beta < \alpha$, then by Lemma 2.3 we have $qk\text{-}\dim M \leq \beta$ which is a contradiction. If M is not quasi critical, then there exists a non-finitely generated submodule N of M such that $qk\text{-}\dim \frac{M}{N} = \alpha$. This implies that M is γ-almost quasi Artinian for some $\gamma \geq \alpha + 1$. But for each non-finitely generated submodule N of M, we have $qk\text{-}\dim \frac{M}{N} \leq \alpha < \alpha + 1$, see [5, Theorem 1]. Therefore M is $\alpha + 1$-almost quasi Artinian. □

Lemma 2.5. *If M is an α-almost quasi Artinian module, then either M is α-quasi critical or $\alpha = qk\text{-}\dim M + 1$. In particular, if M is an α-almost quasi Artinian module, where α is a limit ordinal, then M is α-quasi critical.*

Proof. We infer that M has quasi Krull dimension and $qk\text{-}\dim M \leq \alpha$, by Lemma 2.3. If $qk\text{-}\dim M = \alpha$, then in view of Lemma 2.3, M is α-quasi critical. Now let $qk\text{-}\dim M < \alpha$, then by Lemma 2.4, we get $\alpha = qk\text{-}\dim M + 1$ and we are done. The final part is now evident. □

The following results are now immediate.

Corollary 2.6. *Let M be a $\beta + 1$-almost quasi Artinian module, then either $qk\text{-}\dim M = \beta$ or $qk\text{-}\dim M = \beta + 1$.*

Proposition 2.7. *An R-module M has quasi Krull dimension if and only if M is α-almost quasi Artinian for some ordinal α.*

We recall that an R-module M has finite uniform dimension if it does not contain a direct sum of an infinite number of non-zero submodules. Now in view of [5, Proposition 2], we have the following result.

Corollary 2.8. *Every α-almost quasi Artinian module has finite uniform dimension.*

We continue with the following definition, which is in fact the dual of α-quasi short modules, see [6, Definition 2.7], and in the subsequent results we try to present counterparts of the appropriate results in [6].

Definition 2.9. An R-module M is called α-quasi Krull, if for each non-finitely generated submodule N of M, either $qk\text{-dim}\, N \leq \alpha$ or $qk\text{-dim}\, \frac{M}{N} \leq \alpha$, and α is the least ordinal number with this property.

Now, we cite the following example.

Example 2.10. If $M_1 = M_2 = \mathbb{Z}_{p^\infty}$, then M_1 and M_2 are -1-quasi Krull (resp. 0-almost quasi Artinian) $\mathbb{Z}$-modules such that $M_1 \oplus M_2$ is 0-quasi Krull (resp. 1-almost quasi Artinian). Now let $M_1 = M_2 = \mathbb{Z}$. In this case the $\mathbb{Z}$-module $\mathbb{Z}$ is -1-quasi Krull (resp. -1-almost quasi Artinian), the $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}$ is also -1-quasi Krull (resp. -1-almost quasi Artinian). We should also note that $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}$ is a 0-quasi Krull $\mathbb{Z}$-module which is 1-almost quasi Artinian.

Remark 2.11. If M is an R-module with $qk\text{-dim}\, M = \alpha$, then M is β-quasi Krull for some $\beta \leq \alpha$.

In view of [5, Lemma 2 and Theorem 1], we have the following result.

Remark 2.12. If M is an α-quasi Krull module, then each submodule and each factor module of M is β-quasi Krull for some $\beta \leq \alpha$.

We need the following result.

Lemma 2.13. *If M is an R-module and for each non-finitely submodule N of M, either N or $\frac{M}{N}$ has quasi Krull dimension, then so does M.*

Proof. Let $M_1 \supseteq M_2 \supseteq \cdots$ be any descending chain of non-finitely generated submodules of M. If there exists some i such that M_i has quasi Krull dimension, then each $\frac{M_k}{M_{k+1}}$ has quasi Krull dimension for each $k \geq i$, see [5, Lemma 2]. Otherwise

$\frac{M}{M_i}$ has quasi Krull dimension for each i. Thus in either case there exists some integer k such that each $\frac{M_i}{M_{i+1}}$ has quasi Krull dimension for each $i \geq k$, see [5, Lemma 2]. Consequently M has quasi Krull dimension. □

The previous result and Remark 2.11, immediately yield the next result.

Corollary 2.14. *Let M be an α-quasi Krull module. Then M has quasi Krull dimension and $qk\text{-}\dim M \geq \alpha$.*

Proposition 2.15. *An R-module M has quasi Krull dimension if and only if M is α-quasi Krull for some ordinal α.*

In view of [5, Proposition 2], we have the following result.

Corollary 2.16. *Every α-quasi Krull module has finite uniform dimension.*

Proposition 2.17. *If M is an α-quasi Krull R-module, then either $qk\text{-}\dim M = \alpha$ or $qk\text{-}\dim M = \alpha + 1$.*

Proof. Clearly in view of Remark 2.11 and Corollary 2.14, we have $qk\text{-}\dim M \geq \alpha$. If $qk\text{-}\dim M \neq \alpha$, then $qk\text{-}\dim M \geq \alpha + 1$. Now let $M_1 \supseteq M_2 \supseteq \cdots$ be any descending chain of non-finitely generated submodules of M. If there exists some k such that $qk\text{-}\dim M_k \leq \alpha$, then $qk\text{-}\dim \frac{M_i}{M_{i+1}} \leq qk\text{-}\dim M_i \leq qk\text{-}\dim M_k \leq \alpha$ for each $i \geq k$, [5, Lemma 2]. Otherwise $qk\text{-}\dim \frac{M}{M_i} \leq \alpha$ (note, M is α-quasi Krull) for each i, hence $qk\text{-}\dim \frac{M_i}{M_{i+1}} \leq \alpha$ for each i. Thus in any case there exists an integer k such that for each $i \geq k$, $qk\text{-}\dim \frac{M_i}{M_{i+1}} \leq \alpha$. This shows that $qk\text{-}\dim M \leq \alpha + 1$, i.e., $qk\text{-}\dim M = \alpha + 1$. □

Remark 2.18. An R-module M is -1-quasi Krull if and only if it is Noetherian or 1-atomic (note, an R-module M is called α-atomic, if $n\text{-}\dim M = \alpha$ and $n\text{-}\dim N < \alpha$ for each proper submodules N of M).

Proposition 2.19. *Let M be an R-module, with $qk\text{-}\dim M = \alpha$, where α is a limit ordinal. Then M is α-quasi Krull.*

Proof. We know that M is β-quasi Krull for some $\beta \leq \alpha$. If $\beta < \alpha$, then by Proposition 2.17, $qk\text{-}\dim M \leq \beta + 1 < \alpha$, which is a contradiction. Thus M is α-quasi Krull. □

Proposition 2.20. *Let M be an R-module and $qk\text{-}\dim M = \alpha = \beta + 1$. Then M is either α-quasi Krull or it is β-quasi Krull.*

Proof. We know that M is γ-quasi Krull for some $\gamma \leq \alpha$. If $\gamma < \beta$ then by Proposition 2.17, we have $qk\text{-}\dim M \leq \gamma + 1 < \beta + 1$, which is impossible. Hence we are done. $\square$

For the quasi critical modules we have the following proposition.

Proposition 2.21. *Let M be an $\beta + 1$-quasi critical R-module, where $\alpha = \beta + 1$. Then M is a β-quasi Krull module.*

Proof. Let N be a non-finitely generated submodule of M, then $qk\text{-}\dim \frac{M}{N} < \alpha$. Thus $qk\text{-}\dim \frac{M}{N} \leq \beta$. This shows that for some $\beta' \leq \beta$, M is β'-quasi Krull. If $\beta' < \beta$, then $\beta' + 1 \leq \beta < \alpha$. But $qk\text{-}\dim M \leq \beta' + 1 \leq \beta < \alpha$, by Proposition 2.17, which is a contradiction. Thus $\beta' = \beta$ and we are done. $\square$

The following remark, which is a trivial consequence of the previous fact, shows that the converse of Proposition 2.19, is not true in general.

Remark 2.22. Let M be an $\alpha + 1$-quasi critical R-module, where α is a limit ordinal. Then M is an α-quasi Krull module.

In view of Proposition 2.17 and Lemma 2.4, the following remark is now evident.

Remark 2.23. If M is a β-quasi Krull R-module, then it is an α-almost quasi Artinian module such that $\beta \leq \alpha \leq \beta + 2$, see Proposition 2.17 and Lemma 2.4. We note that every 1-quasi critical module is 0-quasi Krull which is also 1-almost quasi Artinian and every α-quasi critical module, where α is a limit ordinal, is an α-quasi Krull module which is also α-almost quasi Artinian, see Lemma 2.5 and Proposition 2.19.

Proposition 2.24. *Let M be an R-module such that $qk\text{-}\dim M = \alpha + 1$. Then M is either an α-quasi Krull R-module or there exists a non-finitely generated submodule N of M such that $qk\text{-}\dim \frac{M}{N} = qk\text{-}\dim N = \alpha + 1$.*

Proof. We know that M is α-quasi Krull or an $\alpha + 1$-quasi Krull R-module, by Proposition 2.20. Let us assume that M is not an α-quasi Krull R-module, hence there exists a non-finitely generated submodule N of M such that $qk\text{-}\dim N \geq \alpha + 1$ and $qk\text{-}\dim \frac{M}{N} \geq \alpha + 1$. This shows that $qk\text{-}\dim N = \alpha + 1$ and $qk\text{-}\dim \frac{M}{N} = \alpha + 1$ and we are through. $\square$

Proposition 2.25. *Let M be an α-quasi Krull R-module. Then either M is β-almost quasi Artinian for some ordinal $\beta \leq \alpha + 1$ or there exists a non-finitely generated submodule N of M with $qk\text{-}\dim N \leq \alpha$.*

Proof. Suppose that M is not β-almost quasi Artinian for any $\beta \leq \alpha + 1$. This means that there must exist a non-finitely generated submodule N of M such that $qk\text{-dim}\, \frac{M}{N} \nleq \alpha$. Inasmuch as M is α-quasi Krull, we infer that $qk\text{-dim}\, N \leq \alpha$ and we are done. □

3. Properties of α-quasi Krull modules and α-almost quasi Artinian modules

In this section some properties of α-quasi Krull and α-almost quasi Artinian modules over an arbitrary ring R are investigated.

First, in view of Proposition 2.17, we have the following two results.

Proposition 3.1. *Let R be a ring and M be an α-quasi Krull module, which is not a quasi critical module, then M contains a non-finitely generated submodule L such that $qk\text{-dim}\, L \leq \alpha$.*

Proof. Since M is not quasi critical, we infer that there exists a proper non-finitely generated submodule $L \subset M$, such that $qk\text{-dim}\, \frac{M}{L} = qk\text{-dim}\, M$. We know that $qk\text{-dim}\, M = \alpha$ or $qk\text{-dim}\, M = \alpha + 1$, by Proposition 2.17. If $qk\text{-dim}\, M = \alpha$ it is clear that $qk\text{-dim}\, L \leq \alpha$. Hence we may suppose that $qk\text{-dim}\, \frac{M}{L} = qk\text{-dim}\, M = \alpha+1$. Consequently, $qk\text{-dim}\, L \leq \alpha$ and we are done. □

Theorem 3.2. *Let M be an R-module and α be an ordinal number. Let for any non-finitely generated submodule N of M, $\frac{M}{N}$ be γ-quasi Krull for some ordinal number $\gamma \leq \alpha$. Then $qk\text{-dim}\, M \leq \alpha + 2$. In particular M is μ-quasi Krull for some ordinal number $\mu \leq \alpha + 1$.*

Proof. Let $N \subset M$ be a non-finitely generated submodule of M. Since $\frac{M}{N}$ is γ-quasi Krull for some ordinal number $\gamma \leq \alpha$, we infer that $qk\text{-dim}\, \frac{M}{N} \leq \gamma+1 \leq \alpha+1$, by Proposition 2.17. This immediately implies that $qk\text{-dim}\, M \leq \alpha + 2$, see [5, Lemma 4]. Now the last part of theorem is immediate. □

The next result is the dual of Theorem 3.2.

Theorem 3.3. *Let α be an ordinal number and M be an R-module such that every proper non-finitely generated submodule of M is γ-quasi Krull for some ordinal number $\gamma \leq \alpha$. Then $qk\text{-dim}\, M \leq \alpha + 1$. In particular M is μ-quasi Krull for some $\mu \leq \alpha + 1$.*

Proof. Let $N \subset M$ be any proper non-finitely generated submodule of M, such that N is γ-quasi Krull for some ordinal number γ with $\gamma \leq \alpha$. We infer that $qk\text{-dim}\, N \leq \gamma + 1 \leq \alpha + 1$, by Proposition 2.17. But we know that $qk\text{-dim}\, M =$

$\sup\{qk\text{-}\dim N : N \subset M, N \in \mathrm{NF(M)}\}$, see [5, Lemma 3]. This shows that $qk\text{-}\dim M \leq \alpha + 1$. Now the last part of theorem is immediate. □

The next immediate result is the counterparts of Theorems 3.2, 3.3, for α-almost quasi Artinian modules.

Proposition 3.4. *Let M be an R-module and α be an ordinal number. If each proper non-finitely generated submodule N of M (resp. for each proper non-finitely generated submodule N of M, $\frac{M}{N}$) is γ- almost quasi Artinian with $\gamma \leq \alpha$, then M is a μ-almost quasi Artinian module with $\mu \leq \alpha + 1$, $qk\text{-}\dim M \leq \alpha$ (resp. with $\mu \leq \alpha + 1$, $qk\text{-}\dim M \leq \alpha + 1$).*

Clearly every α-almost quasi Artinian (resp. α-quasi Krull) module has quasi Krull dimension (i.e., it has quasi-Noetherian dimension too, for by a nice result due to Lemonnier, every module has quasi-Noetherian dimension if and only if it has quasi Krull dimension, see the comment which follows [5, Lemma 12]). Consequently, we have the following immediate result.

Proposition 3.5. *The following statements are equivalent for a ring R.*

(1) *Every R-module with quasi Krull dimension is Noetherian.*
(2) *Every α-quasi Krull R-module is Noetherian for all α.*
(3) *Every α-almost quasi Artinian R-module is Noetherian for all α.*

Moreover, if R is a right perfect ring (i.e., every R-module is a Loewy module) then every α-quasi Krull (resp. α-almost quasi Artinian) R-module is both Artinian and Noetherian, see [24, Proposition 2.1].

Before concluding this section with our last observation, let us cite the next result which is in [24, Theorem 2.9], see also [18, Theorem 3.2].

Theorem 3.6. *For a commutative ring R the following statements are equivalent.*

(1) *Every R-module with finite Noetherian dimension is Noetherian.*
(2) *Every Artinian R-module is Noetherian.*
(3) *Every R-module with Noetherian dimension is both Artinian and Noetherian.*

Now in view of the above theorem, [14, Proposition 2.21], [12, Proposition 4.18], [6, Proposition 2.24] and also [25, Corollary 2.15], we observe the following result.

Proposition 3.7. *The following statements are equivalent for a commutative ring R.*

(1) *Every Artinian R-module is Noetherian.*

(2) *Every m-Krull module is both Artinian and Noetherian for all integers $m \geq -1$.*
(3) *Every α-Krull module is both Artinian and Noetherian for all ordinals α.*
(4) *Every m-quasi Krull module is both Artinian and Noetherian for all integers $m \geq -1$.*
(5) *Every α-quasi Krull module is both Artinian and Noetherian for all ordinals α.*
(6) *Every m-almost Artinian R-module is both Artinian and Noetherian for all non-negative integers m.*
(7) *Every α-almost Artinian R-module is both Artinian and Noetherian for all ordinals α.*
(8) *Every m-almost quasi Artinian R-module is both Artinian and Noetherian for all non-negative integers m.*
(9) *Every α-almost quasi Artinian R-module is both Artinian and Noetherian for all ordinals α.*
(10) *Every m-quasi short module is both Artinian and Noetherian for all integers $m \geq -1$.*
(11) *Every α-quasi short module is both Artinian and Noetherian for all ordinals α.*
(12) *Every m-almost quasi Noetherian R-module is both Artinian and Noetherian for all non-negative integers m.*
(13) *Every α-almost quasi Noetherian R-module is both Artinian and Noetherian for all ordinals α.*
(14) *Every m-short module is both Artinian and Noetherian for all integers $m \geq -1$.*
(15) *Every α-short module is both Artinian and Noetherian for all ordinals α.*
(16) *Every m-almost Noetherian R-module is both Artinian and Noetherian for all non-negative integers m.*
(17) *Every α-almost Noetherian R-module is both Artinian and Noetherian for all ordinals α.*
(18) *No homomorphic image of R can be isomorphic to a dense subring of a complete local domain of quasi Krull dimension* 1.

Remark 3.8. Since our results in this article are related to the results in [14,21] and there are two minor errors in these referencess (one in each), I strongly recommend the reader to see [15,22] for corrections.

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