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# ON FINITE DIMENSIONAL ALGEBRAS WHICH ARE SUMS OF TWO SUBALGEBRAS

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Abstract. In this paper, we give a general method of the construction of a 3-dimensional associative algebra $R$ over an arbitrary field $F$ that is a sum of two subalgebras $R_1$ and $R_2$ (i.e. $R = R_1 + R_2$).

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## 1. Introduction

Let $R$ be an associative ring. Writing $R = R_1 + R_2$, we mean that $R_1$ and $R_2$ are subrings of $R$ and, for every $r \in R$, there exist $r_1 \in R_1$ and $r_2 \in R_2$ such that $r = r_1 + r_2$.

Many authors studied problems and relationships among properties of $R_1$, $R_2$ and $R$ (see [4], [5], [7], [8], [10], [11], [12], [13]). For instance, the question "is $R$ nil provided that $R_1^2 = 0$ and $R_2$ is nil?" is equivalent to the famous Köthe's nil ideal problem (see [14], [15] and [16]). Let us mention three other open problems in this area.

Suppose that $R_1$ and $R_2$ are rings satisfying a polynomial identity. Does $R$ satisfies a polynomial identity ([1])? The answer to this problem is known in several particular cases (cf. [3], [7], [8], [9], [10], [11], [12], [13]), and a full positive answer in [12].

It is known [14] that if both $R_1$ and $R_2$ are nilpotent rings, then so is $R$. It is also known that there exists a function $f(n, m)$ such that if $R_1^n = 0$ and $R_2^m = 0$, then $R^{f(n,m)} = 0$. However the best such a function is still unknown. It is conjectured that $f(m, n) = mn$.

For the last problem, suppose that $R_1$ is nil and $R_2$ is reduced (it has no nonzero nilpotent elements). Is $R_1$ an ideal of $R$? The answer to this question is known to be "yes" in many particular cases [13], but in general, is still unknown.

We remark that a general method of constructing rings which are sums of two subrings was invited by Kelarev [6]. As we mentioned before, all these problems, on their substantial parts, concern due structure of rings which are sums of two subrings. This is particularly clear for the question 3. Note that it also concerns questions first and second. Namely, if $R$ is a semiprime ring which satisfies a polynomial identity, then the ring of Ore extension of $R$ is isomorphic to a direct sum of matrix rings over division rings. Thus, when studying whether of $R = R_1 + R_2$, and each $R_i$ satisfies a polynomial identity implies $R$ satisfies such an identity in the case of semiprime $R$ to study the structure of subrings of matrix rings of two subrings. Here, we also note that, Bokut [2] proved that every algebra over a field can be embedded into a simple algebra which is a sum of three nilpotent subalgebras. In view of these two facts, we will a general method of constructing 2 and 3 dimensional algebras over a field $F$ which are sums of to subalgebras with respect to the isomorphisms. This general method may give us to describe semiprime rings (even finite dimensional algebras) as a sum of two subrings.

## 2. Some general results

Suppose that $F$ is a field and $R$ is a finite dimensional (non-unitary) associative $F$-algebra. Put $A := F \times R$ and define binary operation $+$ and $\cdot$ on $A$ by the rule

$$(f_1, r_1) + (f_2, r_2) = (f_1 + f_2, r_1 + r_2)$$

and

$$(f_1, r_1) \cdot (f_2, r_2) = (f_1 \cdot f_2, f_1 r_2 + f_2 r_1 + r_1 r_2),$$

where $f_1, f_2 \in F$ and $r_1, r_2 \in R$. It is easy to say that $A$ is an $F$-vector space and we may identify elements $r \in R$ with elements $(0, r)$ in $A$.

**Proposition 2.1.** *$A$ is a finite dimensional $F$-algebra with a unit and $R$ is a two sided ideal of $A$ such that $\dim_F(A/R) = 1$. Moreover, $A/R \cong F$.*

**Proof.** Since $A$ is a vector space over $F$, it is easy to see that

$$\begin{aligned} f(a,b) &= (f,0)\cdot(a,b) \\ &= (fa, fb) \\ &= (a,b)\cdot(f,0), \end{aligned}$$

which shows that $(1, 0)$ is a unit.

For each $(f_i, r_i) \in A$, we get

$$\begin{aligned}(f_1,r_1)[(f_2,r_2)(f_3,r_3)] &= (f_1,r_1)(f_2f_3, f_2r_3+f_3r_2+r_2r_3)\\ &= (f_1f_2f_3, f_1f_2r_3+f_1f_3r_2+f_1r_2r_3+f_2f_3r_1+f_2r_1r_3\\ &+f_3r_1r_2+r_1r_2r_3)\\ &= (f_1f_2, f_1r_2+f_2r_1+r_1r_2)(f_3,r_3)\\ &= [(f_1,r_1)(f_2,r_2)](f_3,r_3).\end{aligned}$$

Similarly

$$\begin{aligned}(f_1,r_1)[(f_2+f_3,r_2+r_3)] &= (f_1f_2+f_1f_3, f_1r_2+f_1r_3+f_2r_1+f_3r_1+r_1r_2+r_1r_3)\\ &= (f_1,r_1)(f_2,r_2)+(f_1,r_1)(f_3,r_3)\end{aligned}$$

and

$$[(f_2+f_3,r_2+r_3)](f_1,r_1) = (f_2,r_2)(f_1,r_1)+(f_3,r_3)(f_1,r_1).$$

Clearly, $R$ is $F$-subspace of $A$ such that $\dim_F(A) = \dim_F(R)+1$ and

$$\begin{aligned}(f,r)(0,s) &= (0, fs+rs)\\ (0,s)(f,r) &= (0, sf+sr)\end{aligned}$$

for every $(f,r) \in A$ and $s \in R$. Hence $R$ is a two-sided ideal of $A$. Finally, $A/R \cong F$. $\square$

For an algebra $R$, the symbol $R^*$ denotes $R$ with an identity adjoined.

**Theorem 2.2.** *If $R$ does not contain a unit, then it is an ideal of finite dimensional $F$-algebra $A$ such that $\dim_F(A) = \dim_F(R)+1$, i.e. there exists a complete orthogonal set of primitive idempotents $e_0, \ldots, e_m$ such that $e_0 \in A \setminus R$, $e_1, \ldots, e_m \in R$, $R = A(1-e_0)A$ and $\dim_F(e_0A/e_0J(A)) = 1$.*

**Proof.** The first part follows from Proposition 2.1. The rest of the claim follows from [17, Proposition VIII.4.1], since every orthogonal set of idempotents of $R + J(A)/J(A)$ can be lifted to an orthogonal set of idempotents of $R$. $\square$

**Remark 2.3.** Fix $A$ from Theorem 2.2 if $R$ does not contain a unit and put $A = R$ otherwise. We know that there exists a complete orthogonal set of primitive idempotents $e_0, \ldots, e_m$ of $A$ such that $e_1, \ldots, e_m \in R$ and $\dim_F(e_0A/e_0J(A)) = 1$ if $A \neq R$. Then $m < \dim_F R$.

**Proof.** Obviously, $m \leq \dim_F R \leq \dim_F A$ and, moreover, $m \neq \dim_F R$ if $R = A$. Let $R \neq A$. If $e_0A$ is simple, then $1 - e_0 \in R$ is a unit of $R$, a contradiction. Thus $\dim_F(e_0A) > 1$, hence $m < \dim_F R$. $\square$

## 3. Two structure theorems

The following observation is easy.

**Lemma 3.1.** *If* $0 \neq R_1$ *and* $0 \neq R_2$ *are proper subalgebras of the algebra* $R$ *over a field* $F$ *and* $\dim_F R = 2$*, then* $\dim_F R_1 = 1 = \dim_F R_2$*.*

Assume that $\dim_F R = 2$ such that $R$ is a sum of subalgebras $R_1$ and $R_2$. By Lemma 3.1, we may assume that $\dim_F R_1 = 1 = \dim_F R_2$. Let $R_1 = Fe$ and $R_2 = Ff$, where $e^2 = e \in R$ and $f^2 = f \in R$, or $e^2 = 0$ or $f^2 = 0$ if $R_1 \cong F^0$ or $R_2 \cong F^0$, where $F^0$ denotes $F$ with zero multiplication. Then we have the following cases on $R_1$ (or $R_2$).

* If $R_1^2 = 0$, then $R_1 \cong F^0$. Hence we have either $R_2 \cong F^0$ or $R_2 \cong F$.

** If $R_1^2 \neq 0$, then $R_1 \cong F$. Hence we have either $R_2 \cong F^0$ or $R_2 \cong F$.

**Case 1.** Let $R_1^2 = 0$, $R_1 \cong F^0$ and $R_2 \cong F^0$. For $\alpha \in R_1$ and $\beta \in R_2$, write

$$ef = \alpha e + \beta f. \tag{1}$$

Note that, in this case, $e^2 = 0 = f^2$. Multiplying the equation (1) by $f$ on the right, we get $\alpha ef = 0$ which implies $\alpha = 0$ or $ef = 0$. If $\alpha = 0$, we get $ef = \beta f$. Thus $\beta ef = 0$. It follows $ef = 0$. Similarly, we obtain that $fe = 0$. Now it is enough to take $R = F^0 \times F^0$, $e = (1, 0)$ and $f = (0, 1)$.

**Case 2.** Let $R_1^2 \neq 0$, $R_1 \cong F$ and $R_2 \cong F$. Then, for $\alpha \in R_1$ and $\beta \in R_2$, we get again the equation (1). Multiplying the equation (1) by $e$ on the left, we get

$$ef = \alpha e + \beta ef. \tag{2}$$

Now, $0 = \beta(ef - f)$ implies either $\beta = 0$ or $ef = f$. If $\beta = 0$, then, by equation (1), $ef = \alpha e$. Since $\alpha e = ef = ef^2 = \alpha ef = \alpha^2 e$, we get either $\alpha = 1$ or $\alpha = 0$. By equation (1), if $\beta = 0$ and $\alpha = 0$, then we have $ef = 0$ and so $fe = 0$, if $\beta = 0$ and $\alpha = 1$ then $ef = e$. Therefore $ef \in \{0, e, f\}$.

By similar reasoning as above we can obtain that $fe \in \{0, e, f\}$. Assume that $ef = 0$. Multiplying the equation (3) by $f$ on the right, we have $\beta_1 f = 0$. So $\beta_1 = 0$. Consequently $0 = efe = \alpha_1 e$ which implies $\alpha_1 = 0$. Hence $fe = 0$. We can take $R = F \times F$ and $e = (1, 0)$, $f = (0, 1)$.

Let $ef = f = fe$. So $e$ is an identity element of $R$. Therefore $e - f$ and $f$ are orthogonal idempotents of $R$. Consequently $R = F \times F$ and we can take $e = (1, 1)$ and $f = (0, 1)$. The case where $ef = e = fe$ can be treated analogously.

In the case where $ef = f$ and $fe = e$, consider $R = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, where $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $f = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

Analogously, if $ef = e$ and $fe = f$, let $R = \begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix}$, where $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $f = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$.

**Case 3.** Let $R_1^2 \neq 0$, $R_1 \cong F$ and $R_2 \cong F^0$. Multiplying the equation (1) by $f$ on the right, we get $0 = \alpha ef$ which implies either $\alpha = 0$ or $ef = 0$.

If $\alpha = 0$, then $ef = \beta f$ by equation (1). Now $ef = e^2 f = eef = e\beta f = \beta ef = \beta^2 f$ which implies either $\beta = 0$ or $\beta = 1$. Clearly, if $\beta = 0$ then $ef = 0$ and if $\beta = 1$, then $ef = f$.

Finally, multiplying the equation (3) by $f$ on the right, we get $\alpha_1 ef = fef$. So $\alpha_1 e = 0$ because $ef = f$ or $ef = 0$. Hence $fe = \beta_1 f = \beta_1^2 f$. It follows that $\beta_1 = 1$ or $\beta_1 = 0$. Consequently, $fe = 0$ or $fe = f$.

In the case where $ef = fe = 0$, we take $R = F \times F^0$, where $e = (1, 0)$ and $f = (0, 1)$. Note that if $R = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, then $ef = 0$ and $fe = f$.

Finally, let $R = (F^0)^*$, $e = (1, 0)$ and $f = (0, 1)$. Clearly, $ef = f = fe$.

As a result of above calculations, we have the following observation.

**Theorem 3.2.** *Assume that $\dim_F R = 2$ and $R = R_1 + R_2$ is a sum of proper subalgebras $R_1$ and $R_2$. Then we have the following cases.*

(I) *If $R_1 \cong F^0$ and $R_2 \cong F^0$, then $R = F^0 \times F^0$.*

(II) *If $R_1 \cong F$ and $R_2 \cong F$, then one of the following cases holds true:*
   (1) $R = F \times F$,
   (2) $R = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$,
   (3) $R = \begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix}$.

(III) *If $R_1 \cong F$ and $R_2 \cong F^0$, then one of the following cases holds true:*
   (1) $R = F \times F^0$,
   (2) $R = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$,

(3) $R = \begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix}$,

(4) $R = (F^0)^*$.

In the next Theorem, we will use the following.

**Remark 3.3.** Let $M$ be an $A$-$A$-bimodule, denote $\mu(M)$ a matrix $\mu(M) = (d_{ij})$ where $d_{ij} = \dim_F(e_i M e_j)$. Denote by $S$ a ring of polynomials in two noncommuting variables $x_1$, $x_2$ over $F$. If $\mathbf{A} = (a_{kl})$ is a matrix where $a_{kl} \in F$, $1 \leq k, l \leq 2$, then define

$$S(\mathbf{A}) = S/\langle a_{22}x_1^2 - a_{11}x_2^2, a_{21}x_1x_2 - a_{12}x_2x_1, a_{22}x_1x_2 - a_{12}x_2^2, a_{22}x_2x_1 - a_{21}x_2^2, x_1^3, x_2^3\rangle,$$

where $a_{11}, a_{12}, a_{21}, a_{22} \neq 0$. Moreover,

$$S(\mathbf{A}) = S/\langle x_1^2, a_{21}x_1x_2 - a_{12}x_2x_1, a_{22}x_1x_2 - a_{12}x_2^2, a_{22}x_2x_1 - a_{21}x_2^2, x_2^3\rangle$$

if $a_{11} = 0$,

$$S(\mathbf{A}) = S/\langle x_1^2, x_1x_2, a_{22}x_2x_1 - a_{21}x_2^2, x_2^3\rangle$$

if $a_{11} = a_{12} = 0$,

$$S(\mathbf{A}) = S/\langle x_1^2, x_1x_2, x_2x_1, x_2^3\rangle$$

if $a_{11} = a_{12} = a_{21} = 0$. Analogously we define $S(\mathbf{A})$ in the case $a_{11} = a_{22} = 0$ or $a_{11} = a_{12} = a_{22} = 0$.

Assume that $\dim_F R = 3$ such that $R$ is a sum of subalgebras order of $R_1$ and $R_2$.

**Theorem 3.4.** *Assume that* $\dim_F R = 3$ *such that* $R$ *is a sum of subalgebras order of* $R_1$ *and* $R_2$.

(I) *Let* $m = 0$ *and* $R = A$. *Then one of the following cases holds true:*

(a) *If* $J(R) = 0$, *then* $R \cong K$, *where* $K$ *is a field of* $F$*-dimension 3 and either* $R_1 = R$ *or* $R_2 = R$.

(b) *If* $R/J(R) \cong F$ *and* $J^2(R) = 0$, *then* $R \cong F[x_1, x_2]/\langle x_1^2, x_1x_2, x_2^2\rangle$ *and*

(1) $R_1 = R$ *and* $R_2$ *is an arbitrary subalgebra of* $R$.

(2) $R_1 \cong F_0 \times F_0$ *and* $R_2 \cong F$.

(3) $R_1 \cong F[x]/\langle x^2\rangle$ *and* $R_2 \cong F_0$.

(4) $R_1 \cong F[x]/\langle x^2\rangle$ *and* $R_2 \cong F_0 \times F_0$.

(5) $R_1 \cong R_2 \cong F[x]/\langle x^2\rangle$.

(c) *If* $R/J(R) \cong F$ *and* $J^2(R) \neq 0$, *then* $R \cong F[x]/\langle x^3\rangle$ *and*

(6) $R_1 = R$ *and* $R_2$ *is an arbitrary subalgebra of* $R$,

(7) $R_1 \cong xF[x]/\langle x^3\rangle$ *and* $R_2 \cong F$,

(8) $R_1 \cong xF[x]/\langle x^3\rangle$, $R_2 \cong F[x]/\langle x^2\rangle$, *where* $F[x]/\langle x^2\rangle \cong F + x^2F \subset R$.

*Furthermore, all the cases in* (1) *are commutative.*

(II) *Let* $m = 1$ *and* $R = A$. *Then one of the following cases holds true:*

(a) *If* $J(R) = 0$, *then* $R \cong F \times G$ *where* $G$ *is a field of* $F$*-dimension* 2 *and*

(1) $R_1 = R$ *and* $R_2$ *is an arbitrary subalgebra of* $R$,

(2) $R_1 \cong G$ *and* $R_2 \cong F$,

(3) $R_1 \cong G$ *and* $R_2 \cong F \times F$.

(b) *If* $J(R) \neq 0$, *then* $R \cong \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ *and we have one of the following cases.*

(4) $R_1 = R$ *and* $R_2$ *is an arbitrary subalgebra of* $R$,

(5) $R_1 \cong F^2$ *and* $R_2 \cong F_0$,

(6) $R_1 \cong \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ *and* $R_2 \cong F$,

(7) $R_1 \cong \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$ *and* $R_2 \cong F$,

(8) $R_1 \cong F^2$ *and* $R_2 \cong \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$,

(9) $R_1 \cong F^2$ *and* $R_2 \cong \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$,

(10) $R_1 \cong \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ *and* $R_2 \cong \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$.

(III) *Let* $m = 2$ *and* $R = A$. *Then* $R \cong F \times F \times F$ *whenever* $m = 2$ *and so we have one of the following cases.*

(a) $R_1 = R$ *and* $R_2$ *is an arbitrary (semisimple) subalgebra of* $R$,

(b) $R_1 \cong F \times F$ *and* $R_2 \cong F$,

(c) $R_1 \cong R_2 \cong F \times F$.

(IV) *Let* $m = 0$ *and* $R \neq A$. *If* $J^2(R) = 0$, *then* $R \cong F_0^3$, *and so* $R_1 \cong F_0^k$ *and* $R_2 \cong F_0^l$ *for* $k, l \leq 3$. *We have one of the following cases.*

(a) *If* $J^2(R) = 0$, *then* $R \cong F_0^3$, *and so* $R_1 \cong F_0^k$ *and* $R_2 \cong F_0^l$ *for* $k, l \leq 3$.

(b) *If* $J^2(R) \neq 0$, $J^3(R) = 0$ *and* $\dim_F(J(R)/J^2(R)) > \dim_F(J^2(R)/J^3(R))$, *then* $R \cong \{x_1, x_2\}S(\mathbf{A})$ *and*

(1) $R_1 = R$ *and* $R_2$ *is an arbitrary subalgebra of* $R$,

(2) $R_1 \cong xF[x]/\langle x^3\rangle$ *and* $R_2 \cong F_0$,

(3) $R_1 \cong F_0^2$ *and* $R_2 \cong F_0$,

(4) $R_1 \cong R_2 \cong xF[x]/\langle x^3\rangle$,

(5) $R_1 \cong xF[x]/\langle x^3\rangle$ *and* $R_2 \cong F_0^2$,

(6) $R_1 \cong F_0^2$ *and* $R_2 \cong F_0^2$.

(c) *If* $J^3(R) \neq 0$, *then* $R = R_1 \cong xF[x]/\langle x^4\rangle$ *and* $R_2$ *is an arbitrary (chain) subalgebra of* $R$.

(V) *Let* $m = 1$ *and* $R \neq A$. *Then* $A/J(R) \cong F \times G$ *where* $G$ *is a field and* $\dim_F(G) \leq 2$. *Let* $G \cong F$. *We have one of the following cases.*

(a) *If* $J^2(R) = 0$ *and* $\mu(J(R)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, *then* $R \cong F_0 \times F[x]/\langle x^2\rangle$ *and*

(1) $R_1 = R$ *and* $R_2$ *is an arbitrary subalgebra of* $R$,

(2) $R_1 \cong F[x]/\langle x^2\rangle$ *and* $R_2 \cong F_0$,

(3) $R_1 \cong F_0 \times F$ *and* $R_2 \cong F_0$,

(4) $R_1 \cong F[x]/\langle x^2\rangle$ *and* $R_2 \cong F_0^2$,

(5) $R_1 \cong F[x]/\langle x^2\rangle$ *and* $R_2 \cong F_0 \times F$.

(b) *If* $J^2(R) = 0$ *and* $\mu(J(R)) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, *then* $R \cong \left\{ \begin{pmatrix} 0 & 0 & b \\ 0 & a & c \\ 0 & 0 & a \end{pmatrix} \mid a, b, c \in F \right\}$ *and*

(6) $R_1 = R$ *and* $R_2$ *is an arbitrary subalgebra of* $R$,

(7) $R_1 = J(R) \cong F_0^2$ *and* $R_2 \cong F$,

(8) $R_1 = \begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix}$ *and* $R_2 \cong F_0$,

(9) $R_1 \cong F[x]/\langle x^2\rangle$ *and* $R_2 \cong F_0$,

(10) $R_1 = J(R) \cong F_0^2$ *and* $R_2 \cong F[x]/\langle x^2\rangle$,

(11) $R_1 = J(R) \cong F_0^2$ *and* $R_2 \cong \begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix}$,

(12) $R_1 \cong \begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix}$ *and* $R_2 \cong F[x]/\langle x^2\rangle$.

(c) *If* $J^2(R) = 0$ *and* $\mu(J(R)) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, *then* $R \cong \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ b & c & a \end{pmatrix} \mid a, b, c \in F \right\}$. *In fact, this case is antiisomorphic to the case (b) (i.e. it has the structure of* $R^{op}$*).*

(d) *If* $J^2(R) = 0$ *and* $\mu(J(R)) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, *then* $R \cong \begin{pmatrix} F & 0 & F \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix}$ *and*

(13) $R_1 = R$ *and* $R_2$ *is an arbitrary subalgebra of* $R$,

(14) $R_1 \cong F_0^2$ *and* $R_2 \cong F$,

(15) $R_1 = \begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix}$ *and* $R_2 \cong F_0$,

(16) $R_1 \cong F \times F_0$ *and* $R_2 \cong F_0$,

(17) $R_1 = J(R) \cong F_0^2$ *and* $R_2 \cong F \times F_0$,

(18) $R_1 = J(R) \cong F_0^2$ *and* $R_2 \cong \begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix}$,

(19) $R_1 \cong \begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix}$ *and* $R_2 \cong F \times F_0$.

(e) *If* $J^2(R) = 0$ *and* $\mu(J(R)) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, *then* $R \cong \begin{pmatrix} F & 0 & 0 \\ 0 & 0 & 0 \\ F & F & 0 \end{pmatrix}$.

(f) *If* $J^2(R) = 0$ *and* $\mu(J(R)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, *then* $R \cong \left\{ \begin{pmatrix} a & b & 0 \\ 0 & 0 & c \\ 0 & 0 & a \end{pmatrix} \mid a, b, c \in F \right\}$

*and*

(20) $R_1 = R$ *and* $R_2$ *is an arbitrary subalgebra of* $R$,

(21) $R_1 \cong F_0^2$ *and* $R_2 \cong F$,

(22) $R_1 = \begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix}$ *and* $R_2 \cong F_0$,

(23) $R_1 \cong \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ *and* $R_2 \cong F_0$,

(24) $R_1 = J(R) \cong F_0^2$ *and* $R_2 \cong \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$,

(25) $R_1 = J(R) \cong F_0^2$ *and* $R_2 \cong \begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix}$,

(26) $R_1 \cong \begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix}$ *and* $R_2 \cong \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$.

(g) *If* $\dim_F G = 2$, *then* $J(R)$ *is not a* $G$*-module, hence* $R \cong G \times F_0$ *and*

(27) $R_1 = R$ *and* $R_2$ *is an arbitrary subalgebra of* $R$,

(28) $R_1 \cong G$ *and* $R_2 \cong F_0$,

(29) $R_1 \cong G$ *and* $R_2 \cong F \times F_0$.

(h) *If* $J^2(R) \neq 0$, *then* $R \cong F \times (xF[x]/\langle x^3 \rangle)$ *and*

(30) $R_1 = R$ *and* $R_2$ *is an arbitrary subalgebra of* $R$,

(31) $R_1 \cong xF[x]/\langle x^3 \rangle$ *and* $R_2 \cong F$,

(32) $R_1 \cong xF[x]/\langle x^3 \rangle$ *and* $R_2 \cong F \times F_0$.

(VI) *Let $m = 2$ and $R \neq A$. Then $A/J(R) \cong F^3$ and we have one of the following cases.*

(a) $R \cong F^2 \times F_0$ *and*

(1) $R_1 = R$ *and* $R_2$ *is an arbitrary subalgebra of* $R$,

(2) $R_1 \cong F^2$ *and* $R_2 \cong F_0$,

(3) $R_1 \cong F \times F_0$ *and* $R_2 \cong F$,

(4) $R_1 \cong F^2$ *and* $R_2 \cong F \times F_0$.

(b) $R \cong F \times \begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix}$ *and*

(5) $R_1 = R$ *and* $R_2$ *is an arbitrary subalgebra of* $R$,

(6) $R_1 \cong F^2$ *and* $R_2 \cong F_0$,

(7) $R_1 \cong \begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix}$ *and* $R_2 \cong F$,

(8) $R_1 \cong F \times F_0$ *and* $R_2 \cong F$,

(9) $R_1 \cong F^2$ *and* $R_2 \cong F \times F_0$,

(10) $R_1 \cong F^2$ *and* $R_2 \cong \begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix}$,

(11) $R_1 \cong \begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix}$ *and* $R_2 \cong F \times F_0$.

(c) $R \cong F \times \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ *and*

(12) $R_1 = R$ *and* $R_2$ *is an arbitrary subalgebra of* $R$,

(13) $R_1 \cong F^2$ *and* $R_2 \cong F_0$,

(14) $R_1 \cong \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ *and* $R_2 \cong F$,

(15) $R_1 \cong F \times F_0$ *and* $R_2 \cong F$,

(16) $R_1 \cong F^2$ *and* $R_2 \cong F \times F_0$,

(17) $R_1 \cong F^2$ *and* $R_2 \cong \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$,

(18) $R_1 \cong \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ *and* $R_2 \cong F \times F_0$.

**Proof.** We can suppose without loss of generality that $\dim_F(R_1) \geq \dim_F(R_2)$.

**(I)** Let $m = 0$ and $R = A$. Then $m = 0$, $R$ is a local algebra and either $R/J(R) \cong F$ or $J(R) = 0$ and $R$ is a field. Note that the case $\dim_F(R/J(R)) = 2$ is excluded, since $J(R)/J^2(R)$ should be an $R/J(R)$-vector space and hence $2 \mid \dim_F(R)$.

If $J(R) = 0$, then $R$ is a field and its subfields of $F$-dimension is either 1 or 3. Hence either $R_1 = R$ or $R_2 = R$.

If $R/J(R) \cong F$ and $J^2(R) = 0$, then $R \cong F[x_1, x_2]/\langle x_1^2, x_1x_2, x_2^2\rangle$. Hence we have one of the possibilities (1)-(5).

If $R/J(R) \cong F$ and $J^2(R) \neq 0$, then $J^3(R) = 0$ and there exists a basis $1, j, j^2$ of $R_F$, where $j \in J(R)$ and $j^2 \in J^2(R)$. Hence $R \cong F[x]/\langle x^3\rangle$. Now we have one of the possibilities (6)-(8).

**(II)** If $m = 1$, $\dim_F(R/J(R)) \geq 2$ and so $\dim_F(J(R)) \leq 1$.

If $J(R) = 0$, then clearly, $R \cong F \times G$, where $G$ is a field of $F$-dimension 2 and we have one of the possibilities (1)-(3).

If $J(R) \neq 0$, then $R/J(R) \cong F \times F$ and $\dim_F(J(R)) = 1$. As $J(R)$ is an $F \times F$-module, we get that $R \cong \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Note that $R \cong \begin{pmatrix} F & 0 \\ F & F \end{pmatrix}$. Hence we have one of the possibilities (4)-(10).

**(III)** This is clear.

**(IV)** We assume that $m = 0$. Then $R = J(R)$ for a local algebra $A$ of the dimension 4. If $J^2(R) = 0$, then $R \cong F_0^3$. So (a) is clear. If $J^2(R) \neq 0$ and $J^3(R) = 0$, then we get $\dim_F(J(R)/J^2(R)) = 2$ and $\dim_F(J^2(R)/J^3(R)) = 1$. Then there exists a basis $\{j_1, j_2, j_3\}$ of $R$ and elements $a_{kl} \in F$, $k, l \in \{1, 2\}$ such that $j_1, j_2 \in J(R) \setminus J^2(R)$, $j_3 \in J^2(R)$ and $j_k j_l = a_{kl} j_3$ for all $k, l \in \{1, 2\}$. Now it is easy to see that $A \cong S((a_{kl}))$ and so we have one of the possibilities (1)-(6).

If $J^3(R) \neq 0$, then $J^4(R) = 0$ and $\dim_F(J(R)/J^2(R)) = \dim_F(J^2(R)/J^3(R)) = \dim_F(J^3(R)) = 1$ and there is a base $\{j, j^2, j^3\}$ of $R$. Hence $R = R_1 \cong xF[x]/\langle x^4\rangle$ and $R_2$ is an arbitrary (chain) subalgebra of $R$.

**(V)** Let $m = 1$ and $R \neq A$. Then either $e_0 J(R) \neq 0$ or $J(R)e_0 \neq 0$ and $A/J(R) \cong F \times G$ where $G$ is a field and $\dim_F(G) \leq 2$. Let $G \cong F$. First, moreover, suppose that $J^2(R) = 0$ and we will discuss cases of $\mu(J(R))$, denote $\{j_1, j_2\}$ a base of $J(R)$ such that $j_k \in e_{a_k} J(R) e_{b_k}$ for a suitable $a_k$ and $b_k$. Note that either the first row or the first column is nonzero and $j_k j_l = 0$. Hence it remains to express multiplications $e_1 j_k$ and $j_k e_1$, which follows immediately from the occurrence of $j_k$ in $e_{a_k} J e_{b_k}$.

If $\mu(J(R)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $j_1 \in e_0 J(R) e_0$, $j_2 \in e_1 J(R) e_1$, $e_1 j_1 = j_1 e_1 = 0$ and $e_1 j_2 = j_2 e_1 = j_2$. Thus $R \cong F_0 \times F[x]/\langle x^2\rangle$ and one of the possibilities (1)-(5) holds true.

If $\mu(J(R)) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, then $j_1 \in e_0J(R)e_1$, $j_2 \in e_1J(R)e_1$, $e_1j_1 = 0$, $j_1e_1 = j_1$ and $e_1j_2 = j_2e_1 = j_2$. Hence $R \cong \left\{ \begin{pmatrix} 0 & 0 & b \\ 0 & a & c \\ 0 & 0 & a \end{pmatrix} \mid a, b, c \in F \right\}$. Hence one of the possibilities (6)-(12) holds true.

If If $\mu(J(R)) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, then $j_1 \in e_0J(R)e_0$, $j_2 \in e_1J(R)e_0$, $e_1j_1 = e_1j_2 = j_1e_1 = 0$ and $j_2e_1 = j_2$. Thus $R \cong \begin{pmatrix} F & 0 & F \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix}$ and one of the possibilities (13)-(19) holds true

If $\mu(J(R)) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, then $j_1 \in e_0J(R)e_0$ and $j_2 \in e_0J(R)e_1$. Hence $e_1j_1 = j_2e_1 = j_1e_1 = 0$ and $e_1j_2 = j_2$, and so $R \cong \begin{pmatrix} F & 0 & 0 \\ 0 & 0 & 0 \\ F & F & 0 \end{pmatrix}$. Similarly as in (c), we need not to describe $R_1$ and $R_2$, since the case is antiisomorphic to the case (d).

If $\mu(J(R)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then $j_1 \in e_0J(R)e_1$, $j_2 \in e_1J(R)e_0$, $e_1j_1 = j_2e_1 = 0$, $j_1e_1 = j_1$ and $e_1j_2 = j_2$, which implies that $R \cong \left\{ \begin{pmatrix} a & b & 0 \\ 0 & 0 & c \\ 0 & 0 & a \end{pmatrix} \mid a, b, c \in F \right\}$. Hence one of the possibilities (20)-(26) holds true.

The the possibilities (27)-(29) are clear.

Finally assume that $J^2(R) \neq 0$. Then $\dim_F(J(R)/J^2(R)) = \dim_F(J^2(R)) = 1$ and there exist a base $j_1, j_2$ of $J(R)$ such that $j_1^2 = j_2 \in J^2(R)$. Note that $j_1 \in e_0J(R)e_0$ since $j_1^2 \neq 0$ and $e_0J(R) \neq 0$. Hence we get $R \cong F \times (xF[x]/\langle x^3 \rangle)$ and the possibilities (30)-(22) are clear.

**(VI)** Let $m = 2$ and $R \neq A$. Clearly, $A/J(R) \cong F^3$ and either $e_0J(R) \neq 0$ or $J(R)e_0 \neq 0$. As a similar calculation in (V), we get that:

If $e_0J(R) \neq 0$ then $R \cong F^2 \times F_0$ and one of the possibilities (1)-(4) holds.

If either $e_0J(R)e_1 \neq 0$ or $e_0J(R)e_2 \neq 0$, then $R \cong F \times \begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix}$ and one of the possibilities (5)-(11) holds.

If either $e_1J(R)e_0 \neq 0$ or $e_2J(R)e_0 \neq 0$, then $R \cong F \times \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ and one of the possibilities (12)-(18) holds. $\square$

We conclude the paper with the following problems for further studies.

**Problem 1.** Describe all finite dimensional algebras over a field which are sums of two subalgebras.

**Problem 2.** Describe all finite dimensional algebras over a field which are sums of two nilpotent subalgebras.

**Problem 3.** Describe all finite dimensional algebras which are sums of two matrix algebras over a field.

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