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n-ABSORBING MONOMIAL IDEALS IN POLYNOMIAL RINGS

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Abstract. In a commutative ring R with unity, given an ideal I of R, Anderson and Badawi in 2011 introduced the invariant $\omega(I)$, which is the minimal integer n for which I is an n-absorbing ideal of R. In the specific case that $R = k[x_1, \ldots, x_n]$ is a polynomial ring over a field k in n variables $x_1, \ldots, x_n$, we calculate $\omega(I)$ for certain monomial ideals I of R.

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1. Introduction

Throughout this paper, we set $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$, and R will denote a commutative ring with unity. Given a nonzero ideal I of R, $\mathrm{Ass}(R/I)$ will denote the set of associated primes of I in R. The primary notion we are interested in this paper is the following:

Definition 1. Let $n \in \mathbb{N}$, R a commutative ring with unity, and I an ideal of R. I is said to be an *n-absorbing ideal* of a ring R if for any $x_1, \ldots, x_{n+1} \in R$ such that $x_1 \cdots x_{n+1} \in I$, there are n of the x_i's whose product is in I. I is said to be a *strongly n-absorbing ideal* of a ring R if for any ideals $I_1, \ldots, I_{n+1}$ of R such that $I_1 \cdots I_{n+1} \subseteq I$, there are n of the I_i's whose product is in I.

(Strongly) 2-absorbing ideals were initially defined and investigated by Badawi in [3] as a generalization of prime ideals, which are precisely the proper 1-absorbing ideals. In 2011, Anderson and Badawi together generalized this further to the notion of a (strongly) n-absorbing ideal for any $n \in \mathbb{N}$ defined above in [1]. For an ideal I in a ring R, we let $\omega(I)$ denote the minimal integer $n \in \mathbb{N}$ such that I is n-absorbing. In a general ring, I may not be n-absorbing for any $n \in \mathbb{N}$, in which case we set $\omega(I) = \infty$. Similarly, we can define the invariant $\omega^\bullet(I)$ to be the smallest integer $n \in \mathbb{N}$ for which an ideal I is strongly n-absorbing, and set $\omega^\bullet(I) = \infty$ if no such integer exists. We set $\omega(R) = \omega^\bullet(R) = 0$. It is easy to see that $\omega(I) \leq \omega^\bullet(I)$ holds for each ideal I of R. In fact, Anderson and Badawi in

Conjecture 1 of [1, page 1669] postulate that $\omega(I) = \omega^\bullet(I)$ holds for any ideal I in an arbitrary ring R; that is, they conjecture that the notion of an n-absorbing ideal and strongly n-absorbing ideal coincide. As of this writing, this problem remains open. However, it is known that the conjecture holds true for any $n \in \mathbb{N}$ if R is a Prüfer domain, i.e., an integral domain such that the set of ideals of R_M is totally ordered under set inclusion for each maximal ideal M of R ([1, Corollary 6.9]), or if R is a commutative algebra over an infinite field ([7]), and for any ring R if $n = 2$ ([3, Theorem 2.13]). The interested reader may refer to the survey article [4, Section 5] for further information on strongly n-absorbing ideals. Anderson and Badawi made two more conjectures in [1] which were investigated by several researchers, and affirmative answers were given, either partial or complete. For example, the second Anderson-Badawi conjecture states that given an ideal I of a ring R and an indeterminate X, $\omega_{R[X]}(I[X]) = \omega_R(I)$ ([1, page 1661]). That is, for each $n \in \mathbb{N}$, I is an n-absorbing ideal of R if and only if $I[X]$ is an n-absorbing ideal of $R[X]$. This conjecture originates from the well-known result that I is a prime ideal (i.e., 1-absorbing ideal) of R if and only if $I[X]$ is a prime ideal of $R[X]$. Anderson and Badawi themselves proved this conjecture for an arbitrary commutative ring when $n = 2$ ([1, Theorem 4.15]), and Nasehpour proves that the second conjecture holds for every $n \in \mathbb{N}$ when R belongs to certain classes of rings, including the class of Prüfer domains ([15]). In [11] Laradji independently proved that the second conjecture holds when R is an arithmetical ring, i.e., when the set of ideals of R_M is totally ordered under set inclusion for each maximal ideal M of R.

Recall that for an ideal I in a ring R, the *Noether exponent* of I, denoted by $e(I)$, is the minimal integer $\mu \in \mathbb{N}$ such that $(\sqrt{I})^\mu \subseteq I$. If such an integer does not exist, we set $e(I) = \infty$. We also set $e(R) = 0$. In a Noetherian ring, since $\sqrt{I}$ is finitely generated for any ideal I, $e(I) < \infty$. Anderson and Badawi in [1] establish a connection between $\omega^\bullet(I)$ and Noether exponents:

Theorem 1.1. [1, Remark 2.2, Theorem 5.3, Section 6, Paragraph 2 on page 1669] *Let $I_1, \dots, I_r$ be ideals of a ring R. Then $\omega(I_1 \cap \cdots \cap I_r) \leq \omega(I_1) + \cdots + \omega(I_r)$ and $\omega^\bullet(I_1 \cap \cdots \cap I_r) \leq \omega^\bullet(I_1) + \cdots + \omega^\bullet(I_r)$. In particular, let I be an ideal in a Noetherian ring R. If $I = Q_1 \cap \cdots \cap Q_n$, where the Q_i are primary ideals, then $\omega(I) \leq \omega^\bullet(I) \leq \sum_{i=1}^{n} e(Q_i)$. Thus every ideal in a Noetherian ring is n-absorbing for some $n \in \mathbb{N}$.*

On the other hand, the third Anderson-Badawi conjecture claims that for each $n \in \mathbb{N}$ and an n-absorbing ideal I of a ring R, $(\sqrt{I})^n \subseteq I$ ([1, Conjecture 2, page

1669]). This conjecture was proved for $n = 2$ by Badawi ([3]), for $n = 3$ by Laradji ([11]), for $n = 3, 4, 5$ by Sihem and Sana ([17]), and for arbitrary n and R by the authors ([6]) and Donadze ([8]), independently. We summarize this as the following theorem in terms of $\omega(I)$ and $e(I)$, along with the result concerning primary ideals ([1, Theorem 6.3(c), Theorem 6.6]).

Theorem 1.2. *Given an ideal I of a ring R, $e(I) \leq \omega(I)$. If Q is a primary ideal of R, then $\omega(Q) = \omega^{\bullet}(Q) = e(Q)$.*

This raises the question then if for an arbitrary ideal I whether $\omega(I)$ can be described purely in terms of Noether exponents or possibly other well-known ring-theoretic invariants. This has been investigated to some extent by others in at least one case. Namely, Moghimi and Naghani [13, Theorem 2.21(1)] show that in a discrete valuation ring R, $\omega(I)$ is precisely the length of the R-module R/I.

In this spirit, we attempt to give in this paper a description of $\omega(I)$ in terms of other ring-theoretic invariants in the special case that I is a monomial ideal of a polynomial ring over a field. In some cases, our arguments are general enough to also give the same results for $\omega^{\bullet}(I)$, and thus as a side-effect we can show that in some cases the notions of a n-absorbing ideal and a strongly n-absorbing ideal coincide as Anderson and Badawi conjecture.

The present paper is divided into two parts. In Section 2, we review some definitions and facts concerning n-absorbing ideals and monomial ideals. Using these, we calculate $\omega(I)$ for primary monomial ideals by computing Noether exponents and the standard primary decomposition of monomial ideals. These results lead to the study of how $\omega(I)$ can be explicitly computed from the generating set of I when I is a monomial ideal of $R = k[x_1, \ldots, x_n]$ with $n \leq 3$ in the following section.

The second part is Section 4, where we define and investigate ω-linear monomial ideals, i.e., monomial ideals I such that $\omega(I^m) = m\omega(I)$ for each $m \in \mathbb{N}$. We give a characterization theorem for primary ω-linear monomial ideals, and in particular show that integrally closed monomial ideals in $R = k[x, y]$ are w-linear, as well as the edge ideal of a cycle.

2. Some background

As a prerequisite of the main section of this paper, we briefly review some of the basic material excerpted from [10] regarding monomial ideals, and show that $\omega(I)$ can be directly calculated from the generators of I when I is a primary monomial ideal.

Let k be a field and $R = k[x_1, \ldots, x_n]$ be the polynomial ring with n variables over k. An element of R of the form $x_1^{a_1} \cdots x_n^{a_n}$ with $a_i \in \mathbb{N}_0$ is called a *monomial*, and an ideal of R generated by monomials is called a *monomial ideal*. The degree of $f = x_1^{a_1} \cdots x_n^{a_n}$, denoted by $\deg(f)$, is defined to be $a_1 + \cdots + a_n$. $G(I)$ will denote the set of monomials in I which are minimal with respect to divisibility. Any element of R can be written uniquely as a k-linear combination of monomials; that is, given $f \in R$, we may write $f = \sum a_u u$ where the sum is taken over the monomial ideals of R and $a_u \in k$ for each monomial u. Then the *support* of f, denoted by $\mathrm{supp}(f)$, is the set of monomials u such that $a_u \neq 0$. An ideal I of a ring R is *irreducible* if there are no ideals I_1, I_2 of R such that $I = I_1 \cap I_2$ and $I \subsetneq I_1$, $I \subsetneq I_2$. We denote by $\mathfrak{m}$ the unique maximal homogeneous ideal of R.

Lemma 2.1. [10, Chapter 1] *Let $R = k[x_1, \ldots, x_n]$ and I a monomial ideal of R generated by monomials $u_1, \ldots, u_r$ of R. Then the following hold:*

(i) *Given a monomial $f \in I$, there exists $i \in \{1, \ldots, r\}$ so $u_i | f$.*

(ii) *$G(I)$ is the unique minimal set of monomial generators of I.*

(iii) *I can be written as a finite intersection of ideals of the form $(x_{i_1}^{d_1}, \ldots, x_{i_m}^{d_m})$. An irredundant presentation of this form is unique ($I = Q_1 \cap \cdots \cap Q_r$ is irredundant if none of the ideals Q_i can be omitted).*

(iv) *I is irreducible if and only if I is of the form $(x_{i_1}^{d_1}, \ldots, x_{i_m}^{d_m})$. Moreover, every irreducible monomial ideal of the form $(x_{i_1}^{d_1}, \ldots, x_{i_m}^{d_m})$ is $(x_{i_1}, \ldots, x_{i_m})$-primary.*

(v) *If J is another monomial ideal of R, then*

$$I \cap J = (\{lcm(u, v) \mid u \in G(I), v \in G(J)\}).$$

In particular, if a and b are coprime monomials of R and I is a monomial ideal of R, then $(ab, I) = (a, I) \cap (b, I)$.

(vi) *An ideal I' of R is monomial if and only if for each $f \in I'$,* $\mathrm{supp}(f) \subseteq I'$.

By Lemma 2.1(iv), the irredundant unique decomposition of Lemma 2.1(iii) is also a primary decomposition of I, which is known as the *standard decomposition* of I (see [10, P. 12]). We will also need the following characterization of primary monomial ideals:

Lemma 2.2. [9, Exercise 3.6] *Let $R = k[x_1, \ldots, x_n]$ and $P = (x_{i_1}, \ldots, x_{i_r})$ a monomial prime ideal of R. Then given a P-primary monomial ideal Q, $G(Q)$ consists of monomials of the ring $k[x_{i_1}, \ldots, x_{i_r}]$ and there exists $a_1, \ldots, a_r \in \mathbb{N}$*

so $\{x_{i_1}^{a_1}, \ldots, x_{i_r}^{a_r}\} \subseteq G(Q)$. *Conversely, every monomial ideal of this form is a* P*-primary ideal.*

Proof. Let $f \in G(Q)$. If $f \notin k[x_{i_1}, \ldots, x_{i_r}]$, then $x_j | f$ for some $x_j \notin P$ and $g = \dfrac{f}{x_j} \in Q$ since Q is a P-primary ideal, but this contradicts the minimality of $G(Q)$. Hence $f \in k[x_{i_1}, \ldots, x_{i_r}]$. On the other hand, given $j \in \{1, \ldots, r\}$ there exists $a_j \in \mathbb{N}$ so $x_{i_j}^{a_j} \in G(Q)$, since $\sqrt{Q} = P$.
To prove the converse, let Q be a monomial ideal such that $G(Q)$ consists of monomials of the ring $k[x_{i_1}, \ldots, x_{i_r}]$ and there exists $a_1, \ldots, a_r \in \mathbb{N}$ so $\{x_{i_1}^{a_1}, \ldots, x_{i_r}^{a_r}\} \subseteq G(Q)$. Then $\sqrt{Q} = P$ by [10, Proposition 1.2.4]. On the other hand, if $P_1 \in \text{Ass}(R/Q) \setminus \{P\}$, then $P_1 = Q : f$ for some monomial f of R ([10, Corollary 1.3.10]). Now choose $d \in \{1, \ldots, n\}$ so $x_d \in P_1 \setminus P$. Then $x_d f \in Q$, and $f \in Q$ by Lemma 2.1(i). But then $P_1 = R$, a contradiction. Hence $\text{Ass}(R/Q) = \{P\}$ and Q is a P-primary monomial ideal. □

Corollary 2.3. *Let* P *be a prime monomial ideal and* I, J *be* P*-primary monomial ideals of* R*. Then both* $I \cap J$ *and* IJ *are* P*-primary monomial ideals. Moreover,* $I : J$ *is a* P*-primary monomial ideal provided* $J \not\subset I$*.*

Proof. This is an immediate consequence of Lemma 2.1(v) and Lemma 2.2. □

We can now calculate $\omega(I)$, where I is an irreducible monomial ideal.

Lemma 2.4. *Let* $R = k[x_1, \ldots, x_n]$ *denote a polynomial ring over a field* k*. Let* $I = (x_{i_1}^{d_1}, \ldots, x_{i_m}^{d_m})$*, where* $d_1, \ldots, d_n \in \mathbb{N}$ *and* $1 \leq i_1 < i_2 < \cdots < i_m \leq n$*. Then* $\omega(I) = \omega^\bullet(I) = e(I) = d_1 + \cdots + d_m - m + 1$*.*

Proof. Since I is an $(x_{i_1}, x_{i_2}, \ldots, x_{i_m})$-primary ideal by Lemma 2.2, the first two equalities follow from Theorem 1.2. Thus it suffices to show that $e(I) = r$, where $r = d_1 + \cdots + d_m - m + 1$. We have $\sqrt{I} = (x_{i_1}, \ldots, x_{i_m})$. For $N \in \mathbb{N}$, $(\sqrt{I})^N \subseteq I$ if and only if for every $c_1, \ldots, c_m \in \mathbb{N}_0$ with $c_1 + \cdots + c_m = N$, we have $x_{i_1}^{c_1} \cdots x_{i_m}^{c_m} \in I$. By Lemma 2.1(i), the latter happens precisely when $c_i \geq d_i$ for some $1 \leq i \leq m$. Thus $e(I) = r$. □

Next, we produce a way to calculate $\omega(I)$ when I is a monomial primary ideal not necessarily generated by pure powers.

Lemma 2.5. *Let* I *be an ideal of a ring* R*. Suppose there is* $P \in \text{Spec}(R)$ *such that* $I = J_1 \cap \cdots \cap J_r$*, where* J_i *are ideals of* R *with* $\sqrt{J_i} = P$ *for each* $i \in \{1, \ldots, r\}$*. Then* $e(I) = \max_{1 \leq i \leq r}\{e(J_i)\}$*.*

Proof. Note that $\sqrt{I} = \sqrt{J_1} \cap \cdots \cap \sqrt{J_r} = P$. Thus given $\mu \in \mathbb{N}$, $(\sqrt{I})^\mu \subseteq I$ if and only if $(\sqrt{J_i})^\mu \subseteq J_i$ for each $i \in \{1, \ldots, r\}$, from which the conclusion of the lemma follows. □

Corollary 2.6. *Let $R = k[x_1, \ldots, x_n]$ denote a polynomial ring over a field k. If Q is a monomial primary ideal of R and $Q = \bigcap_{i=1}^r Q_i$ is its standard decomposition, then*

$$\omega(Q) = \omega^\bullet(Q) = \max_{1 \le i \le r} \{e(Q_i)\}.$$

Example 2.7. Let $R = k[x, y, z]$ with a field k and $I = (x^4, y^3, z^2, xy, y^2z)$. Then repeatedly applying Lemma 2.1(v), we obtain the standard decomposition $I = (x, y^2, z^2) \cap (x^4, y, z^2) \cap (x, y^3, z)$. Thus by Lemma 2.4 and Corollary 2.6,

$$\omega(I) = \omega^\bullet(I) = \max\{1 + 2 + 2 - 3 + 1, 4 + 1 + 2 - 3 + 1, 1 + 3 + 1 - 3 + 1\} = 5.$$

3. When I is a monomial ideal of $R = k[x_1, \ldots, x_n]$ with $n \le 3$

In this section we show that when I is a monomial ideal of $R = k[x_1, \ldots, x_n]$ with $n \le 3$, then $\omega(I)$ can be explicitly calculated from $G(I)$. We first prove a theorem analogous to [2, Theorem 2.5]. Note that by $a_1 \cdots \widehat{a_i} \cdots a_n$ we mean $\prod_{1 \le j \le n, j \ne i} a_j$.

Lemma 3.1. *Let R be a UFD and p an irreducible element of R. Then given $n \in \mathbb{N}$, I is an n-absorbing ideal of R if and only if pI is an $(n+1)$-absorbing ideal of R. In particular, $\omega(pI) = \omega(I) + 1$.*

Proof. Suppose that I is n-absorbing. Let $f_1, \ldots, f_{n+2} \in R$ and $f_1 \cdots f_{n+2} \in pI$. Then since p is irreducible, $p \mid f_i$ for some i. Without loss of generality, suppose that $p \mid f_1$. Then $f_1/p \in R$, and so $(f_1/p)f_2 \cdots f_{n+2} \in I$. Since I is n-absorbing, and hence $(n+1)$-absorbing as well, we have that either $(f_1/p)f_2 \cdots \widehat{f_i} \cdots f_{n+2} \in I$ for some $i \in \{2, \ldots, n+2\}$, in which case $f_1 f_2 \cdots \widehat{f_i} \cdots f_{n+2} \in pI$ and we are done, or $f_2 \cdots f_{n+2} \in I$. This is a product of length $n+1$, so that since I is n-absorbing, for some j with $2 \le j \le n+2$, we have $f_2 \cdots \widehat{f_j} \cdots f_{n+2} \in I$. Thus $pf_2 \cdots \widehat{f_j} \cdots f_{n+2} \in pI$, and so $f_1 f_2 \cdots \widehat{f_j} \cdots f_{n+2} \in pI$. This shows that pI is then $(n+1)$-absorbing, and $\omega(pI) \le \omega(I) + 1$.

To show the converse, suppose that pI is an $(n+1)$-absorbing ideal. If I is not an n-absorbing ideal, then there exists $f_1, \ldots, f_{n+1} \in R$ such that $f = f_1 \cdots f_{n+1} \in I$ but $f_1 \cdots \widehat{f_i} \cdots f_{n+1} \notin I$ for each i. Since pI is $(n+1)$-absorbing and $pf \in pI$, it follows that either $pf_1 \cdots \widehat{f_i} \cdots f_{n+1} \in pI$ for some i or $f \in pI$. But the former is impossible by our choice of f_i's, and without loss of generality we may assume that

$p|f_1$. Now $(f_1/p)f_2\cdots f_n \in I$, and neither $\widehat{(f_1/p)}f_2\cdots f_{n+1}$ nor $(f_1/p)f_2\cdots \widehat{f_i}\cdots f_{n+1}$ is an element of I for each $i \geq 2$. Therefore, since R is a UFD, we may assume that none of f_i are divisible by p. Now $pf_1\cdots f_{n+1} \in pI$, but $pf_1\cdots \widehat{f_i}\cdots f_{n+1} \notin pI$ and $f_1\cdots f_{n+1} \notin pI$, which contradicts the assumption that pI is an $(n+1)$-absorbing ideal. Hence I is an n-absorbing ideal and $\omega(pI) \geq \omega(I)+1$. □

The following corollary is now immediate.

Corollary 3.2. *Given a monomial f and an ideal I of $R = k[x_1,\ldots,x_n]$, $\omega(fI) =$deg$(f)+\omega(I)$. In particular, $\omega(fR) =$deg(f).*

Given a monomial ideal I with the standard decomposition $I = \bigcap_{\ell=1}^t T_\ell$, we can define an equivalence relation on $\{1,\ldots,t\}$ by defining $i \sim j$ iff $\sqrt{T_i} = \sqrt{T_j}$, and set $\{S_i\}_{i=1}^r$ to be the corresponding equivalence classes. Then $Q_i = \bigcap_{\ell\in S_i} T_\ell$ is a monomial primary ideal for each $i \in \{1,\ldots,r\}$, and $I = \bigcap_{i=1}^r Q_i$ is an irredundant primary decomposition of I. We will call this decomposition the *canonical primary decomposition* of I.

Theorem 3.3. *Let $R = k[x_1,\ldots,x_n]$. Let I be a monomial ideal with canonical primary decomposition $I = \bigcap_{i=1}^r Q_i$. If there exists $k \in \{1,\ldots,r\}$ such that $\sqrt{Q_i} \subseteq \sqrt{Q_k}$ for all $i \in \{1,\ldots,r\}$, then $\omega(I) = \max\{e(Q_k),\omega(\bigcap_{1\leq i\leq r, i\neq k} Q_i)\}$ and $\omega^\bullet(I) = \max\{e(Q_k),\omega^\bullet(\bigcap_{1\leq i\leq r, i\neq k} Q_i)\}$.*

Proof. Let $t = \max\{e(Q_k),\omega(\bigcap_{1\leq i\leq r,i\neq k} Q_i)\}$. We will first show that I is t-absorbing. If not, then there are $f_1,\ldots,f_{t+1} \in R$ such that $f = \prod_{j=1}^{t+1} f_j \in I$ but $g_j := f/f_j \notin I$ for each $j \in \{1,\ldots,t+1\}$. Hence given any $i \in \{1,\ldots,t+1\}$, there exists $\ell \in \{1,\ldots,r\}$ such that $g_i \notin Q_\ell$, and since $f_i g_i = f \in I \subseteq Q_\ell$, we must have $f_i \in \sqrt{Q_\ell} \subseteq \sqrt{Q_k}$. Therefore, $g_j \in (\sqrt{Q_k})^t \subseteq (\sqrt{Q_k})^{e(Q_k)} \subseteq Q_k$ for all $j \in \{1,\ldots,t+1\}$. On the other hand, $\bigcap_{1\leq i\leq r,i\neq k} Q_i$ is t-absorbing and $f \in \bigcap_{1\leq i\leq r,i\neq k} Q_i$, so that we conclude $g_j \in \bigcap_{1\leq i\leq r,i\neq k} Q_i$ for some $j \in \{1,\ldots,t+1\}$ and thereby $g_j \in I$, a contradiction. Thus $\omega(I) \leq t$. Next, we show that $\omega(I) \geq t$; that is, I is not $(t-1)$-absorbing. We now consider two cases.

Case 1: $t = \omega(\bigcap_{1\leq i\leq r,i\neq k} Q_i)$. Since $\bigcap_{1\leq i\leq r,i\neq k} Q_i$ is not $(t-1)$-absorbing, there are $h_1,\ldots,h_t \in R$ such that $h = \prod_{i=1}^t h_i \in \bigcap_{1\leq i\leq r,i\neq k} Q_i$ and $\ell_j := h/h_j \notin \bigcap_{1\leq i\leq r,i\neq k} Q_i$ for each $j \in \{1,\ldots,t\}$. By an argument similar to the first paragraph of this proof, $h_i \in \sqrt{Q_k}$ for each $i \in \{1,\ldots,t\}$, and so $h \in Q_k$. Hence $h \in I$ and $\ell_j \notin I$ for each $j \in \{1,\ldots,t\}$, so that I is not $(t-1)$-absorbing.

Case 2: $t = e(Q_k)$. Consider the standard decomposition of I, and choose an irreducible component T of I such that $e(T) = e(Q_k)$ and $\sqrt{T} = \sqrt{Q_k}$. Since we obtained the canonical primary decomposition $I = \bigcap_{i=1}^r Q_i$ from the standard decomposition, we can choose a monomial $g \in (\bigcap_{1\le i\le r, i\ne k} Q_i) \setminus T$ by Lemma 2.1(vi). Now $T = (x_{i_1}^{a_1}, \dots, x_{i_l}^{a_l})$ for some $a_j \in \mathbb{N}$ and $1 \le i_1 < \cdots < i_l \le n$. Note that we may assume that $g = \prod_{j=1}^l x_{i_j}^{c_j}$ for some $c_j \in \mathbb{N}_0$ such that $c_j < a_j$ for each $j \in \{1, \dots, l\}$. Set

$$f := x_{i_1}^{a_1-1} \cdots x_{i_l}^{a_l-1}(x_{i_1} + \cdots + x_{i_l}).$$

Then f is a product of $e(T)$ elements of $\sqrt{T}$ by Lemma 2.4, and so $f \in (\sqrt{T})^{e(T)} = (\sqrt{Q_k})^{e(Q_k)} \subseteq Q_k$. Since $g \mid f$ it also follows that $f \in \bigcap_{1\le i\le r, i\ne k} Q_i$. Hence $f \in I$. However, given $j \in \{1, \dots, l\}$, $\dfrac{f}{x_{i_j}} \notin T$. Indeed, $x_{i_1}^{a_1-1} \cdots x_{i_l}^{a_l-1} \in \operatorname{supp}\Big(\dfrac{f}{x_{i_j}}\Big) \setminus T$ by Lemma 2.1(i), and $\dfrac{f}{x_{i_j}} \notin T$ by Lemma 2.1(vi). Similarly $x_{i_1}^{a_1-1} \cdots x_{i_l}^{a_l-1} = \dfrac{f}{x_{i_1} + \cdots + x_{i_l}} \notin T$. Therefore I is not $(e(Q_k)-1)$-absorbing, and $\omega(I) \ge e(Q_k) = t$. Hence we have shown that $\omega(I) = \max\{e(Q_k), \omega(\bigcap_{1\le i\le r, i\ne k} Q_i)\}$. The proof of $\omega^\bullet(I) = \max\{e(Q_k), \omega^\bullet(\bigcap_{1\le i\le r, i\ne k} Q_i)\}$ can be obtained in a similar manner, and is omitted. □

The following corollary is immediate.

Corollary 3.4. *Let $R = k[x_1, \dots, x_n]$ and I a monomial ideal of R with standard decomposition $I = \bigcap_{i=1}^r T_i$. Then $\omega(I) = \omega^\bullet(I) = \max_{1\le i\le r}\{e(T_i)\}$ if Ass(R/I) is totally ordered under set inclusion.*

In the next proposition, we give a characterization of when the upper bound of $\omega(I)$ from Theorem 1.1 is sharp.

Proposition 3.5. *Let I be a monomial ideal of $R = k[x_1, \dots, x_n]$ with an irredundant primary decomposition $I = Q_1 \cap \cdots \cap Q_r$. Then $\omega(I) = \omega^\bullet(I) = \sum_{i=1}^r e(Q_i)$ if and only if I has no embedded associated primes.*

Proof. Set $P_i = \sqrt{Q_i}$ for each $i = 1, \dots, r$.
$\Rightarrow$: We prove the contrapositive; assume that $P_1, \dots, P_r$ are not incomparable prime ideals. Then without loss of generality we may assume that $P_1 \subsetneq P_2$, and we have $\omega(Q_1 \cap Q_2) = \max\{e(Q_1), e(Q_2)\}$ by Corollary 3.4. Therefore by Theorem

1.1 we have

$$\begin{aligned}\omega(I) &= \omega\Big(Q_1 \cap Q_2 \cap (\bigcap_{i\neq 1,2} Q_i)\Big)\\ &\le \omega\Big(Q_1 \cap Q_2\Big) + \omega\Big(\bigcap_{i\neq 1,2} Q_i\Big)\\ &= \max\{e(Q_1), e(Q_2)\} + \omega\Big(\bigcap_{i\neq 1,2} Q_i\Big)\\ &\le \max\{e(Q_1), e(Q_2)\} + \sum_{i\neq 1,2} e(Q_i)\\ &< \sum_{i=1}^{r} e(Q_i).\end{aligned}$$

$\Leftarrow$: Assume that $P_1, \ldots, P_r$ are incomparable prime ideals. The case when $r = 1$ follows from Theorem 1.2, so we may assume that $r \ge 2$. Since $\omega(I) \le \omega^\bullet(I) \le \sum_{i=1}^{r} e(Q_i)$ by Theorem 1.1, it suffices to show that I is not $(\sum_{i=1}^{r} e(Q_i) - 1)$-absorbing. Now given $i \in \{1, \ldots, r\}$, choose T_i to be an irreducible component of I with $\sqrt{T_i} = P_i$ and $e(T_i) = e(Q_i)$. Write $T_i = (x_{i_1}^{a_1}, \ldots, x_{i_{s_i}}^{a_{s_i}})$ with $1 \le i_1 < \cdots < i_{s_i} \le n$ and $a_1, \ldots, a_{s_i} \in \mathbb{N}$. For $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, s_i\}$, set

$$f_{i,j} = x_{i_j} + \sum_{t\neq j} x_{i_t}^2 \text{ and } f_i = \Big(\sum_{l=1}^{s_i} x_{i_l}\Big)\Big(\prod_{j=1}^{s_i} f_{i,j}^{a_j-1}\Big).$$

It follows that $f_i \in P_i^{e(T_i)} = (\sqrt{Q_i})^{e(Q_i)} \subseteq Q_i$. Thus $f := \prod_{i=1}^{r} f_i \in I$, and f is a product of $\sum_{i=1}^{r} e(Q_i)$ elements of R. We wish to show that $\dfrac{f}{\sum_{l=1}^{s_i} x_{i_l}} \notin I$ and $\dfrac{f}{f_{i,j}} \notin I$ for each $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, s_i\}$. Without loss of generality, we let $i = 1$. Note that $\dfrac{f_1}{f_{1,j}} \notin T_1$, since $\prod_{t=1}^{s_1} x_{1_t}^{a_t-1} \in \operatorname{supp}\Big(\dfrac{f_1}{f_{1,j}}\Big) \setminus T_1$. On the other hand, $\sum_{l=1}^{s_i} x_{i_l} \notin P_1$ and $f_{i,l} \notin P_1$ for each $i \neq 1$ and $l \in \{1, \ldots, s_i\}$. Therefore $f_i \notin P_1$ for each $i \neq 1$, and $\dfrac{f}{f_1} = \prod_{i=2}^{r} f_i \notin P_1$. Hence $\dfrac{f}{f_{1,j}} = \Big(\dfrac{f}{f_1}\Big)\Big(\dfrac{f_1}{f_{1,j}}\Big) \notin Q_1$. The proof that $\dfrac{f}{\sum_{l=1}^{s_1} x_{1_l}} \notin Q_1$ follows similarly. Hence we have $\omega(I) = \sum_{i=1}^{r} e(Q_i)$. $\square$

Theorem 3.3 and Proposition 3.5 yield the following corollary.

Corollary 3.6. *Let I be a monomial ideal of $R = k[x_1, \ldots, x_n]$ with $dim(R/I) = 1$. Let $I = \bigcap_{i=1}^{r} Q_i$ be the canonical primary decomposition of I. Then*

$$\omega(I) = \omega^{\bullet}(I) = \begin{cases} \max\{e(Q_k), \sum_{i \neq k} e(Q_i)\} & \text{if } \sqrt{Q_k} = \mathfrak{m} \text{ for some } k \in \{1, \ldots, r\}. \\ \sum_{i=1}^{r} e(Q_i) & \text{otherwise.} \end{cases}$$

Corollary 3.7. *Let f be a monomial of R. Then $\omega^{\bullet}(fR) = deg(f)$. In particular, $\omega(fR) = \omega^{\bullet}(fR)$.*

Proof. Let $f = \prod_{k=1}^{r} x_{i_k}^{a_k}$ for some $a_1, \ldots, a_r \in \mathbb{N}$ and $1 \leq i_1 < i_2 < \cdots < i_r \leq n$. Then $fR = x_{i_1}^{a_1} R \cap \cdots \cap x_{i_r}^{a_r} R$, and by Lemma 2.4 and Proposition 3.5 we have $\omega(fR) = \omega^{\bullet}(fR) = \sum_{i=1}^{r} e(x_{i_k}^{a_k} R) = \sum_{i=1}^{r} a_k = \deg(f)$. □

Given a monomial ideal I of $R = k[x, y, z]$, we can produce an algorithm that can compute $\omega(I)$. If I is principal, then Corollary 3.7 says that $\omega(I)$ is equal to the degree of a generator for I. Otherwise, $I = hJ$ for some monomial h and a monomial ideal J with $\dim(R/J) \leq 1$. Now, $\omega(J)$ can be calculated explicitly using Corollary 2.6 or Corollary 3.6 after obtaining a canonical primary decomposition of J, and we have $\omega(I) = deg(h) + \omega(J)$ by Corollary 3.2.

Example 3.8. Let $R = k[x, y, z]$ and $I = (x^3y^4, x^2y^5, x^4y^3z^2, x^5y^3z, x^2y^4z^2)$. Then $I = x^2y^3J$ with canonical primary decomposition $J = (x^2, y) \cap (y, z) \cap (x^3, y^2, z^2, xy)$. By Lemma 2.4 and Corollary 2.6, the standard decomposition $(x^3, y^2, z^2, xy) = (x, y^2, z^2) \cap (x^3, y, z^2)$ yields that $e((x^3, y^2, z^2, xy)) = 4$. Thus by Corollary 3.6,

$$\begin{aligned} \omega(I) &= deg(x^2y^3) + \omega(J) \\ &= 5 + \max\{e((x^3, y^2, z^2, xy)), e((x^2, y)) + e((y, z))\} \\ &= 5 + \max\{4, 2 + 1\} \\ &= 9. \end{aligned}$$

Another interesting result that follows from Lemma 3.1 and Theorem 3.3 is a formula of $\omega(I)$ and $\omega^{\bullet}(I)$ for monomial ideals of $R = k[x, y]$ where k is a field and x, y are indeterminates over k.

Theorem 3.9. *Let $R = k[x, y]$ and J a monomial ideal of R. Write $J = (x^{a_1}y^{b_1}, \ldots, x^{a_r}y^{b_r})$, where $\{a_i\}$ is strictly decreasing and $\{b_i\}$ is strictly increasing. Then*

$$\omega(J) = \omega^{\bullet}(J) = \begin{cases} a_1 + b_1 & \text{if } r = 1. \\ \max_{1 \leq i \leq r-1}\{a_i + b_{i+1}\} - 1 & \text{if } r > 1. \end{cases}$$

Proof. The case when $r = 1$ follows from Corollary 3.7. For $r > 1$, first observe the standard decomposition of J is $J = x^{a_r}R \cap y^{b_1}R \cap (x^{a_1}, y^{b_2}) \cap (x^{a_2}, y^{b_3}) \cap \cdots \cap (x^{a_{r-1}}, y^{b_r})$ ([12, Proposition 3.2]). The case $b_1 = a_r = 0$ follows from Corollary 2.6. Suppose that at least one of a_r and b_1 is nonzero. Then by Lemma 2.4 and Corollary 3.6,

$$\begin{aligned}\omega(J) = \omega^\bullet(J) &= \max\{e((x^{a_1}, y^{b_2}) \cap (x^{a_2}, y^{b_3}) \cap \cdots \cap (x^{a_{r-1}}, y^{b_r})), e(x^{a_r}R) + e(y^{b_1}R)\} \\ &= \max\{\max_{1\le i\le r-1}\{e((x^{a_i}, y^{b_{i+1}}))\}, a_r + b_1\} \\ &= \max\{\max_{1\le i\le r-1}\{a_i + b_{i+1} - 1\}, a_r + b_1\} \\ &= \max_{1\le i\le r-1}\{a_i + b_{i+1}\} - 1.\end{aligned}$$

□

Example 3.10. If $R = k[x, y]$ and $J = (x^{11}y^4, x^8y^5, x^7y^9, x^4y^{10}, x^2y^{16})$, then by Theorem 3.9,

$$\omega(J) = \omega^\bullet(J) = \max\{11 + 5, 8 + 9, 7 + 10, 4 + 16\} - 1 = 19.$$

4. ω-Linear ideals

Given an ideal I of a ring R, we will say that I is an *ω-linear ideal* if $\omega(I^m) = m\omega(I)$ for each $m \in \mathbb{N}$. Perhaps the most common example of ω-linear ideals can be found amongst those $P \in \mathrm{Spec}(R)$ for which P^n is P-primary for each $n \in \mathbb{N}$ ([1, Theorem 3.1, Theorem 5.7]). For instance,

1. R is a Prüfer domain and $P^2 \neq P$.
2. R is a Noetherian ring and P is a maximal ideal that contains a nonzerodivisor.
3. $R = k[x_1, \ldots, x_n]$ and P is a monomial ideal.

In this section, we investigate the properties of ω-linear ideals. Again, we will restrict our concern to monomial ideals of a polynomial ring $R = k[x_1, \ldots, x_n]$ where k is a field.

We first consider a few useful inequalities regarding monomial ideals.

Lemma 4.1. *Let I be a monomial ideal of $R = k[x_1, \ldots, x_n]$. Then $\omega(I) \geq \max\{\deg(f) \mid f \in G(I)\}$.*

Proof. Let $f \in G(I)$. Then $f = \prod_{k=1}^{r} x_{i_k}^{a_k}$ for some $a_1, \ldots, a_r \in \mathbb{N}$ and $1 \leq i_1 < i_2 < \cdots < i_r \leq n$. Since $f \in I$ but $\dfrac{f}{x_{i_k}} \notin I$ for each $k \in \{1, \ldots, r\}$ by minimality of $G(I)$, we have that I is not $(\deg(f) - 1)$-absorbing. Hence $\omega(I) \geq \deg(f)$, and since f was chosen arbitrarily, we have the desired conclusion. □

Lemma 4.2. *Let $I \subseteq J$ be ideals of a ring R. If $\sqrt{I} = \sqrt{J}$, then $e(J) \leq \omega(I)$. In particular, if I and J are both P-primary ideals of a prime ideal P of R, then $\omega(J) \leq \omega(I)$.*

Proof. Since $\sqrt{I} = \sqrt{J}$, $(\sqrt{J})^{\omega(I)} \subseteq (\sqrt{I})^{e(I)} \subseteq I \subseteq J$ by Theorem 1.2 and $e(J) \leq \omega(I)$. The second statement follows immediately since $e = \omega$ for primary ideals. □

Lemma 4.3. *Let P be a prime monomial ideal of $R = k[x_1, \ldots, x_n]$. If I, J are P-primary monomial ideals of R, then $\omega(I+J) \leq \min\{\omega(I), \omega(J)\} \leq \max\{\omega(I), \omega(J)\} = \omega(I \cap J) \leq \omega(IJ) \leq \omega(I) + \omega(J)$. Moreover, $\omega(I : J) \geq \omega(I) - \omega(J)$.*

Proof. Note that by Corollary 2.3, $IJ \subseteq I \cap J \subseteq I + J$ are all P-primary monomial ideals. Therefore $\omega(I + J) \leq \min\{\omega(I), \omega(J)\} \leq \max\{\omega(I), \omega(J)\} \leq \omega(I \cap J) \leq \omega(IJ)$ by Lemma 4.2. On the other hand, let $I = \bigcap_{i=1}^{r} Q_i$ and $J = \bigcap_{j=1}^{s} T_j$ be the standard decompositions of I and J, respectively. Then $I \cap J = (\bigcap_{i=1}^{r} Q_i) \cap (\bigcap_{j=1}^{s} T_j)$ is an irreducible decomposition of $I \cap J$, and by throwing away any redundant components, there are $A \subseteq \{1, \ldots, r\}$ and $B \subseteq \{1, \ldots, s\}$ so that $I \cap J = (\bigcap_{i \in A} Q_i) \cap (\bigcap_{j \in B} T_j)$ is the standard decomposition of $I \cap J$. Thus by Corollary 2.6,

$$\begin{aligned}\omega(I \cap J) &= \max\{\max_{i \in A}\{e(Q_i)\}, \max_{j \in B}\{e(T_j)\}\} \\ &\leq \max\{\max_{1 \leq i \leq r}\{e(Q_i)\}, \max_{1 \leq j \leq s}\{e(T_j)\}\} \\ &= \max\{\omega(I), \omega(J)\}.\end{aligned}$$

Moreover, $(\sqrt{IJ})^{e(I)+e(J)} = P^{e(I)+e(J)} = P^{e(I)}P^{e(J)} = (\sqrt{I})^{e(I)}(\sqrt{J})^{e(J)} \subseteq IJ$, and so $e(IJ) \leq e(I)+e(J)$. Combined with Theorem 1.2, we have $\omega(IJ) \leq \omega(I)+\omega(J)$. It remains to show that $\omega(I : J) \geq \omega(I) - \omega(J)$. When $J \subseteq I$, then we have $I : J = R$ and $\omega(I : J) = 0 \geq \omega(I) - \omega(J)$ by Lemma 4.2. If $J \not\subseteq I$, then $I : J$ is P-primary by Corollary 2.3, and since $J(I : J) \subseteq I$, we have $\omega(I : J)+\omega(J) \geq \omega(I)$ by the first part of this lemma, hence the claim. □

As Anderson and Badawi pointed out ([1, Example 2.7]), the conclusion of Lemma 4.3 does not hold in every ring R. We add, that even in a polynomial ring over a field, the conclusion of the above lemma may fail if we drop any part of the hypothesis.

Example 4.4. Let $R = k[x, y, z]$ and $I = (x^2, xy, y^2, xz^2)$ and $J = (x^2, xy, y^2, yz^3)$, so that neither I nor J are primary ideals. The standard decompositions of I, J

and $I \cap J$ are

$$\begin{aligned} I &= (x^2, y, z^2) \cap (x, y^2) \\ J &= (x, y^2, z^3) \cap (x^2, y) \\ I \cap J &= (x, y^2) \cap (x^2, y) \\ I + J &= (x, y) \cap (x, y^2, z^3) \cap (x^2, y, z^2). \end{aligned}$$

Thus we have $\omega(I) = 3$, $\omega(J) = 4$, $\omega(I \cap J) = 2$ and $\omega(I + J) = 4$, so that $\omega(I \cap J) < \omega(I + J) = \max\{\omega(I), \omega(J)\}$.

Example 4.5. Let $R = k[x, y, z]$ and $I = (x, y)$ and $J = (y, z^2)$, so that I and J are both primary, but $\sqrt{I} \neq \sqrt{J}$. Then we have $\omega(I) = 1$, $\omega(J) = 2$ and $\omega(I \cap J) = 3$, so that $\omega(I \cap J) > \max\{\omega(I), \omega(J)\}$.

Corollary 4.6. *Let I be a primary monomial ideal of $R = k[x_1, \ldots, x_n]$. Then for each $m \in \mathbb{N}$ we have $\omega(I^m) \leq m\omega(I)$.*

Proof. Follows immediately by induction on m and Lemma 4.3. □

Next, we derive a characterization of primary monomial ω-linear ideals.

Lemma 4.7. *Let $R = k[x_1, \ldots, x_n]$ and Q a primary monomial ideal of R, so that $G(Q)$ consists of monomials of the ring $k[x_{i_1}, \ldots, x_{i_r}]$ for some $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ and there exists $a_1, \ldots, a_r \in \mathbb{N}$ so $\{x_{i_1}^{a_1}, \ldots, x_{i_r}^{a_r}\} \subseteq G(Q)$. Choose $s \in \{1, \ldots, r\}$ so $a_s = \max_{1 \leq j \leq r}\{a_j\}$.*

(1) *If $G(Q) = \{x_{i_1}^{a_1}, \ldots, x_{i_r}^{a_r}\}$, then $\omega(Q^m) = (m-1)a_s + \omega(Q)$ for each $m \in \mathbb{N}$.*

(2) *Q is ω-linear if and only if $\omega(Q) = a_s$.*

Proof. (1) Let $Q = (x_{i_1}^{a_1}, \ldots, x_{i_r}^{a_r})$. Then given $m \in \mathbb{N}$, set $S_m = \{(k_1, \ldots, k_r) \in \mathbb{N}^r \mid \sum\limits_{j=1}^{r} k_j = m + r - 1\}$ and $Q_k = (x_{i_1}^{k_1 a_1}, \ldots, x_{i_r}^{k_r a_r})$ for each $k = (k_1, \ldots, k_r) \in S_m$. Then $Q^m = \bigcap_{k \in S_m} Q_k$ ([14, Theorem 6.2.4]). Now by Corollary 2.6 and Lemma 2.4,

$$\omega(Q^m) = \max_{k \in S_m}\{e(Q_k)\} = \max_{k \in S_m}\{\sum_{j=1}^{r} k_j a_j\} - r + 1 = (m-1)a_s + \omega(Q).$$

(2) Fix $m \in \mathbb{N}$ and set

$$I_1 = (x_{i_1}^{a_1}, \ldots, x_{i_r}^{a_r})^m, I_2 = (x_{i_1}, \ldots, x_{i_{s-1}}, x_{i_s}^{m a_s}, x_{i_{s+1}}, \ldots, x_{i_r}).$$

It follows that $I_1 \subseteq Q^m \subseteq I_2$ are $(x_{i_1}, \ldots, x_{i_r})$-primary ideals, so we have $m a_s = \omega(I_2) \leq \omega(Q^m) \leq \omega(I_1) = (m-1)a_s + \sum_{j=1}^{r} a_j - r + 1$ by Corollary 4.2, Lemma 2.4 and part 1 of this lemma. Therefore if Q is ω-linear, then $\omega(Q) = \lim_{m\to\infty} \frac{m\omega(Q)}{m} = \lim_{m\to\infty} \frac{\omega(Q^m)}{m} = a_s$. Conversely, suppose that $\omega(Q) = a_s$ and fix $m \in \mathbb{N}$. Then since $x_{i_s}^{ma_s} \in G(Q^m)$ we have $\omega(Q^m) \geq ma_s = m\omega(Q)$ by Lemma 4.1. Hence $\omega(Q^m) = m\omega(Q)$ by Corollary 4.6 and so Q is ω-linear. □

Corollary 4.8. *Let I be an irreducible monomial ideal of $R = k[x_1, \dots, x_n]$ so that $I = (x_{i_1}^{a_1}, \dots, x_{i_r}^{a_r})$ for some $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ and $a_1, \dots, a_n \in \mathbb{N}$. Set $a_s = \max_{1\leq j\leq r}\{a_j\}$. Then the following are equivalent.*

(1) *I is ω-linear.*
(2) *$\omega(I^m) = m\omega(I)$ for some $m > 1$.*
(3) *$\omega(I) = a_s$.*
(4) *$a_i = 1$ for each $i \neq s$.*

Proof. (1) $\Rightarrow$ (2) Obvious.
(2) $\Rightarrow$ (3) Suppose that $\omega(I^m) = m\omega(I)$ for some $m > 1$. By Lemma 4.7(1) we have $\omega(I^m) = (m-1)a_s + \omega(I)$. Hence $\omega(I) = a_s$.
(3) $\Leftrightarrow$ (4) Immediate consequence of Lemma 2.4.
(3) $\Leftrightarrow$ (1) Follows from Lemma 4.7(2). □

Lemma 4.9. *Let P be a monomial prime ideal of R. If I, J are P-primary ω-linear monomial ideals of R, then so is $I \cap J$.*

Proof. Without loss of generality we may assume that $\omega(I) \geq \omega(J)$. By Lemma 4.7(2), there is $j \in \{1, \dots, r\}$ so that $x_{i_j}^{\omega(I)} \in G(I)$. There exists $a \in \mathbb{N}$ so $x_{i_j}^{a} \in G(J)$. Then again, by Lemma 4.7(2), $a \leq \omega(J)$. Now, $x_{i_j}^{\omega(I)} = lcm(x_{i_j}^{\omega(I)}, x_{i_j}^{a}) \in G(I \cap J)$. On the other hand, $\omega(I \cap J) = \omega(I)$ by Lemma 4.3. Hence $I \cap J$ is ω-linear by Lemma 4.7(2). □

Given a monomial ideal I of $R = k[x, y]$ we will write $I = (x^{a_1}y^{b_1}, \dots, x^{a_r}y^{b_r})$ where $\{a_i\}$ and $\{b_i\}$ are strictly decreasing and strictly increasing sequences of non-negative integers, respectively. Similarly, if J is a monomial ideal of R we write $J = (x^{c_1}y^{d_1}, \dots, x^{c_s}y^{d_s})$ where $\{c_i\}$ and $\{d_i\}$ are strictly decreasing and strictly increasing sequence of non-negative integers, respectively. Hence $b_1 = a_r = 0$ iff I is (x, y)-primary, and $d_1 = c_s = 0$ iff J is (x, y)-primary.

Lemma 4.10. *Let $R = k[x, y]$ and I, J be (x, y)-primary monomial ideals with $\omega(I) \geq \omega(J)$. Then $\omega(IJ) \leq \omega(I) + \max\{c_1, d_s\}$.*

Proof. We may assume that $c_1 \geq d_s$. Then $e(I) = \omega(I) \geq \omega(J) \geq c_1$ by Lemma 4.1, so $(x,y)^{e(I)+c_1} = (x,y)^{e(I)}(x^{c_1}, y^{c_1}) = (\sqrt{I})^{e(I)}(x^{c_1}, y^{c_1}) \subseteq IJ$ are (x,y)-primary ideals. Therefore $\omega(IJ) \leq \omega((x,y)^{e(I)+c_1}) = e(I) + c_1 = \omega(I) + c_1$ by Lemma 4.2. □

We now classify ω-linear monomial ideals I in $R = k[x,y]$.

Proposition 4.11. *Let $R = k[x,y]$ and $I = (x^{a_1}y^{b_1}, \ldots, x^{a_r}y^{b_r})$ be a monomial ideal of R. Then the following are equivalent.*

(1) *I is ω-linear.*
(2) *$\omega(I^m) = m\omega(I)$ for some $m > 1$.*
(3) *$\omega(I) = \max\{a_1 + b_1, a_r + b_r\}$.*

Proof. Note that given $m \in \mathbb{N}$ and a monomial f of R, by Lemma 3.1 we have

$$\begin{aligned} \omega(I^m) &= m\omega(I) \\ \Leftrightarrow m(deg(f)) + \omega(I^m) &= m(deg(f)) + m\omega(I) \\ \Leftrightarrow deg(f^m) + \omega(I^m) &= m(deg(f) + \omega(I)) \\ \Leftrightarrow \omega((fI)^m) &= m\omega(fI). \end{aligned}$$

Moreover, if I is a principal ideal, then I satisfies all of $1, 2$, and 3 by Corollary 3.2. Hence we may assume that I is a (x,y)-primary monomial ideal of R. That is, $a_r = b_1 = 0$.
(1) $\Rightarrow$ (2) is trivial.
(2) $\Rightarrow$ (3) Suppose that $\omega(I^m) = m\omega(I)$ for some $m > 1$. Note that $\omega(I^{m-1}) + \omega(I) \geq \omega(I^m) = m\omega(I)$ by Lemma 4.3 and $\omega(I^{m-1}) \leq (m-1)\omega(I)$ by Corollary 4.6, and thereby $\omega(I^{m-1}) = (m-1)\omega(I)$. Hence we must have $\omega(I^2) = 2\omega(I)$. Since $\omega(I^2) \leq \omega(I) + \max\{a_1, b_r\}$ by Lemma 4.10, $\omega(I) = \omega(I^2) - \omega(I) \leq \max\{a_1, b_r\}$. On the other hand, $\omega(I) \geq \max\{a_1, b_r\}$ by Lemma 4.1. Therefore $\omega(I) = \max\{a_1, b_r\}$.
(3) $\Rightarrow$ (1) Follows from Lemma 4.7(2). □

Lemma 4.12. *The set of monomial ω-linear ideals of $R = k[x,y]$ is multiplicatively closed.*

Proof. Let I and J be monomial ω-linear ideals of R. By Lemma 3.1 we may assume that I and J are (x,y)-primary ideals of R. Then $\omega(I) = \max\{a_1, b_r\}$, $\omega(J) = \max\{c_1, d_s\}$ by Proposition 4.11. Now, $x^{a_1+c_1}$ and $y^{b_r+d_s}$ are elements of $G(IJ)$. Hence by Lemma 4.7(2) and Lemma 4.1, to show that IJ is ω-linear it suffices to show that $\omega(IJ) \leq \max\{a_1 + c_1, b_r + d_s\}$. Suppose that $\omega(I) = a_1$ and

$\omega(J) = c_1$. Then all we have to show is $\omega(IJ) \leq a_1 + c_1$, which follows from Lemma 4.3. The case when $\omega(I) = b_r$ and $\omega(J) = d_s$ can be derived in the exact same manner. Therefore, without loss of generality we may assume that $\omega(I) = a_1 > b_r$ and $\omega(J) = d_s > c_1$. Observe now that $Ix^{c_1} + Jy^{b_r}$ is an (x, y)-primary ideal contained in IJ. Thus by Lemma 4.2 and Theorem 3.9 we have

$$\begin{aligned}\omega(IJ) &\leq \omega(Ix^{c_1} + Jy^{b_r}) \\ &= \max\{\max_{1\leq i\leq r-1}\{a_i + b_{i+1} + c_1\} - 1, \max_{1\leq j\leq s-1}\{c_j + d_{j+1} + b_r\} - 1\} \\ &= \max\{\omega(I) + c_1, \omega(J) + b_r\} \\ &= \max\{a_1 + c_1, b_r + d_s\}.\end{aligned}$$

□

Recall that given an ideal I of a commutative ring R, an element $f \in R$ is said to be *integral* over I if there is some $k \in \mathbb{N}$ and $c_i \in I^i$ for each $i \in \{1, \ldots, k\}$ so that

$$f^k + c_1 f^{k-1} + \cdots + c_{k-1} f + c_k = 0.$$

The set of elements of R integral over I is called the *integral closure* of I and is denoted by $\overline{I}$. I is said to be *integrally closed* if $I = \overline{I}$.

Corollary 4.13. *Every integrally closed monomial ideal of $R = k[x, y]$ is ω-linear.*

Proof. Let I be an integrally closed monomial ideal of R. It is well known that R is an *integrally closed domain* (i.e., R is an integral domain that contains every nonzero element of the quotient field of R that is integral over R), and that each principal ideal of R is integrally closed, and the product of an integrally closed ideal of R and a nonzero element of R yields another integrally closed ideal of R. Hence by Lemma 3.1 we may assume that I is (x, y)-primary. Now by [16, Proposition 2.6] there are monomial ideals $I_1 = (\{x^{r-i}y^{b_i}\}_{i=0}^r)$ and $I_2 = (\{x^{a_i}y^i\}_{i=0}^r)$ of R with $0 = b_0 < b_1 < \cdots < b_r$ and $a_0 > a_1 > \cdots > a_r = 0$ so $I = I_1 I_2$. Thus by Lemma 4.12, it suffices to show that I_1 and I_2 are ω-linear. By Theorem 3.9, $\omega(I_1) = \max_{0\leq i\leq r-1}\{c_i\}$, where $c_i = r - i + b_{i+1} - 1$ for each $i \in \{0, 1, \ldots, r-1\}$. Since $c_{i+1} - c_i = b_{i+1} - (b_i + 1) \geq 0$ for each $i \in \{0, 1, \ldots, r-1\}$, we have $\omega(I_1) = c_{r-1} = b_r = \max\{r, b_r\}$ and I_1 is ω-linear by Proposition 4.11. The proof that I_2 is ω-linear follows similarly. □

Remark 4.14. (1) Even if I and J are ω-linear monomial primary ideals such that $\sqrt{I} = \sqrt{J}$, we may have $\omega(I \cap J) < \omega(IJ) < \omega(I) + \omega(J)$. Indeed, set $R = k[x, y]$,

$I = (x^3, xy, y^2)$ and $J = (x^2, xy, y^3)$. Then both I and J are ω-linear (x, y)-primary ideals of R. However, $IJ = (x^5, x^3y, x^2y^2, xy^3, y^5)$, so $\omega(IJ) = 5 < 6 = \omega(I)+\omega(J)$. On the other hand, $\omega(I \cap J) = \max\{\omega(I), \omega(J)\} = 3$ by Corollary 2.6.

(2) Not every ω-linear monomial ideal of $R = k[x, y]$ is integrally closed. For example, set $I = (x^3, xy^2, y^4)$. Then $\omega(I) = 4$ by Theorem 3.9, and I is ω-linear by Proposition 4.11. However, $(x^2y)^2 = x^3(xy^2) \in I^2$ and $x^2y \notin I$. Thus I is not integrally closed ([10, Theorem 1.4.2]).

So far, we have considered only ω-linear monomial ideals of the form fI where f is a monomial and I is a primary ideal, and most of the proof is solely based on the fact that $e(I) = \omega(I)$ when I is a primary ideal. We now show that there exists a class of (integrally closed) nonprimary ω-linear monomial ideals. In fact, some of the squarefree monomial ideals are ω-linear. Recall that a monomial $f = x_{i_1}^{a_1} \cdots x_{i_r}^{a_r}$ is said to be *squarefree* if $a_1 = \cdots = a_r = 1$. A monomial ideal generated by squarefree monomials is said to be a *squarefree monomial ideal*.

Lemma 4.15. *Let I be a squarefree monomial ideal. Then $\omega(I^m) \geq m\omega(I)$ for each $m \in \mathbb{N}$.*

Proof. Let $P_1, \ldots, P_r$ be minimal prime ideals of I. Then $I = \bigcap_{i=1}^r P_i$ and $\omega(I) = r$ by Proposition 3.5. Set $f_i = \sum_{x_j \in G(P_i)} x_j$ for each $i \in \{1, \ldots, r\}$. Then $f := \prod_{i=1}^r f_i \in \prod_{i=1}^r P_i \subseteq I$, so $f^m \in I^m$. However, $\dfrac{f^m}{f_i} \notin P_i^m$, so $\dfrac{f^m}{f_i} \notin I^m$([10, Proposition 1.4.4]). Thus I^m is not $(mr-1)$-absorbing and $\omega(I^m) \geq m\omega(I)$. $\square$

Recall that a graph G consists of a set of vertices $V = \{v_1, ..., v_n\}$ and a set of edges $E \subseteq \{v_iv_j | v_i, v_j \in V\}$, and is called *bipartite* if there exists two disjoint subsets U_1, U_2 of V such that $E \subseteq \{v_iv_j \mid v_i \in U_1, v_j \in U_2\}$. The *edge ideal* of G is defined to be the ideal $I = (\{x_ix_j | v_iv_j \in E\})$ of $R = k[x_1, ..., x_d]$, where k be a field and d is the number of vertices of G. Given a graph $G = (V, E)$, a subset W of V is said to be a *vertex cover* if given $v_iv_j \in E$, either $v_i \in W$ or $v_j \in W$. A vertex cover W of G is said to be a *minimal vertex cover* if each proper subset of W is not a vertex cover of G.

If I is an edge ideal of a graph, then it is a squarefree monomial ideal and a monomial prime ideal P is a minimal ideal of I if and only if the set of vertices that corresponds to P is a minimal vertex cover. Also, a graph is bipartite if and only if it has no cycle of odd length as its subgraph.

Our first example of a nonprimary ω-linear ideal is the edge ideal of a bipartite graph.

Lemma 4.16. *Let $R = k[x_1, \ldots, x_n]$. If I is an ideal of R that is also the edge ideal of a bipartite graph G, then I is ω-linear.*

Proof. Let I be an edge ideal of a graph G and let $P_1, \ldots, P_r$ be the set of (incomparable) minimal prime ideals of I. Recall that a graph G is bipartite if and only if

$$I^m = \bigcap_{P \text{ is a minimal prime of } I} P^m$$

for each $m \in \mathbb{N}$ ([18, Theorem 5.9]). Hence if G is bipartite, then by Proposition 3.5, $\omega(I^m) = \sum_{i=1}^{r} e(P_i^m) = \sum_{i=1}^{r} m = mr$ for each $m \in \mathbb{N}$. Therefore the conclusion follows. □

There are nonbipartite graphs whose edge ideals are ω-linear.

Theorem 4.17. *Let $R = k[x_1, \ldots, x_n]$. Let $I = (x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1)$ (that is, I is the edge ideal of a cycle graph of length n). Then I is ω-linear.*

Proof. Since a cycle of even length is bipartite, by Lemma 4.16 we may assume that $n = 2l+1$ for some $l \in \mathbb{N}$. Fix $m \in \mathbb{N}$. I is a squarefree monomial ideal, so $I = P_1 \cap \cdots \cap P_r$ where $P_1, \ldots, P_r$ are the minimal prime ideals of I ([10, Lemma 1.3.5]). Thus by Proposition 3.5 we have $\omega(I) = \sum_{i=1}^{r} e(P_i) = r$, and we only need to show that $\omega(I^m) = mr$. Note that since I is an edge ideal of a cycle of length $2l+1$, $\mathrm{Ass}(R/I^m) = \{P_1, \ldots, P_r\}$ if $m \leq l$ and $\mathrm{Ass}(R/I^m) = \{P_1, \ldots, P_r, \mathfrak{m}\}$ if $m > l$ ([5, Lemma 3.1]). Hence if $m \leq l$, then $I^m = \bigcap_{i=1}^{r} P_i^m$ and $\omega(I^m) = \sum_{i=1}^{r} e(P_i^m) = mr$ by Proposition 3.5, so we are done. Assume that $m > l$. Then $I^m = (\bigcap_{i=1}^{r} P_i^m) \cap Q$ is the canonical primary decomposition of I^m, where Q is an $\mathfrak{m}$-primary monomial ideal of R ([10, Proposition 1.4.4]). Now, $Q = (x_1^{a_1}, \ldots, x_n^{a_n}, f_1, \ldots, f_t)$ for some $a_i \in \mathbb{N}$ and monomials f_i. Since I is a squarefree monomial ideal and Q is a primary component of I^m, we must have $a_i \leq m$ for each $i \in \{1, \ldots, n\}$, and thus $e(Q) \leq e((x_1^{a_1}, \ldots, x_n^{a_n})) \leq mn - n + 1 \leq mr$ by Lemma 2.4 and since $n \leq r$. It follows that $\omega(I^m) = \max\{\sum_{i=1}^{r} e(P_i^m), e(Q)\} = \max\{mr, e(Q)\} = mr$ by Theorem 3.3 and Proposition 3.5. □

We close the section with the following question: Is every integrally closed monomial ideal ω-linear? Integrally closed monomial ideals considered in this note (certain monomial ideals in $R = k[x, y]$, irreducible monomial ideals, or edge ideal of bipartite graphs) were all ω-linear. Note also that if this question has an affirmative answer, then it follows that every edge ideal is ω-linear.

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References

[1] D. F. Anderson and A. Badawi, *On n-absorbing ideals of commutative rings*, Comm. Algebra, 39(5) (2011), 1646-1672.

[2] D. F. Anderson and S. T. Chapman, *How far is an element from being prime?*, J. Algebra Appl., 9(5) (2010), 779-789.

[3] A. Badawi, *On 2-absorbing ideals of commutative rings*, Bull. Austral. Math. Soc., 75(3) (2007), 417-429.

[4] A. Badawi, *n-Absorbing ideals of commutative rings and recent progress on three conjectures: a survey*, In: Fontana, M., Frisch, S., Glaz, S., Tartarone, F., Zanardo, P. (eds) Rings, Polynomials, and Modules, Springer, Cham, (2017), 33-52.

[5] J. Chen, S. Morey and A. Sung, *The stable set of associated primes of the ideal of a graph*, Rocky Mountain J. Math., 32(1) (2002), 71-89.

[6] H. S. Choi and A. Walker, *The radical of an n-absorbing ideal*, arXiv:1610.10077 [math.AC], (2016), J. Commutative Algebra, accepted.

[7] G. Donadze, *The Anderson-Badawi Conjecture for commutative algebras over infinite fields*, Indian J. Pure Appl. Math., 47(4) (2016), 691-696.

[8] G. Donadze, *A proof of the Anderson-Badawi $rad(I)^n \subseteq I$ formula for n-absorbing ideals*, Proc. Indian Acad. Sci. Math. Sci., 128(1) (2018), Art. 6 (6 pp).

[9] D. Eisenbud, Commutative Algebra, with a View Toward Algebraic Geometry, Graduate Texts in Mathematics, 150, Springer-Verlag, New York, 1995.

[10] J. Herzog and T. Hibi, Monomial Ideals, Graduate Texts in Mathematics, 260, Springer-Verlag London, Ltd., London, 2011.

[11] A. Laradji, *On n-absorbing rings and ideals*, Colloq. Math., 147(2) (2017), 265-273.

[12] E. Miller and B. Sturmfels, Combinatorial Commutative Algebra, Graduate Texts in Mathematics, 227, Springer-Verlag, New York, 2005.

[13] H. F. Moghimi and S. R. Naghani, *On n-absorbing ideals and the n-Krull dimension of a commutative ring*, J. Korean Math. Soc., 53(6) (2016), 1225-1236.

[14] W. F. Moore, M. Rogers and S. Sather-Wagstaff, Monomial Ideals and Their Decompositions, Universitext, Springer, Cham, 2018, www.ndsu.edu/pubweb/ ssatherw/DOCS/monomial.pdf.

[15] P. Nasehpour, *On the Anderson-Badawi* $\omega_{R[X]}(I[X]) = \omega_R(I)$ *conjecture*, Arch. Math. (Brno), 52(2) (2016), 71-78.

[16] V. C. Quiñonez, *Integral closure and other operations on monomial ideals*, J. Commut. Algebra, 2(3) (2010), 359-386.

[17] S. Sihem and H. Sana, *On Anderson-Badawi conjectures*, Beitr. Algebra Geom., 58(4) (2017), 775-785.

[18] A. Simis, W. V. Vasconcelos and R. H. Villarreal, *On the ideal theory of graphs*, J. Algebra, 167 (1994), 389-416.

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