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LOCAL T_2 EXTENDED PSEUDO-QUASI-SEMI METRIC SPACES

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Abstract. In this paper, we characterize various local T_2 extended pseudo-quasi-semi metric spaces and investigate the relationships among these various forms. Finally, we give some invariance properties of these local T_2 extended pseudo-quasi-semi metric spaces.

1. Introduction

There are various ways of generalizing the notions of metric spaces [2, 18, 19, 20, 21, 22, 25]. In 1988, E.Lowen and R.Lowen [21] introduced the category of extended pseudo-quasi-semi metric spaces and non-expansive maps which is the most general and best behaved of the category of metric spaces in which one can take arbitrary products, coproducts and form quotient objects within it.
In 1991, Baran [3] introduced local separation properties T_0, T_1 and $PreT_2$ (Pre-Hausdorff) in set-based topological categories. Local T_1 separation property is used to define the notion of strongly closed subobject of an object of a topological category which is used in the notions of compactness, perfectness [9] and connectedness [10]. Local T_0 and $PreT_2$ are used to define various forms of local T_2 objects [3] in arbitrary topological categories. Furthermore, local T_1 and $PreT_2$ are used to define local regular and normal objects in [3, 8].

In this paper, we characterize local T_2 extended pseudo-quasi-semi metric spaces, i.e., T_2 extended pseudo-quasi-semi metric space at a point p, where T_2 is one of $\overline{T_2}$, T_2', LT_2, and KT_2 and investigate the relationships among them. Furthermore, we show that each of the full subcategories $\mathbf{Pre\overline{T}_2(pqsMet)}$, $\mathbf{T_2'(pqsMet)}$, and $\mathbf{LT_2(pqsMet)}$ of $\mathbf{pqsMet}$, the category of extended pseudo-quasi-semi metric

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spaces and non-expansive maps is epireflective, i.e., a full and isomorphism-closed, closed under the formation of products and extremal subobjects (i.e., subspaces) and the full subcategory $\mathbf{LT_2}(\mathbf{pqsMet})$ of **pqsMet** is bireflective, i.e., it is epireflective and contains $\mathcal{I}$, the subcategory of all indiscrete extended pseudo-quasi-semi metric spaces.

2. Preliminaries

Recall, [1, 16, 24] that a functor $\mathcal{U} : \mathcal{E} \to \mathbf{Set}$, the category of sets and functions, is said to be topological or that $\mathcal{E}$ is a topological category over **Set** if $\mathcal{U}$ is faithful and amnestic (i.e., if $\mathcal{U}(f) = id$ and f is an isomorphism, then $f = id$), has small fibers, and for which every $\mathcal{U}$-source has an initial lift or, equivalently, for which each $\mathcal{U}$-sink has a final lift. A topological functor $\mathcal{U} : \mathcal{E} \to \mathbf{Set}$ is said to be normalized if subterminals have a unique structure.

An extended pseudo-quasi-semi metric space is a pair (X, d), where X is a set $d : X \times X \to [0, \infty]$ is a function fulfills the following condition $d(x, x) = 0$ for all $x \in X$ [20, 21, 23].

A map $f : (X, d) \to (Y, e)$ between extended pseudo-quasi-semi metric spaces is said to be a non-expansive if it fulfills the property $e(f(x), f(y)) \leq d(x, y)$ for all $x, y \in X$.

The construct of extended pseudo-quasi-semi metric spaces and non-expansive maps is denoted by **pqsMet**.

2.1 Let $\{(X_i, d_i), i \in I\}$ be a class of extended pseudo-quasi-semi metric spaces, X a nonempty set, and $\{f_i : X \to X_i, i \in I\}$ be a source in **Set**. A source $\{f_i : (X, d) \to (X_i, d_i), i \in I\}$ in **pqsMet** is an initial lift if and only if $d = \sup\limits_{i \in I}(d_i \circ (f_i \times f_i))$, i.e., for all $x, y \in X, d(x, y) = \sup\limits_{i \in I}(d_i(f_i(x), f_i(y)))$ [20, 23].

2.2.. Let $(X_i, d_i), i \in I\}$ be a class of extended pseudo-quasi-semi metric spaces and X be a nonempty set. A sink $\{f_i : (X_i, d_i) \to (X, d), i \in I\}$ is final in **pqsMet** if and only if for all $x, y \in X$,
$d(x, y) = \inf\limits_{i \in I}\{(d_i(f_i(x_i), f_i(y_i))) :$ there exist $x_i, y_i \in X_i$ such that $f_i(x_i) = x$ and $f_i(y_i) = y$ $\}$ [20, 23].
In particular, let $X = \coprod_{i \in I} X_i$. Define

$$d_2((k, x), (j, y)) = \begin{cases} d_k(x, y) & \text{if } k = j \\ \infty & \text{if } k \neq j \\ \end{cases}$$

for all $(k, x), (j, y) \in X$. (X, d_2) is the coproduct of $\{(X_i, d_i), i \in I\}$ extended pseudo-quasi-semi metric spaces, i.e., $\{k_i : (X_i, d_i) \to (X, d_2), i \in I\}$ is a final lift of $\{k_i : X_i \to X, i \in I\}$, where k_i are the canonical injection maps [18].

2.3.. The discrete extended pseudo-quasi-semi metric structure d on X is given by

$$d_{dis}(a,b) = \begin{cases} 0 & \text{if } a = b \\ \infty & \text{if } a \neq b \end{cases}$$

for all $a, b \in X$.
Note that the category **pqsMet** is a Cartesian closed and hereditary topological [21].

Let B be a set and $p \in B$. Let $B \vee_p B$ be the wedge at p [3], i.e., two disjoint copies of B identified at p. A point $x \in B \vee_p B$ will be denoted by x_1 (x_2) if x is in the first (respectively, the second) component of $B \vee_p B$. Note that $p_1 = p_2$. Let $i_1, i_2 : B \to B \coprod B$ be the canonical injection maps and $q : B \coprod B \to B \vee_p B$ be the quotient map. The principal p-axis map $A_p : B \vee_p B \to B^2$ is given by $A_p(x_1) = (x, p)$ and $A_p(x_2) = (p, x)$. The skewed p-axis map $S_p : B \vee_p B \to B^2$ is given by $S_p(x_1) = (x, x)$ and $S_p(x_2) = (p, x)$ [3]. The fold map at p, $\nabla_p : B \vee_p B \to B$ is given by $\nabla_p(x_i) = x$ for $i = 1, 2$ [3].

Definition 1. *Let (X, τ) be a topological space and $p \in X$.*
(1) If for each point x of X distinct from p, there exists a neighborhood of p missing x or there exists a neighborhood of x missing p, then (X, τ) is called T_0 at p [3, 6].
(2) If for each point x distinct from p, there exists a neighborhood of p missing x and there exists a neighborhood of x missing p, then (X, τ) is called T_1 at p [3, 6].
(3)(X, τ) is called pre-Hausdorff space at p, denoted by $PreT_2$ at p, if for each point x distinct from p, the subspace $\{x, p\}$ is not indiscrete, then the points x and p have disjoint neighborhoods [3, 6].

The following result is given in [6].

Theorem 2. *Let (X, τ) be a topological space and $p \in X$.*
(1) The followings are equivalent:
(i.) (X, τ) is $PreT_2$ at p,
(ii.) The initial topology induced from $A_p : X \vee_p X \to (X^2, \tau_*)$ and $S_p : X \vee_p X \to (X^2, \tau_*)$ are the same, where τ_* is the product topology on X^2.
(iii.) The induced (initial) topology from $S_p : X \vee_p X \to (X^2, \tau_*)$ and the co-induced (final) topology from $q \circ i_k : (X, \tau) \to X \vee_p X, k = 1, 2$ are the same, where i_1 and i_2 are the canonical injection maps and $q : X \coprod X \to X \vee_p X$ is the quotient map.
(2) The followings are equivalent:
(i.) (X, τ) is T_0 at p
(ii.) The initial topology induced from $A_p : X \vee_p X \to (X^2, \tau_*)$ and $\nabla_p : X \vee_p X \to (X, P(X))$ is discrete.
(iii.) The induced (initial) topology from $S_p : X \vee_p X \to (X^2, \tau_*)$ and the co-induced (final) topology from $q \circ i_k : (X, \tau) \to X \vee_p X, k = 1, 2$ are the same.
(3) A topological space (X, τ) is T_2 at p if and only if (X, τ) is T_0 at p and (X, τ) is $PreT_2$ at p.

3. Local T_2 Extended Pseudo-Quasi-Semi Metric Spaces

Definition 3. *(cf. [3]) Let $\mathcal{U} : \mathcal{E} \to$ **Set** be topological, X an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$ and $p \in \mathcal{U}(X)$.*
(1) If the initial lift of the U -source $\{A_p : B \vee_p B \to U(X) = B$ and $\nabla_p : B \vee_p B \to U(D(B)) = B\}$ is discrete, then X is called $\overline{T}_0$ at p, where D is the discrete functor which is a left adjoint to U.
(2) If the initial lift of the U-source $\{id : B \vee_p B \to U(B \vee_p B, (B \vee_p B)') = B \vee_p B$ and $\bigtriangledown_p : B \vee_p B \to UD(B) = B$ $\}$ is discrete, then X is called T_0' object at p, where $(B \vee_p B)'$ is the final lift of the U-sink $\{q \circ i_1, q \circ i_2 : U(X) = B \to B \vee_p B\}$, the maps i_1, i_2 are the canonical injection maps and $q : B \coprod B \to B \vee_p B$ is the quotient map.
(3) If the initial lift of the $\mathcal{U}$-source $S_p : B \vee_p B \to \mathcal{U}(X^2) = B^2$ and the final lift of the $\mathcal{U}$-sink $\{i_1, i_2 : \mathcal{U}(X) = B \to B \vee_p B\}$ coincide, then X is called a $PreT_2'$ object at p.
(4) If the initial lift of the $\mathcal{U}$-sources $S_p : B \vee_p B \to \mathcal{U}(X^2) = B^2$ and $A_p : B \vee_p B \to \mathcal{U}(X^2) = B^2$ coincide, then X is called a $Pre\overline{T}_2$ object at p.

Theorem 4. *(1) An extended pseudo-quasi-semi metric space (X, d) is $Pre\overline{T}_2$ at $p \in X$ if and only if the following conditions are satisfied.*
(i) For all $x \in X$ with $x \neq p$, $d(x, p) = d(p, x)$.
(ii) For any two distinct points x, y of X with $x \neq p \neq y$, we have either $d(x, p) = d(p, y) \geq d(x, y), d(y, x)$ or $d(p, y) = d(x, y) = d(y, x) \geq d(x, p)$ or $d(x, p) = d(x, y) = d(y, x) \geq d(p, y)$.
(2) An extended pseudo-quasi-semi metric space (X, d) is $PreT_2'$ at $p \in X$ if and only if for all $x \in X$ with $x \neq p$, $d(x, p) = \infty$ and $d(p, x) = \infty$.
(3) An extended pseudo-quasi-semi metric space (X, d) is $\overline{T}_0$ at $p \in X$ if and only if for all $x \in X$ with $x \neq p$, $d(x, p) = \infty$ or $d(p, x) = \infty$.
(4) Every extended pseudo-quasi-semi metric space (X, d) is $T_0^{'}$ at $p \in X$.

Proof. The proof of (1) and (2) are given in [14].
(3) Suppose (X, d) is $\overline{T}_0$ at $p \in X$ and for $x \in X$ with $x \neq p$. Since $x_1 \neq x_2$, by Definition 3,

$$\begin{aligned} \infty &= \sup\{d_{dis}(\nabla_p(x_1), \nabla_p(x_2)), d(\pi_1 A_p(x_1), \pi_1 A_p(x_2)), d(\pi_2 A_p(x_1), \pi_2 A_p(x_2))\} \\ &= \sup\{d(x, p), d(p, x)\} \end{aligned}$$

and consequently, $d(x, p) = \infty$ or $d(p, x) = \infty$.
The other side of proof is similar to the proof of Theorem 3.4 of [12] replacing the skewed p-axis map S_p by the principal p-axis map A_p.
(4) Let (X, d) be an extended pseudo-quasi-semi metric space, $p \in X$, and $\bar{d}$ be the initial structure on $X \vee_p X$ induced from $id : X \vee_p X \to (X \vee_p X, d_1)$ and $\bigtriangledown_p : X \vee_p X \to (X, d_{dis})$, where d_{dis} is the discrete extended pseudo-quasi-semi metric structure on X and d_1 is the quotient extended pseudo-quasi-semi metric

structure on $X \vee_p X$ induced from $q : (X \coprod X, d_2) \to X \vee_p X, d_2$ is the final (coproduct) structure on $X \coprod X$ induced from the canonical injection maps i_1 and i_2.
Let u and v be any points in $X \vee_p X$.
If $u = v$, then $\bar{d}(u, v) = 0$. If $\bigtriangledown_p(u) = p = \bigtriangledown_p(v)$, then $u = p_k = v, k = 1, 2$ and consequently, $\bar{d}(u, v) = 0$.
Suppose that $u \neq v$ and $\bigtriangledown_p(u) = x = \bigtriangledown_p(v)$ for some $x \in X$ with $x \neq p$. It follows that $u = x_1, v = x_2$ or $u = x_2, v = x_1$.
If $u = x_1$ and $v = x_2$, then, by 2.2, $d_1(u, v) = d_2(x_1, x_2) = \infty$. Hence, by 2.1, $\bar{d}(u, v) = \sup\{d_{dis}(\bigtriangledown_p(u), \bigtriangledown_p(v)), d_1(u, v)\} = \sup\{0, \infty\} = \infty$. Similarly, if $u = x_2$ and $v = x_1$, then $\bar{d}(u, v) = \infty$.
If $u \neq v$ and $\bigtriangledown_p(u) \neq \bigtriangledown_p(v)$, then, by 2.3, $d_{dis}(\bigtriangledown_p(u), \bigtriangledown_p(v)) = \infty$, and by 2.1, $\bar{d}(u, v) = \sup\{d_{dis}(\bigtriangledown_p(u), \bigtriangledown_p(v)), d_1(u, v)\} = \sup\{\infty, d_1(u, v)\} = \infty$.
Hence, for any points u and v in $X \vee_p X$

$$\bar{d}(u, v) = \begin{cases} 0 & \text{if } u = v \\ \infty & \text{if } u \neq v \end{cases}$$

and consequently, by 2.1, 2.3, and Definition 3, (X, d) is $T_0^{'}$ at p. □

Definition 5. *(cf. [3, 5, 7]) Let $\mathcal{U} : \mathcal{E} \to$ **Set** be topological, X an object in $\mathcal{E}$ and $p \in \mathcal{U}(X)$.*
(1) If X is $\overline{T}_0$ at p and $Pre\overline{T}_2$ at p, then X is called $\overline{T}_2$ at p.
(2) If X is T_0' at p and $PreT_2'$ at p, then X is called T_2' at p.
(3) If X is T_0' at p and $Pre\overline{T}_2$ at p, then X is called KT_2 at p.
(4) If X is $\overline{T}_0$ at p and $PreT_2'$ at p, then X is called LT_2 at p.

Theorem 6. *(1) An extended-pseudo-quasi-semi metric space (X, d) is $\overline{T}_2$ (resp. T_2', LT_2) at $p \in X$ if and only if for all $x \in X$ with $x \neq p$, $d(x, p) = \infty$ and $d(p, x) = \infty$.*
(2) An extended-pseudo-quasi-semi metric space (X, d) is KT_2 if and only if the following conditions are satisfied.
(i) For all $x \in X$ with $x \neq p$, $d(x, p) = d(p, x)$.
(ii) For any two distinct points x, y of X with $x \neq p \neq y$, we have either $d(x, p) = d(p, y) \geq d(x, y), d(y, x)$ or $d(p, y) = d(x, y) = d(y, x) \geq d(x, p)$ or $d(x, p) = d(x, y) = d(y, x) \geq d(p, y)$.

Proof. (1) Suppose that (X, d) is $\overline{T}_2$ and $x \in X$ with $x \neq p$. By Theorem 4(1) and Definition 3, $d(x, p) = d(p, x)$. Since (X, d) is $\overline{T}_0$ at p, by Theorem 4(3) $d(x, p) = \infty = d(p, x)$.
Conversely, if $d(x, p) = \infty = d(p, x)$ for all $x \in X$ with $x \neq p$, then by Parts (1) and (3) of Theorem 4, (X, d) is $\overline{T}_0$ at p and $Pre\overline{T}_2$ at p. By definition 5, (X, d) is $\overline{T}_2$.
(2) It follows from Theorem 4 and Definition 5. □

Theorem 7. *Let $\mathcal{U} : \mathcal{E} \to$ **Set** be normalized and X be an object in $\mathcal{E}$ with $p \in U(X)$. Then*
(i) If X is $\overline{T}_2$ at p or T_2' at p, then X is KT_2 at p.
(ii) If X is LT_2 at p, then X is KT_2 at p, $\overline{T}_2$ at p, and T_2' at p.

Proof. (i) Suppose that X is $\overline{T}_2$ at p, i.e., X is $\overline{T}_0$ at p and $Pre\overline{T}_2$ at p. Since $\mathcal{U} : \mathcal{E} \to$ **Set** is normalized, by Corollary 2.11 of [4], X is T_0' at p. Hence, by Definition 5, X is KT_2 at p.
If X is T_2' at p, then by Corollary 2.7 of [4] and Theorem 3.1 of [7], X is $Pre\overline{T}_2$ at p and by Definition 5, X is KT_2 at p.
The proof of (ii) is similar to the proof of (i) by using Definition 5, Theorem 3.1 of [7], and Corollaries 2.7 and 2.11 of [4].

□

Remark 8. *(1) Let $\mathcal{U} : \mathcal{E} \to$ **Set** be normalized and X be an object in $\mathcal{E}$ with $p \in U(X)$. If X is an $Pre\overline{T}_2$ object at p, then by Corollary 2.7 of* [4] *and Theorem 3.5 of* [11]*, X is $\overline{T}_0$ at p if and only if X is $\overline{T}_2$ at p.*
(2) For the category **Top** *of topological spaces, by Theorem 2, all of LT_2 at p, KT_2 at p, $\overline{T}_2$ at p, and T_2' at p are equivalent.*
(3) Let (X, d) be an extended pseudo-quasi-semi metric space and $p \in X$.
(A) By Theorem 4 and Theorem 6, the followings are equivalent.
(i) (X, d) is $\overline{T}_2$ at p.
(ii) (X, d) is T_2' at p.
(iii) (X, d) is LT_2 at p.
(iv) For all $x \in X$ with $x \neq p$, $d(x, p) = \infty$ and $d(p, x) = \infty$.
(B) By Theorems 4,6 and 7, each of LT_2 at p, $\overline{T}_2$ at p, and T_2' at p implies KT_2 at p but the reverse implication is not true, in general. For example, take X be a set with $|X| \geq 2$ and d be the indiscrete extended pseudo-quasi-semi metric structure on X, i.e., $d(a, b) = 0$ for all $a, b \in X$. By Theorem 6, (X, d) is KT_2 at p for all $p \in X$ but (X, d) is not any of LT_2 at p, $\overline{T}_2$ at p, and T_2' at p.
(C) The relationship between each of T_0' at p, $\overline{T}_0$ at p, T_1 at p and general T_0', $\overline{T}_0$, T_1 are investigated in [13]*.*
(4) Note, also, that all objects of a set-based arbitrary topological category may be KT_2 at p. For example, by Theorem 3.2 and Theorem 3.4 of [17]*, it is shown that all Cauchy spaces* [15] *are T_0' at p and $Pre\overline{T}_2$ at p, and consequently, all Cauchy spaces are KT_2 at p.*

Now we give some invariance properties of local LT_2, KT_2, $\overline{T}_2$, and T_2' extended pseudo-quasi-semi metric spaces.

Let (X, d) be an extended-pseudo-quasi-semi metric space and F be a none empty subset of X. Let $q : X \to X/F$ be the epi map that is the identity on $X \backslash F$ and identifying F with a point $*$.

Theorem 9. *Let (X, d) be an extended-pseudo-quasi-semi metric space, $p \in X$ and F be a none empty subset of X. If (X, d) is $\overline{T}_2$ (resp. T_2', LT_2) at p and $d(F, x) = \infty = d(x, F)$ for all $x \in X$ with $x \notin F$, then $(X/F, d_1)$ $\overline{T}_2$ (resp. T_2', LT_2) at $q(p)$, where d_1 is the quotient structure on X/F induced from the epi map q.*

Proof. Suppose that (X, d) is $\overline{T}_2$ (resp. T_2', LT_2) at p and $d(F, x) = \infty = d(x, F)$ for all $x \in X$ with $x \notin F$. Let $a \in X/F$ with $a \neq q(p)$. Suppose $p \notin F$ and $a \neq *$. By 2.2, $d_1(a, p) = d(a, p)$ and $d_1(p, a) = d(p, a)$. Since (X, d) is $\overline{T}_2$ (resp. T_2', LT_2) at p, by Theorem 6, $d(a, p) = \infty = d(p, a)$ and consequently, $d_1(a, p) = \infty = d_1(p, a)$. If $a = *$, then by 2.2, $d_1(a, p) = d_1(*, p) = d(F, p)$ and $d_1(a, p) = d_1(p, *) = d(p, F)$. Since $p \notin F$, by assumption, we get $d(F, p) = \infty = d(p, F)$ and consequently, $d_1(*, p) = \infty = d_1(p, *)$.
If $p \in F$, then $q(p) = *$. Since $a \neq *$, $a \notin F$ and by 2.2,

$$d_1(a, q(p)) = d_1(a, *) = d(a, F)$$

and

$$d_1(q(p), a) = d_1(*, a) = d(F, a)$$

Since $a \notin F$, by assumption, we get $d(F, a) = \infty = d(a, F)$ and consequently, $d_1(*, a) = \infty = d_1(a, *)$.
Hence, $(X/F, d_1)$ $\overline{T}_2$ (resp. T_2', LT_2) at $q(p)$. □

Theorem 10. *Let (X, d) be an extended-pseudo-quasi-semi metric space and F be a none empty subset of X. If (X, d) is KT_2 at p for all $p \in X$, then $(X/F, d_1)$ is KT_2 at $q(p)$.*

Proof. Suppose that (X, d) is KT_2 at p for all $p \in X$. Let $a \in X/F$ with $a \neq q(p)$. Suppose $p \notin F$ and $a \neq *$. By 2.2, $d_1(a, p) = d(a, p)$ and $d_1(p, a) = d(p, a)$. Since (X, d) is KT_2 at p, by Theorem 6, $d(a, p) = d(p, a)$ and consequently, $d_1(a, p) = d_1(p, a)$. If $a = *$, then by 2.2, $d_1(a, p) = d_1(*, p) = d(F, p)$ and $d_1(a, p) = d_1(p, *) = d(p, F)$. Since $p \notin F$ and (X, d) is KT_2 at p for all $p \in X$, in particular, $d(x, p) = d(p, x)$ for all $x \in F$. It follows that $d(F, p) = d(p, F)$ and consequently, $d_1(*, p) = d_1(p, *)$.
If $p \in F$, then $q(p) = *$. Since $a \neq *$, $a \notin F$ and by 2.2,

$$d_1(a, q(p)) = d_1(a, *) = d(a, F)$$

and

$$d_1(q(p), a) = d_1(*, a) = d(F, a)$$

Since $a \notin F$ and (X, d) is KT_2 at p for all $p \in X$, by Theorem 6, in particular, $d(x, a) = d(a, x)$ for all $x \in F$ which implies $d(F, a) = d(a, F)$ and consequently, $d_1(*, a) = d_1(a, *)$.
Suppose that for any two distinct points a, b of X/F with $a \neq q(p) \neq b$. Suppose

$q(p) \neq *$, i.e., $p \notin F$ and $a \neq * \neq b$. It follows that $a, b, p \notin F$ and $a \neq p \neq b$. Since (X, d) is KT_2 at p, by Theorem 6,

$$d(a,p) = d(p,b) \geq d(a,b), d(b,a)$$

or

$$d(p,b) = d(a,b) = d(b,a) \geq d(a,p)$$

or

$$d(a,p) = d(a,b) = d(b,a) \geq d(p,b)$$

Note that, by 2.2, $d_1(a,p) = d(a,p), d_1(a,b) = d(a,b), d_1(p,b) = d(p,b)$ and consequently,

$$d_1(a,p) = d_1(p,b) \geq d_1(a,b), d_1(b,a)$$

or

$$d_1(p,b) = d_1(a,b) = d_1(b,a) \geq d_1(a,p)$$

or

$$d_1(a,p) = d_1(a,b) = d_1(b,a) \geq d_1(p,b)$$

Suppose $a = * \neq p \neq b$, then $p \notin F$. Since (X, d) is KT_2 at p for all $p \in X$, in particular, for all $x \in F$ with $x \neq p \neq b$, by Theorem 6, we have

$$d(x,p) = d(p,b) \geq d(x,b), d(b,x)$$

or

$$d(p,b) = d(x,b) = d(b,x) \geq d(x,p)$$

or

$$d(x,p) = d(x,b) = d(b,x) \geq d(p,b)$$

for all $x \in F$.
It follows that

$$d_1(*,p) = d(F,p) = d_1(p,b) \geq d_1(*,b) = d(F,b), d_1(b,a) = d(b,F)$$

or

$$d_1(p,b) = d_1(*,b) = d(F,b) = d_1(b,a) = d(b,F) \geq d_1(a,p) = d(F,p)$$

or

$$d_1(*,p) = d(F,p) = d_1(*,b) = d(F,b) = d_1(b,*) = d(b,F) \geq d_1(p,b)$$

The case $a \neq p \neq * = b$ can be done similarly.
Suppose $a \neq q(p) = * \neq b$, i.e., $p \in F$. Since (X, d) is KT_2 at p for all $p \in X$, in particular, for all $p \in F$ with $a \neq p \neq b$, by Theorem 6, we get

$$d(a,p) = d(p,b) \geq d(a,b), d(b,a)$$

or

$$d(p,b) = d(a,b) = d(b,a) \geq d(a,p)$$

or

$$d(a,p) = d(a,b) = d(b,a) \geq d(p,b)$$

Note that for all $p \in F$ with $a \neq p \neq b$,

$$d(a,F) = d(F,b) \geq d(a,b), d(b,a)$$

or

$$d(F,b) = d(a,b) = d(b,a) \geq d(a,F)$$

or

$$d(a,F) = d(a,b) = d(b,a) \geq d(F,b)$$

and consequently,

$$d_1(a,*) = d_1(*,b) \geq d_1(a,b), d_1(b,a)$$

or

$$d_1(*,b) = d_1(a,b) = d_1(b,a) \geq d_1(a,*)$$

or

$$d_1(a,*) = d_1(a,b) = d_1(b,a) \geq d_1(*,b)$$

Hence, by Theorem 6, $(X/F, d_1)$ is KT_2 at $q(p)$ for all $q(p) \in X/F$. □

Let $\mathcal{U} : \mathcal{E} \to \mathbf{Set}$ be topological. Recall, [24], that a full and isomorphism-closed subcategory $\mathcal{S}$ of $\mathcal{E}$ is
a) epireflective in $\mathcal{E}$ if and only if it is closed under the formation of products and extremal subobjects, i.e., subspaces;
b) bireflective in $\mathcal{E}$ if and only if it is epireflective and contains $\mathcal{I}$, the subcategory of all indiscrete objects.

Let $\mathbf{T_2}(\mathbf{pqsMet})$ be the full subcategory of $\mathbf{pqsMet}$ consisting of all local T_2 extended pseudo-quasi-semi metric spaces, i.e., T_2 extended pseudo-quasi-semi metric space at p for each $p \in X$, where T_2 is one of $\overline{T_2}$, T_2', LT_2, or KT_2.

Theorem 11. *(1) The full subcategory* $\mathbf{T_2}(\mathbf{pqsMet})$ *is epireflective in* $\mathbf{pqsMet}$*, where* T_2 *is one of* $\overline{T_2}$*,* T_2'*, or* LT_2*.*
(2) The full subcategory $\mathbf{KT_2}(\mathbf{pqsMet})$ *is bireflective in* $\mathbf{pqsMet}$*.*

Proof. (1) Note that the full subcategory $\mathbf{T_2}(\mathbf{pqsMet})$ is isomorphism-closed subcategory of $\mathbf{pqsMet}$, where T_2 is one of $\overline{T_2}$, T_2', LT_2 since if (X,d) is an extended pseudo-quasi-semi metric space, (Y,e) is local T_2 extended pseudo-quasi-semi metric space, and $f : (X,d) \to (Y,e)$ is an isomorphism, then it follows easily that $d(x,p) = e(f(x), f(p))$ for all $x, p \in X$, and consequently, by Theorem 6, (X,d) is a T_2 extended pseudo-quasi-semi metric space at p, where T_2 is one of $\overline{T_2}$, T_2', LT_2. By Theorems 6 and 7 of [14], the subcategory $\mathbf{T_2}(\mathbf{pqsMet})$ is closed under the formation of products and subspaces. Hence, $\mathbf{T_2}(\mathbf{pqsMet})$ is epireflective in $\mathbf{pqsMet}$.
(2) Note that the full subcategory $\mathbf{KT_2}(\mathbf{pqsMet})$ is isomorphism-closed. By Theorems 6 and 7 of [14] and Theorem 7, the full subcategory $\mathbf{KT_2}(\mathbf{pqsMet})$ is epireflective in $\mathbf{pqsMet}$. Finally, if (X,d) is the indiscrete extended pseudo-quasi-semi

metric space, then by Theorem 6 and Remark 8(3), (X,d) is KT_2 at p for each $p \in X$. Hence, $\mathbf{KT_2}(\mathbf{pqsMet})$ is bireflective in **pqsMet**. $\square$

Remark 12. *(1) By Theorem 11 and Proposition 21.37 of* [1], $\mathbf{KT_2}(\mathbf{pqsMet})$ *is a topological category. Moreover,* $\mathbf{KT_2}(\mathbf{pqsMet})$ *and* $\mathbf{Pre\overline{T}_2}(\mathbf{pqsMet})$ *are isomorphic.*
(2) By Theorems 6 and 11, the following categories are isomorphic.
(i) $\mathbf{\overline{T}_2}(\mathbf{pqsMet})$,
(ii) $\mathbf{T'_2}(\mathbf{pqsMet})$,
(iii) $\mathbf{LT_2}(\mathbf{pqsMet})$.
(3) By Theorem 2 and Remark 8, the following categories are isomorphic.
(i) $\mathbf{\overline{T}_2}(\mathbf{Top})$,
(ii) $\mathbf{T'_2}(\mathbf{Top})$,
(iii) $\mathbf{LT_2}(\mathbf{Top})$,
(iv) $\mathbf{KT_2}(\mathbf{Top})$.

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