(α, m1, m2)-CONVEXITY AND SOME INEQUALITIES OF
HERMITE-HADAMARD TYPE

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Abstract. In this paper, we introduce a new class of extended (α, m1, m2)-convex functions. Some algebraic properties of these class functions have been investigated. Some new Hermite-Hadamard type inequalities are derived. Results represent significant refinement and improvement of the previous results. Also, the author establish a new integral identity and, by this identity, Hölder’s and power mean inequality, discover some new Hermite-Hadamard type inequalities for functions whose first derivatives are (α, m1, m2)-convex. Our results are new and coincide with the previous results in special cases.

1. INTRODUCTION

Definition 1. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in [0, 1]$. If this inequality reverses, then the function $f$ is said to be concave on interval $I \neq \emptyset$.

This definition is well known in the literature. One of the most important integral inequalities for convex functions is the Hermite-Hadamard inequality. The following double inequality is well known as the Hermite-Hadamard inequality.

**Definition 2.** $f : [a, b] \to \mathbb{R}$ be a convex function, then the inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}$$

is known as the Hermite-Hadamard inequality.

Some refinements of the Hermite-Hadamard inequality on convex functions have been investigated by [3, 9, 14] and the Authors obtained a new refinement of the Hermite-Hadamard inequality for convex functions.

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Definition 3. The function \( f : [0, b] \to \mathbb{R} \), \( b > 0 \), is said to be \( m \)-convex function, where \( m \in [0, 1] \); if we have
\[
f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)
\]
for all \( x, y \in [0, b] \) and \( t \in [0, 1] \). We say that the function \( f \) is \( m \)-concave function if \((-f) \) is \( m \)-convex.

Obviously, for \( m = 1 \) the above definition recaptures the concept of standard convex functions on \([a, b] \); and for \( m = 0 \) the concept star-shaped functions.

The interested reader can find more about partial ordering of convexity in [13]. For many papers connected with \( m \)-convex and \((\alpha, m)\)-convex functions see (11, 14, 15) and the references therein. There are similar inequalities for \( \alpha \)-convex functions on \([0, 1] \) and \((\alpha, m)\)-convex functions in [7] and [14], respectively.

Definition 4. The function \( f : [0, b] \to \mathbb{R} \), \( b > 0 \) is sad to be \((\alpha, m)\)-convex, where \((\alpha, m) \in [0, 1]^2 \); if we have
\[
f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t)^\alpha f(y)
\]
for all \( x, y \in [0, b] \) and \( t \in [0, 1] \). Denote by \( K^\alpha_m(b) \) the class of all \((\alpha, m)\)-convex functions on \([0, b] \) for which \( f(0) \leq 0 \).

It can be easily seen that for \((\alpha, m) = (1, m) \), \((\alpha, m)\)-convexity reduces to \( m \)-convexity; \((\alpha, m) = (a, 1) \), \((\alpha, m)\)-convexity reduces to \( \alpha \)-convexity and for \((\alpha, m) = (1, 1) \), \((\alpha, m)\)-convexity reduces to the concept of usual convexity defined on \([0, b] \), \( b > 0 \).

Definition 5. The function \( f : [0, b] \to \mathbb{R} \), \( b > 0 \), is said to be \((m_1, m_2)\)-convex, if
\[
f(m_1tx + m_2(1-t)y) \leq m_1 tf(x) + m_2(1-t)f(y)
\]
for all \( x, y \in I \), \( t \in [0, 1] \) and \((m_1, m_2) \in (0, 1]^2 \).

Definition 6. Let \( f : [0, b] \to \mathbb{R} \). If \( f(tx) \leq tf(x) \) is valid for all \( x \in [0, b] \), then we say that \( f(x) \) is a starshaped function on \([0, b] \).

Definition 7. Let \( f : [0, b] \to \mathbb{R} \) and \( m_1 \in (0, 1] \). If \( f(m_1tx) \leq m_1 tf(x) \) is valid for all \( x \in [0, b] \) and \( t \in [0, 1] \), then we say that the function \( f(x) \) is a \( m_1 \)-starshaped function on \([0, b] \). Specially, for \( m_1 = 1 \), we have \( f(tx) \leq tf(x) \).

In [4], Kadakal proved the following theorem for \((m_1, m_2)\)-convex functions.

Theorem 8. Let the function \( f : [0, b^*] \to \mathbb{R} \), \( b^* > 0 \), be a \((m_1, m_2)\)-convex functions with \( m_1, m_2 \in (0, 1]^2 \). If \( 0 \leq a < b < b^* \) and \( f \in L[a, b] \), then the following inequalities holds:
\[
\frac{1}{b-a} \int_a^b f(x)dx \leq \min \left\{ \frac{m_1 f \left( \frac{a}{m_1} \right) + m_2 f \left( \frac{b}{m_2} \right)}{2}, \frac{m_1 f \left( \frac{b}{m_1} \right) + m_2 f \left( \frac{a}{m_2} \right)}{2} \right\}.
\]
2. \((\alpha, m_1, m_2)\)-CONVEX FUNCTIONS AND SOME PROPERTIES

In this section, we will begin by setting some algebraic properties for \((\alpha, m_1, m_2)\)-convex functions.

**Definition 9.** \(f : [0, b] \rightarrow \mathbb{R}, b > 0,\) is said to be \((\alpha, m_1, m_2)\)-convex function, if
\[
f(m_1tx + m_2(1 - t)y) \leq m_1t^\alpha f(x) + m_2(1 - t^\alpha)f(y)
\]
for all \(x, y \in I, t \in [0, 1]\) and \((\alpha, m_1, m_2) \in (0, 1]^3\).

We will denote by \(K^\alpha_{m_1,m_2}(b)\) the class of all \((\alpha, m_1, m_2)\)-convex functions on interval \(I\) for which \(f(0) \leq 0\). Also, we note that, for any \(t \in [0, 1]\) and \((\alpha, m_1, m_2) \in (0, 1]^3\), we have
\[
f(0) = f(m_1t^\alpha.0 + m_2(1 - t^\alpha).0) \\
\leq m_1t^\alpha f(0) + m_2(1 - t^\alpha)f(0) \\
0 \leq [m_1t^\alpha + m_2(1 - t^\alpha) - 1] f(0).
\]

Since \(m_1t^\alpha + m_2(1 - t^\alpha) - 1 \leq 0\), we get \(f(0) \leq 0\).

It can be easily seen that for \((\alpha, m_1, m_2) \in \{(0, 0, 0) , (\alpha, 1, 0) , (1, 1, 0) , (1, 1, m) , (1, 1, 1) , (\alpha, 1, 1) , (\alpha, 1, m)\}\) one obtains the following classes of functions: increasing, \(\alpha\)-starshaped, starshaped, \(m\)-convex, convex, \(\alpha\)-convex and \((\alpha, m)\)-convex functions respectively.

**Definition 10.** Let \(f : [0, b] \rightarrow \mathbb{R}\) and \((\alpha, m_1) \in (0, 1]^2\). If
\[
f(m_1tx) \leq m_1t^\alpha f(x)
\]
is valid for all \(x \in [0, b]\) and \(t \in [0, 1]\), then we say that \(f(x)\) is \((\alpha, m_1)\)-starshaped function on interval \([0, b]\). Specially, for \(m_1 = 1\) and \(\alpha = 1\), we have \(f(tx) \leq tf(x)\).

**Remark 11.** In Definition 9, if we choose \(m_2 = 0\), we get the concept of \((\alpha, m_1)\)-starshaped functions on interval \([0, b]\).

**Proposition 12.** If the function \(f\) is in the class \(K^\alpha_{m_1,m_2}(b)\), then it is \((\alpha, m_1)\)-starshaped.

**Proof.** For any \(x \in [0, b], t \in [0, 1]\) and \((\alpha, m_1, m_2) \in (0, 1]^3\), we have
\[
f(m_1tx) = f(m_1tx + m_2(1 - t)x) \\
\leq m_1t^\alpha f(x) + m_2(1 - t^\alpha)f(0) \\
\leq m_1t^\alpha f(x).
\]
Specially, for \(m_1 = 1\), we have \(f(tx) \leq t^\alpha f(x)\). \(\square\)

**Theorem 13.** Let \(f, g : [0, b] \rightarrow \mathbb{R}\). If \(f\) and \(g\) are \((\alpha, m_1, m_2)\)-convex, then

(i) \(f + g\) is \((\alpha, m_1, m_2)\)-convex,
(ii) For \(c \in \mathbb{R} (c \geq 0)\) \(cf\) is \((\alpha, m_1, m_2)\)-convex.
Theorem 14. Let $f$ be a $(\alpha, m_1, m_2)$-convex function. If the function $g$ is a $(\alpha, m_1, m_2)$-convex and increasing, then the function $gof$ is a $(\alpha, m_1, m_2)$-convex.

Proof. (i) For $x, y \in I$ and $t \in [0, 1]$, we have
\[
(f + g) (m_1tx + m_2(1-t)y) = f (m_1tx + m_2(1-t)y) + g (m_1tx + m_2(1-t)y) \
\leq m_1f(x) + m_2(1-t^\alpha)f(y) + m_1t^\alpha g(x) + m_2(1-t^\alpha)g(y) \
\leq m_1t^\alpha (f + g) (x) + m_2(1-t^\alpha)(f + g)(y).
\]

(ii) For $c \in \mathbb{R}$ ($c \geq 0$), we obtain
\[
(cf) (m_1tx + m_2(1-t)y) \leq c \left[ m_1t^\alpha f(x) + m_2(1-t^\alpha)f(y) \right] \
\leq m_1t^\alpha(cf)(x) + m_2(1-t^\alpha)(cf)(y).
\]
This completes the proof of theorem. \qed

Theorem 15. Let $f, g : I \rightarrow \mathbb{R}$ be both nonnegative and monotone increasing. If $f$ and $g$ are $(\alpha, m_1, m_2)$-convex functions, then $fg$ is a $(\alpha, m_1, m_2)$-convex.

Proof. If $x \leq y$ (the case $y \leq x$ is similar) then $[f(x) - f(y)] [g(y) - g(x)] \leq 0$ which implies
\[
f(x)g(y) + f(y)g(x) - f(x)g(y) - f(y)g(x) = 0 \leq f(x)g(x) + f(y)g(y).
\]
On the other hand for $x, y \in I$ and $t \in [0, 1]$, \[
(fg) (m_1tx + m_2(1-t)y) = f (m_1tx + m_2(1-t)y) g (m_1tx + m_2(1-t)y) \
\leq [m_1t^\alpha f(x) + m_2(1-t^\alpha)f(y)] [m_1t^\alpha g(x) + m_2(1-t^\alpha)g(y)] \
= m_1m_2t^{2\alpha} f(x)g(x) + m_1m_2t^\alpha (1-t^\alpha) f(x)g(y) + m_2m_1t^\alpha (1-t^\alpha) f(y)g(x) \
+ m_2m_2(1-t^\alpha)^2 f(y)g(y) \
= m_1m_2t^{2\alpha} f(x)g(x) + m_1m_2t^\alpha (1-t^\alpha) [f(x)g(y) + f(y)g(x)] + m_2(1-t^\alpha)^2 f(y)g(y).
\]
Using now (2.1), we obtain,
\[
(fg) (m_1tx + m_2(1-t)y) \leq m_1m_2t^{2\alpha} f(x)g(x) + m_1m_2t^\alpha (1-t^\alpha) [f(x)g(y) + f(y)g(x)] \
+ m_2(1-t^\alpha)^2 f(y)g(y)
\]
\[ m_1 t^\alpha [m_1 t^\alpha + m_2 (1 - t^\alpha)] f(x) g(x) + m_2 (1 - t^\alpha) [m_1 t^\alpha + m_2 (1 - t^\alpha)] f(y) g(y). \]

Since \( m_1 t^\alpha + m_2 (1 - t^\alpha) \leq m \leq 1 \), where \( m = \max \{m_1, m_2\} \). Therefore, we get
\[
(fg) (m_1 t x + m_2 (1 - t) y) \leq m_1 t^\alpha f(x) g(x) + m_2 (1 - t^\alpha) f(y) g(y)
= m_1 t^\alpha (fg) (x) + m_2 (1 - t^\alpha) (fg) (y).
\]

This completes the proof of theorem. \( \square \)

**Theorem 16.** Let \( f : [0, b^*] \to \mathbb{R} \) a finite function on \( \frac{a}{m_1}, \frac{b}{m_2} \in [0, b^*] \), \((\alpha, m_1, m_2)\)-convex with \( \alpha, m_1, m_2 \in (0, 1] \). Then \( f \) is on bounded any closed interval \([a, b]\).

**Proof.** Let
\[
M = \max \left\{ m_1 f \left( \frac{a}{m_1} \right), m_2 f \left( \frac{b}{m_2} \right), m_2 f \left( \frac{a}{m_1} \right), m_1 f \left( \frac{b}{m_1} \right) \right\},
\]
so for any \( z = ta + (1 - t)b \) in interval \([a, b]\), we get
\[
f(z) = f(ta + (1 - t)b)
= f \left( m_1 t \frac{a}{m_1} + m_2 (1 - t) \frac{b}{m_2} \right)
\leq m_1 t^\alpha f \left( \frac{a}{m_1} \right) + m_2 (1 - t^\alpha) f \left( \frac{b}{m_2} \right)
\leq M.
\]

Thus, the function \( f \) is upper bounded in interval \([a, b]\).

Now we notice that any \( z \in [a, b] \) can be written as \( \frac{a + b}{2} + t \) for \( |t| \leq \frac{b - a}{2} \), hence
\[
f \left( \frac{a + b}{2} \right) = f \left( \frac{1}{2} \left( \frac{a + b}{2} + t \right) + \frac{1}{2} \left( \frac{a + b}{2} - t \right) \right)
\leq m_1 \frac{1}{2^\alpha} f \left( \frac{a + b}{m_1} + t \right) + m_2 \left( 1 - \frac{1}{2^\alpha} \right) f \left( \frac{b - t}{m_2} \right).
\]

In other word, we get
\[
f \left( \frac{a + b}{m_1} \right) \geq \frac{2^\alpha}{m_1} f \left( \frac{a + b}{2} \right) - \frac{2^\alpha}{m_1} m_2 \left( 1 - \frac{1}{2^\alpha} \right) f \left( \frac{a + b - t}{m_2} \right)
\geq \frac{2^\alpha}{m_1} f \left( \frac{a + b}{2} \right) - \frac{2^\alpha}{m_1} \left( 1 - \frac{1}{2^\alpha} \right) M
= \frac{2^\alpha}{m_1} f \left( \frac{a + b}{2} \right) - \frac{2^\alpha - 1}{m_1} M
\]
and similarly,
\[
f \left( \frac{a + b}{2} \right) = f \left( \frac{1}{2} \left( \frac{a + b}{2} + t \right) + \frac{1}{2} \left( \frac{a + b}{2} - t \right) \right)
\]
\[
\leq m_2 \frac{1}{2^\alpha} f \left( \frac{a + b + t}{m_2} \right) + m_1 \left( 1 - \frac{1}{2^\alpha} \right) f \left( \frac{a + b - t}{m_1} \right),
\]

hence, we obtain

\[
\begin{align*}
f \left( \frac{a + b + t}{m_2} \right) & \geq \frac{2^\alpha}{m_2} f \left( \frac{a + b}{2} \right) - \frac{2^\alpha}{m_2} m_1 \left( 1 - \frac{1}{2^\alpha} \right) f \left( \frac{a + b - t}{m_1} \right) \\
& \geq \frac{2^\alpha}{m_1} f \left( \frac{a + b}{2} \right) - \frac{2^\alpha}{m_1} \left( 1 - \frac{1}{2^\alpha} \right) M \\
& = \frac{2^\alpha}{m_1} f \left( \frac{a + b}{2} \right) - \frac{2^\alpha - 1}{m_1} M,
\end{align*}
\]

and since \( \frac{a + b}{2} + t \) is arbitrary in \([a, b]\), the function \( f \) is also bounded below in \([a, b]\).

This completes the proof of theorem. \( \square \)

**Theorem 17.** Let \( \alpha, m_1, m_2 \in [0, 1] \), \( b > 0 \) and \( f_\beta : [0, b] \to \mathbb{R} \) be an arbitrary family of \((\alpha, m_1, m_2)\)-convex functions and let \( f(x) = \sup_\beta f_\beta (x) \). If

\[ J = \left\{ u \in [0, b] : \frac{u}{m_1}, \frac{u}{m_2} \in [0, b] \text{ and } f(u), f \left( \frac{u}{m_1} \right), f \left( \frac{u}{m_2} \right) < \infty \right\} \]

is nonempty, then \( J \) is an interval and \( f \) is \((\alpha, m_1, m_2)\)-convex on \( J \).

**Proof.** Let \( t \in [0, 1] \) and \( x, y \in J \) be arbitrary. Then

\[
f (tx + (1 - t)y) = \sup_\beta f_\beta (x) \left( m_1 t \frac{x}{m_1} + m_2 (1 - t) \frac{y}{m_2} \right) \\
\leq \sup_\beta \left[ m_1 t^\alpha f_\beta \left( \frac{x}{m_1} \right) + m_2 (1 - t^\alpha) f_\beta \left( \frac{y}{m_2} \right) \right] \\
\leq m_1 t^\alpha \sup_\beta f_\beta \left( \frac{x}{m_1} \right) + m_2 (1 - t^\alpha) \sup_\beta f_\beta \left( \frac{y}{m_2} \right) \\
= m_1 t^\alpha f \left( \frac{x}{m_1} \right) + m_2 (1 - t^\alpha) f \left( \frac{y}{m_2} \right) < \infty.
\]

This shows simultaneously that \( J \) is an interval since it contains every point between any two of its points.

Now, we show that the function \( f \) is \((m_1, m_2)\)-convex on \( J \): If \( t \in [0, 1] \) and \( x, y \in J \), then

\[
f (m_1 tx + m_2 (1 - t)y) = \sup_\beta f_\beta (m_1 tx + m_2 (1 - t)y) \\
\leq \sup_\beta \left[ m_1 t^\alpha f_\beta (x) + m_2 (1 - t^\alpha) f_\beta (y) \right] \\
\leq m_1 t^\alpha \sup_\beta f_\beta (x) + m_2 (1 - t^\alpha) \sup_\beta f_\beta (y) \\
= m_1 t^\alpha f (x) + m_2 (1 - t^\alpha) f (y)
\]
and that the function $f$ is $(\alpha,m_1,m_2)$-convex on $J$.
This completes the proof of theorem. 

3. HERMITE-HADAMARD INEQUALITY FOR $(\alpha,m_1,m_2)$-CONVEX FUNCTIONS

The goal of this paper is to develop concepts of the $(\alpha,m_1,m_2)$-convex functions and to establish some inequalities of Hermite-Hadamard type for these classes of functions.

Theorem 18. Let $f : I \to \mathbb{R}$ be a $(\alpha,m_1,m_2)$-convex function with $(\alpha,m_1,m_2) \in (0,1]^3$. If $0 \leq a < b < \infty$ and $f \in L[a,b]$, then the following inequalities hold:

i. $f\left(\frac{a+b}{2}\right) \leq \frac{b-a}{2}\int_{a}^{b} f(m_1x) \, dx + \left(1 - \frac{1}{2^{\alpha}}\right) m_2^2 \int_{a}^{b} f(m_2y) \, dy$

ii. $\frac{1}{b-a} \int_{a}^{b} f(x) \, dx \leq \min\left\{\frac{m_1f\left(\frac{a}{m_1}\right) + \alpha m_2f\left(\frac{b}{m_2}\right)}{\alpha + 1}, \frac{m_1f\left(\frac{a}{m_1}\right) + \alpha m_2f\left(\frac{b}{m_2}\right)}{\alpha + 1}\right\}$

Proof. i. By the $(\alpha,m_1,m_2)$-convexity of the function $f$, we have

$$f\left(\frac{a+b}{2}\right) = f\left(\frac{m_1 t a m_1 + m_2 (1-t) b m_2}{2}\right)$$

$$= f\left(\frac{1}{2} m_1 \left[a m_1 + m_2 (1-t) b m_2\right] + \left[m_1 (1-t) a m_1 + m_2 t b m_2\right]\right)$$

$$\leq \frac{1}{2^\alpha} m_1 f\left(t \frac{a}{m_1} + m_2 (1-t) b \frac{m_2}{m_1}\right)$$

$$+ \left(1 - \frac{1}{2^\alpha}\right) m_2 f\left(t \frac{b}{m_2} + \frac{m_1}{m_2} (1-t) a \frac{1}{m_1}\right).$$

Now, if we take integral the last inequality on $t \in [0,1]$ and choose $m_1 x = ta + (1-t)b$ and $m_2 y = tb + (1-t)a$, we deduce

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2^\alpha} \frac{m_1^2}{b-a} \int_{a}^{b} f(m_1 x) \, dx + \left(1 - \frac{1}{2^\alpha}\right) \frac{m_2^2}{b-a} \int_{a}^{b} f(m_2 y) \, dy.$$

ii. By using the $(\alpha,m_1,m_2)$-convexity of the function $f$, if the variable is changed as $u = ta + (1-t)b$

$$\int_{0}^{1} f(ta + (1-t)b) \, dt = \frac{1}{b-a} \int_{0}^{1} f(u) \, du$$

$$\leq \int_{0}^{1} \left[t^\alpha m_1 f\left(\frac{a}{m_1}\right) + (1-t^\alpha) m_2 f\left(\frac{b}{m_2}\right)\right] \, dt$$

$$= \frac{m_1 f\left(\frac{a}{m_1}\right) + \alpha m_2 f\left(\frac{b}{m_2}\right)}{\alpha + 1}.$$
and similarly for $z = tb + (1 - t)a$, then
\[
\int_0^1 f(tb + (1 - t)a) \, dt = \frac{1}{b-a} \int_0^1 f(z) \, dz \\
\leq \int_0^1 \left[ t^\alpha m_1 f\left(\frac{b}{m_1}\right) + (1 - t^\alpha) m_2 f\left(\frac{a}{m_2}\right) \right] \, dt \\
= \frac{m_1 f\left(\frac{b}{m_1}\right) + \alpha m_2 f\left(\frac{a}{m_2}\right)}{\alpha + 1}.
\]

So, we have
\[
\frac{1}{b-a} \int_a^b f(x) \, dx \leq \min \left\{ \frac{m_1 f\left(\frac{a}{m_1}\right) + \alpha m_2 f\left(\frac{b}{m_2}\right)}{\alpha + 1}, \frac{m_1 f\left(\frac{b}{m_1}\right) + \alpha m_2 f\left(\frac{a}{m_2}\right)}{\alpha + 1} \right\}.
\]

This completes the proof of theorem. \qedsymbol

**Remark 19.** Under the conditions of Theorem 18, if $m_1 = 1$, $m_2 = m$, then, the following inequality holds:
\[
\frac{1}{b-a} \int_a^b f(x) \, dx \leq \min \left\{ \frac{f(a) + mf\left(\frac{b}{m}\right)}{2}, \frac{mf\left(\frac{b}{m}\right) + f(a)}{2} \right\}.
\]

This inequality is the Hermite-Hadamard inequality for the $m$-convex functions \[15\].

**Remark 20.** Under the conditions of Theorem 18

i) If $\alpha = m_1 = m_2 = 1$, then, the following inequality holds:
\[
\frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}.
\]

This inequality is the Hermite-Hadamard inequality for the convex functions \[5\].

(ii) If $\alpha = m_1 = 1$, $m_2 = m$, then, the following inequality holds:
\[
\frac{1}{b-a} \int_a^b f(x) \, dx \leq \min \left\{ \frac{f(a) + mf\left(\frac{b}{m}\right)}{2}, \frac{f(b) + mf\left(\frac{a}{m}\right)}{2} \right\}.
\]

This inequality is the Hermite-Hadamard inequality for the $m$-convex functions \[2\].

(iii) If $\alpha = s$, $m_1 = m_2 = 1$, then, the following inequality holds:
\[
f\left(\frac{a + b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \min \left\{ \frac{f(a) + sf\left(\frac{b}{s}\right)}{s + 1}, \frac{f(b) + sf\left(\frac{a}{s}\right)}{s + 1} \right\}.
\]

This inequality is the Hermite-Hadamard inequality for the $s$-convex functions in the first sense \[10\].

(iv) If $m_1 = 1$, $m_2 = m$, then, the following inequality holds:
\[
\frac{1}{b-a} \int_a^b f(x) \, dx \leq \min \left\{ \frac{f(a) + mf\left(\frac{b}{m}\right)}{\alpha + 1}, \frac{f(b) + mf\left(\frac{a}{m}\right)}{\alpha + 1} \right\}.
\]
This is the Hermite-Hadamard inequality for the \((\alpha, m)\)-convex functions \([12]\).

**Theorem 21.** Let the function \(f : [0, b] \to \mathbb{R}, \ b > 0, be \ (\alpha, m_1, m_2)\)-convex functions with \(\alpha, m_1, m_2 \in (0, 1]^3\). If \(m = \min \{m_1, m_2\}, 0 < a < b < \frac{b}{m} < b^*\) and \(f \in L [a, \frac{b}{m}]\), then the following inequalities holds:

\[
f\left(\frac{a + b}{2}\right) \leq \frac{1}{2^m} \int_a^b f\left(\frac{x}{m_1}\right) dx + \left(1 - \frac{1}{2^m}\right) \frac{m_2}{b - a} \int_a^b f\left(\frac{y}{m_2}\right) dy
\]

**Proof.** By the \((\alpha, m_1, m_2)\)-convexity of the function \(f\), we have

\[
f\left(\frac{a + b}{2}\right) = f\left(\frac{m_1 \frac{a}{m_1} + m_2 (1 - t) \frac{b}{m_2}}{2}\right)
\]

\[
= f\left(\frac{1}{2} m_1 \left(\frac{a}{m_1} + \frac{m_2 (1 - t) b}{m_2}\right) + \frac{1}{2} m_2 \left(\frac{b}{m_2} + \frac{m_1 (1 - t) a}{m_1}\right)\right)
\]

\[
\leq \frac{1}{2^m} \int_a^b f\left(\frac{ta + (1 - t)b}{m_1}\right) + \left(1 - \frac{1}{2^m}\right) \frac{m_2}{b - a} \int_a^b f\left(\frac{y}{m_2}\right) dy
\]

Now, if we take integral the last inequality on \(t \in [0, 1]\) and choose \(x = ta + (1 - t)b\) and \(y = tb + (1 - t)a\), we deduce

\[
f\left(\frac{a + b}{2}\right) \leq \frac{1}{2^m} \int_a^b f\left(\frac{x}{m_1}\right) dx + \left(1 - \frac{1}{2^m}\right) \frac{m_2}{b - a} \int_a^b f\left(\frac{y}{m_2}\right) dy
\]

This completes the proof of theorem. \(\Box\)

**Remark 22.** Under the conditions of Theorem 21, if \(\alpha = m_1 = m_2 = 1\), then, the following inequality holds:

\[
f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(x) dx
\]

This inequality is the left hand side of Hermite-Hadamard inequality for the convex functions \([13]\).

4. SOME NEW INEQUALITIES FOR \((\alpha, m_1, m_2)\)-CONVEXITY

In \([6]\), Kirmaci used the following lemma to prove Theorems.

**Lemma 23.** Let \(f : I^* \subset \mathbb{R} \to \mathbb{R}\) be a differentiable mapping on \(I^*\), \(a, b \in I^*\) (\(I^*\) is the interior of \(I\)) with \(a < b\). If \(f' \in L [a, b]\), then we have

\[
\frac{1}{b - a} \int_a^b f(x) dx - f\left(\frac{a + b}{2}\right)
\]

\[
= (b - a) \left[\int_0^1 t f' (ta + (1 - t)b) dt + \int_1^0 (t - 1) f' (ta + (1 - t)b) dt\right]
\]
The main purpose of this section is to establish new estimations and refinements of the Hermite-Hadamard inequality for functions whose first derivatives in absolute value are \((\alpha, m_1, m_2)\)-convex. For this, we will use the following lemma.

**Lemma 24.** Let \(f : I^0 \subseteq \mathbb{R} \to \mathbb{R}\) be a differentiable on \(I^0\), \(m_1a, m_2b \in I^0\) with \(m_1a < m_2b\). If \(f' \in L [m_1a, m_2b]\), then the following equality

\[
\frac{1}{m_2b - m_1a} \int_{m_1a}^{m_2b} f(x) dx - f \left( \frac{m_1a + m_2b}{2} \right)
= (m_2b - m_1a) \left[ \int_0^{\frac{1}{2}} tf'(m_1ta + m_2(1-t)b) dt + \int_{\frac{1}{2}}^1 (t-1)f'(m_1ta + m_2(1-t)b) dt \right]
\]

holds for \(t \in [0, 1]\) and \(m_1, m_2 \in (0, 1]^2\).

**Proof.** By integration by parts and then by changing of variable \(x = m_1ta + m_2(1-t)b\), we get

\[
\int_0^{\frac{1}{2}} tf'(m_1ta + m_2(1-t)b) dt + \int_{\frac{1}{2}}^1 (t-1)f'(m_1ta + m_2(1-t)b) dt
= - f \left( \frac{m_1ta + m_2(1-t)b}{m_2b - m_1a} \right) \bigg|_0^{\frac{1}{2}} + \frac{1}{m_2b - m_1a} \int_0^{\frac{1}{2}} f \left( \frac{m_1ta + m_2(1-t)b}{2} \right) dt
- \frac{f \left( \frac{m_1ta + m_2(1-t)b}{m_2b - m_1a} \right) (t-1) \bigg|_\frac{1}{2}^1 + \frac{1}{m_2b - m_1a} \int_{\frac{1}{2}}^1 f \left( \frac{m_1ta + m_2(1-t)b}{2} \right) dt
= \frac{1}{(m_2b - m_1a)^2} \int_{m_1a}^{m_2b} f(x) dx - \frac{1}{m_2b - m_1a} f \left( \frac{m_1a + m_2b}{2} \right).
\]

Thus, the proof of lemma is completed. \(\square\)

**Theorem 25.** Let \(f : I^0 \subseteq \mathbb{R} \to \mathbb{R}\) be a differentiable mapping on \(I^0\), \(m_1a, m_2b \in I^0\) with \(m_1a < m_2b\) and \(f' \in L [m_1a, m_2b]\). If \(|f'|\) is \((\alpha, m_1, m_2)\)-convex on the interval \([m_1a, m_2b]\), then the following equality

\[
\frac{1}{m_2b - m_1a} \int_{m_1a}^{m_2b} f(x) dx - f \left( \frac{m_1a + m_2b}{2} \right)
\leq (m_2b - m_1a) \left\{ \left[ \frac{1}{(\alpha + 1)(\alpha + 2)} \left( 1 - \frac{1}{2\alpha + 1} \right) \right] m_1 |f'(a)|
+ \left[ \frac{1}{4} + \frac{1}{(\alpha + 1)(\alpha + 2)} \left( \frac{1}{2\alpha + 1} - 1 \right) \right] m_2 |f'(b)| \right\}
\]

holds for \(t \in [0, 1]\) and \(\alpha, m_1, m_2 \in (0, 1]^3\).
\textbf{Proof.} Using Lemma 24 and the inequality

\[ |f'(m_1ta + m_2(1-t)b)| \leq m_1 t^\alpha |f'(a)| + m_2 (1 - t^\alpha) |f'(b)| \]

we get

\[
\left| \frac{1}{m_2 b - m_1 a} \int_{m_1 a}^{m_2 b} f(x) dx - f \left( \frac{m_1 a + m_2 b}{2} \right) \right|
\leq (m_2 b - m_1 a)
\times \left[ \int_0^{\frac{1}{2}} tf' (m_1 ta + m_2 (1-t)b) dt + \int_{\frac{1}{2}}^1 (t - 1)f' (m_1 ta + m_2 (1-t)b) dt \right]
\leq (m_2 b - m_1 a)
\left[ \int_0^{\frac{1}{2}} |t| (m_1 t^\alpha |f'(a)| + m_2 (1 - t^\alpha) |f'(b)|) dt + \int_{\frac{1}{2}}^1 (m_1 t^{\alpha+1} |f'(a)| + m_2 (t - t^{\alpha+1}) |f'(b)|) dt \right]
\]

\[
= (m_2 b - m_1 a)
\left[ \frac{m_1 |f'(a)|}{\alpha + 2} + \frac{1}{2} - \frac{1}{(\alpha + 2)(\alpha + 3)2^{\alpha+2}} \right] m_2 |f'(b)|
+ \left( \frac{1}{\alpha + 1} \frac{1}{\alpha + 2} - \frac{\alpha + 3}{\alpha + 1} \frac{1}{\alpha + 2} \right) m_1 |f'(a)|
\left[ \frac{1}{\alpha + 1} \frac{1}{\alpha + 2} \left( 1 - \frac{1}{2^{\alpha+1}} \right) \right] m_1 |f'(a)|
\left[ \frac{1}{\alpha + 1} \frac{1}{\alpha + 2} \left( \frac{1}{2^{\alpha+1}} - 1 \right) \right] m_2 |f'(b)|
\]

where

\[
\int_0^{\frac{1}{2}} t^{\alpha+1} dt = \frac{1}{(\alpha + 2)2^{\alpha+2}},
\int_0^{\frac{1}{2}} (t - t^{\alpha+1}) dt = \frac{1}{8} - \frac{1}{(\alpha + 2)2^{\alpha+2}},
\]
\[
\int_{1/2}^1 (t^\alpha - t^{\alpha+1}) \, dt = \frac{1}{(\alpha + 1)(\alpha + 2)} - \frac{\alpha + 3}{(\alpha + 1)(\alpha + 2)2^{\alpha+2}},
\]
\[
\int_{1/2}^1 (1-t)(1-t^\alpha) \, dt = \frac{1}{8} - \frac{1}{(\alpha + 1)(\alpha + 2)} + \frac{\alpha + 3}{(\alpha + 1)(\alpha + 2)2^{\alpha+2}}.
\]

This completes the proof of theorem. \(\square\)

**Remark 26.** Under the conditions of Theorem 25 if we take \(\alpha = m_1 = m_2 = 1\), then our result coincides with [6].

**Theorem 27.** Let \(f : I^o \subseteq \mathbb{R} \rightarrow \mathbb{R}\) be a differentiable mapping on \(I^o\), \(m_1a, m_2b \in I^o\) with \(m_1a < m_2b\) and \(f' \in L[m_1a, m_2b]\), and let \(q > 1\). If \(|f'|\) is \((\alpha, m_1, m_2)\)-convex on the interval \([m_1a, m_2b]\), then the following equality
\[
\left| \frac{1}{m_2b - m_1a} \int_{m_1a}^{m_2b} f(x)dx - f \left( \frac{m_1a + m_2b}{2} \right) \right|
\leq (m_2b - m_1a) \left( \frac{1}{(p+1)2^{p+1}} \right)^{\frac{1}{p}} \left\{ \left[ \frac{m_1 |f'(a)|^q - m_2 |f'(b)|^q}{(\alpha + 1)2^{\alpha+1}} + m_2 |f'(b)|^q \right]^{\frac{1}{q}} \right\}
\]
holds for \(t \in [0, 1]\) and \(\alpha, m_1, m_2 \in (0, 1]^3\), where \(\frac{1}{p} + \frac{1}{q} = 1\).

**Proof.** Using Lemma 24 Hölder’s integral inequality and the inequality
\[
|f'(m_1ta + m_2(1-t)b)|^q \leq m_1 t^\alpha |f'(a)|^q + m_2 (1-t^\alpha) |f'(b)|^q,
\]
we obtain
\[
\left| \frac{1}{m_2b - m_1a} \int_{m_1a}^{m_2b} f(x)dx - f \left( \frac{m_1a + m_2b}{2} \right) \right|
\leq (m_2b - m_1a) \left[ \int_0^{1/2} |t| |f'(m_1ta + m_2(1-t)b)| \, dt 
\right]
+ \int_{1/2}^1 |t - 1| |f'(m_1ta + m_2(1-t)b)| \, dt 
\leq (m_2b - m_1a) \left[ \left( \int_0^{1/2} t^\alpha \, dt \right)^{\frac{1}{p}} \left( \int_0^{1/2} |f'(m_1ta + m_2(1-t)b)|^q \, dt \right)^{\frac{1}{q}} 
\right]
+ \left( \int_{1/2}^1 |t - 1|^p \, dt \right)^{\frac{1}{p}} \left( \int_{1/2}^1 |m_1 t^\alpha f'(a) + m_2 (1-t^\alpha) f'(b)|^q \, dt \right)^{\frac{1}{q}}
\[ \left( m_2 b - m_1 a \right) \times \left[ \left( \int_0^{\frac{1}{2}} t^p \, dt \right)^{\frac{1}{p}} \left( \int_0^{\frac{1}{2}} \left( m_1 t^\alpha \left| f'(a) \right|^q + m_2 (1 - t^\alpha) \left| f'(b) \right|^q \right) \, dt \right)^{\frac{1}{q}} \\
+ \left( \int_{\frac{1}{2}}^1 |t - 1|^p \, dt \right)^{\frac{1}{p}} \left( \int_{\frac{1}{2}}^1 \left( m_1 t^\alpha \left| f'(a) \right|^q + m_2 (1 - t^\alpha) \left| f'(b) \right|^q \right) \, dt \right)^{\frac{1}{q}} \right] \]

\[ = \left( m_2 b - m_1 a \right) \left( \frac{1}{(\alpha + 1)2^{\alpha+1}} \right)^{\frac{1}{2}} \left\{ \frac{\left| m_1 \left| f'(a) \right|^q - m_2 \left| f'(b) \right|^q \right|}{(\alpha + 1)2^{\alpha+1}} + \frac{m_2 \left| f'(b) \right|^q}{2} \right\}^{\frac{1}{2}} \]

where

\[ \int_0^{\frac{1}{2}} t^p \, dt = \int_{\frac{1}{2}}^1 |t - 1|^p \, dt = \frac{1}{(p + 1)2^{p+1}} \]
\[ \int_0^{\frac{1}{2}} t^\alpha \, dt = \frac{1}{(\alpha + 1)2^{\alpha+1}}, \quad \int_0^{\frac{1}{2}} (1 - t^\alpha) \, dt = \frac{1}{2} - \frac{1}{(\alpha + 1)2^{\alpha+1}}, \]
\[ \int_{\frac{1}{2}}^1 t^\alpha \, dt = \frac{1}{(\alpha + 1)2^{\alpha+1}}, \quad \int_{\frac{1}{2}}^1 (1 - t^\alpha) \, dt = \frac{1}{2} - \frac{1}{(\alpha + 1)2^{\alpha+1}} \]

Remark 28. Under the conditions of Theorem 25, if we take \( \alpha = m_1 = m_2 = 1 \), then our result coincides with 6.

Theorem 29. Let \( f : I^0 \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a differentiable mapping on \( I^0 \), \( m_1 a, m_2 b \in I^0 \) with \( m_1 a < m_2 b \) and \( f' \in L[m_1 a, m_2 b] \), and let \( q \geq 1 \). If \( f' \) is \( (\alpha, m_1, m_2) \)-convex on the interval \([m_1 a, m_2 b] \), then the following equality

\[ \left| \frac{1}{m_2 b - m_1 a} \int_{m_1 a}^{m_2 b} f(x) \, dx - f \left( \frac{m_1 a + m_2 b}{2} \right) \right| \]

\[ \leq \left( m_2 b - m_1 a \right) 2^{\frac{3}{p} - 3} \left[ \left( \frac{m_1 \left| f'(a) \right|^q - m_2 \left| f'(b) \right|^q}{(\alpha + 2)2^{\alpha+2}} + \frac{m_2 \left| f'(b) \right|^q}{8} \right)^{\frac{1}{q}} \right. \]

\[ + \left. \left( \frac{m_1 \left| f'(a) \right|^q - m_2 \left| f'(b) \right|^q}{(\alpha + 1)2^{\alpha+2}} + \frac{m_2 \left| f'(b) \right|^q}{8} \right)^{\frac{1}{q}} \right] \]

holds for \( t \in [0, 1] \) and \( \alpha, m_1, m_2 \in (0, 1]^3 \), where \( \frac{1}{p} + \frac{1}{q} = 1 \).
Proof. Using Lemma 24, well-known power mean inequality and the inequality

\[ |f'(m_1ta + m_2(1-t)b)|^q \leq m_1t^\alpha |f'(a)|^q + m_2(1-t^\alpha)|f'(b)|^q, \]

we get

\[
\left| \frac{1}{m_2 b - m_1 a} \int_{m_1 a}^{m_2 b} f(x) dx - f \left( \frac{m_1 a + m_2 b}{2} \right) \right|
\]

\[
\leq (m_2 b - m_1 a) \left[ \int_0^{\frac{1}{2}} |t| |f'(m_1 ta + m_2 (1-t)b)| dt + \int_{\frac{1}{2}}^1 |t-1| |f'(m_1 ta + m_2 (1-t)b)| dt \right]
\]

\[
\leq (m_2 b - m_1 a) \left[ \left( \int_0^{\frac{1}{2}} |t| dt \right)^{\frac{1}{2}} \left( \int_0^{\frac{1}{2}} |t| |f'(m_1 ta + m_2 (1-t)b)|^q dt \right)^{\frac{1}{q}} \right.
\]

\[
+ \left( \int_{\frac{1}{2}}^1 |t-1| dt \right)^{\frac{1}{q}} \left( \int_{\frac{1}{2}}^1 |t-1| |m_1 t^\alpha f'(a) + m_2 (1-t^\alpha)f'(b)|^q dt \right)^{\frac{1}{q}}
\]

\[
\leq (m_2 b - m_1 a) \left[ \left( \int_0^{\frac{1}{2}} t dt \right)^{\frac{1}{2}} \left( \int_0^{\frac{1}{2}} t |m_1 t^\alpha f'(a)|^q + m_2 (1-t^\alpha)f'(b)|^q | dt \right)^{\frac{1}{2}} \right.
\]

\[
+ \left( \int_{\frac{1}{2}}^1 |t-1| dt \right)^{\frac{1}{q}} \left( \int_{\frac{1}{2}}^1 |t-1| |m_1 t^\alpha f'(a)|^q + m_2 (1-t^\alpha)f'(b)|^q | dt \right)^{\frac{1}{q}}
\]

\[
= (m_2 b - m_1 a) \left[ \frac{1}{2} \right]^{\frac{1}{q}} \left( m_1 |f'(a)|^q \int_0^{\frac{1}{2}} t^{\alpha+1} dt + m_2 |f'(b)|^q \int_{\frac{1}{2}}^1 (t - t^{\alpha+1}) dt \right)^{\frac{1}{q}}
\]

\[
+ \left( \frac{1}{2} \right)^{\frac{1}{q}} \left( m_1 |f'(a)|^q \int_{\frac{1}{2}}^1 (1-t)t^\alpha dt + m_2 |f'(b)|^q \int_{\frac{1}{2}}^1 (1-t)(1-t^\alpha) dt \right)^{\frac{1}{q}}
\]

\[
= (m_2 b - m_1 a) 2^{\frac{3}{q} - 3} \left[ \left( \frac{m_1 |f'(a)|^q - m_2 |f'(b)|^q}{(\alpha + 2)^{2\alpha+2}} + \frac{m_2 |f'(b)|^q}{8} \right)^{\frac{1}{2}} \right]
\]

\[
+ \left( \frac{m_1 |f'(a)|^q - m_2 |f'(b)|^q}{(\alpha + 1)(\alpha + 2)} \left( 1 - \frac{\alpha + 3}{2^{\alpha+2}} \right) + \frac{m_2 |f'(b)|^q}{8} \right)^{\frac{1}{q}}
\]

where integrals can be calculated as above. \hfill \square
Corollary 30. Under the conditions of Theorem 29, if we take $q = 1$, then the following inequality holds:

$$\left| \frac{1}{m_2b - m_1a} \int_{m_1a}^{m_2b} f(x)dx - f\left( \frac{m_1a + m_2b}{2} \right) \right| \leq (m_2b - m_1a) \left[ \frac{m_1|f'(a)|}{(\alpha + 1)(\alpha + 2)} \left( 1 - \frac{1}{2^{\alpha+1}} \right) + m_2 |f'(b)| \left( \frac{1}{4} - \frac{1}{(\alpha + 1)(\alpha + 2)} \left( 1 - \frac{1}{2^{\alpha+1}} \right) \right) \right]$$

References


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