AN ALMOST ORTHOSYMMETRIC BILINEAR MAP

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Abstract. In this paper, as a generalization of the concept of pseudo-almost $f$-algebra, we define a new concept of almost orthosymmetric bilinear map on a vector lattice and prove that the Arens triadjoint of a positive almost orthosymmetric bilinear map is positive almost orthosymmetric. This also extends results on the order bidual of pseudo-almost $f$-algebras.

1. Introduction

We studied in [16] a new class of pseudo-almost $f$-algebra (a lattice ordered algebra $A$ in which $a \land b = 0$ in $A$ implies $ab \land ba = 0$) and presented its relation with the certain lattice ordered algebras; $f$-algebras [5], almost $f$-algebras [6] and $d$-algebras [12].

In [17], concentrating on the Arens multiplications [2, 3] in the algebraic bidual of pseudo-almost $f$-algebras (so-called $r$-algebra in [17]), we prove that the order continuous bidual of an Archimedean pseudo-almost $f$-algebra is again a Dedekind complete (and hence Archimedean) pseudo-almost $f$-algebra. This is a generalization of a result of Bernau and Huijsmans in [4] in which they prove that the order continuous bidual of an almost $f$-algebra (respectively $d$-algebra) is again an almost $f$-algebra (respectively $d$-algebra).

In this paper, as an extension of the notion of pseudo-almost $f$-algebra, we introduce a new concept of almost orthosymmetric bilinear map and prove that if $A, B$ are vector lattices and $T : A \times A \to B$ is a positive almost orthosymmetric bilinear map, then the triadjoint $T^{***} : (A')' \times (A')' \to (B')'$ is a positive almost orthosymmetric bilinear map. This also generalizes results on the order bidual of pseudo-almost $f$-algebras in [17].

The Arens multiplication introduced in [3] on the bidual of various lattice ordered algebras has been well documented (see, e.g., [4]). The more general question about Arens triadjoints of bilinear maps on products of vector lattices has recently aroused

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considerable interest (see, e.g., [7]). In this direction, as the extensions of the notions of classes of almost f-algebra, f-algebra, d-algebra and pseudo-almost f-algebra, we have studied the Arens triadjoints of some classes of bilinear maps on vector lattices; mainly, orthosymmetric bilinear maps, bi-orthomorphisms, d-bimorphisms and almost orthomorphism bilinear maps (see [15] [14]):

**Definition 1.** Let A and B be vector lattices. A bilinear map \( T : A \times A \to B \) is said to be

1. **orthosymmetric** if \( x \wedge y = 0 \) implies \( T(x, y) = 0 \) for all \( x, y \in A \) (first appeared in a paper by G. Buskes and A. van Rooij in [9] in 2000).
2. **a bi-orthomorphism** if it is a separately order bounded bilinear map such that \( x \wedge y = 0 \) in \( A \) implies \( T(z, x) \wedge y = 0 \) for all \( z \in A^+ \), when \( A = B \) (first appears in a paper by G. Buskes, R. Page Jr and R. Yilmaz in [10] in 2009).
3. **a d-bimorphism** if \( x \wedge y = 0 \) in \( A \) implies \( T(z, x) \wedge T(z, y) = 0 \) for all \( z \in A^+ \) (first appears in a paper R. Yilmaz in [14] in 2017).
4. **almost orthosymmetric** if \( x \wedge y = 0 \) implies \( T(x, y) \wedge T(y, x) = 0 \) for all \( x, y \in A \).

The following theorem is obvious from the above definitions.

**Theorem 2.**

1. Every bi-orthomorphism is both orthosymmetric and a d-bimorphism.
2. Every orthosymmetric bilinear map is almost orthosymmetric.

From here on, let \( A, B, C \) be Archimedean vector lattices and \( A', B', C' \) be their respective duals.

A bilinear map \( T : A \times B \to C \) can be extended in a natural way to the bilinear map \( T^{**} : A'' \times B'' \to C'' \) constructed in the following stages:

\[
T^* : C' \times A \to B', \quad T^*(f, x)(y) = f(T(x, y))
\]
\[
T^{**} : B'' \times C' \to A', \quad T^{**}(G, f)(x) = G(T^*(f, x))
\]
\[
T^{***} : A'' \times B'' \to C'', \quad T^{***}(F, G)(f) = F(T^{**}(G, f))
\]

for all \( x \in A, y \in B, f \in C', F \in A', G \in B' \) (so-called the first Arens adjoint of \( T \)).

Another extension of a bilinear map \( T : A \times B \to C \) is the map \( **T : A'' \times B'' \to C'' \) constructed in the following stages:

\[
**T : C' \times A'' \to B', \quad **T(f, F)(y) = F(**T(y, f))
\]
\[
***T : A'' \times B'' \to C'', \quad ***T(F, G)(f) = G(**T(f, F))
\]

for all \( x \in A, y \in B, f \in C', F \in A'', G \in B'' \) (so-called the second Arens adjoint of \( T \)) [3].

In this work we shall concentrate on the first Arens adjoint: that is, we prove that \( T^{***} : (A')_n' \times (A')_n' \to (B')_n' \) is positive almost orthosymmetric whenever \( T : A \times A \to B \) is so. Similar results hold for the second.

For the elementary theory of vector lattices and terminology not explained here we refer to [1] [13] [18].
2. The triadjoint of an almost orthosymmetric bilinear map

In this section we prove that the extension $T^{***}$ of a positive almost orthosymmetric bilinear map $T : A \times A \to B$ is again positive almost orthosymmetric. We first recall some relevant notions. The canonical image $\widehat{a}$ of a vector lattice $A$ into its order bidual $A''$ is defined by $\widehat{a}(f) = f(a)$ for all $f \in A'$. For each $a \in A$, $\widehat{a}$ defines an order continuous algebraic lattice homomorphism on $A'$ and the canonical image $\widehat{A}$ of $A$ is a subalgebra of $(A')'_c$. Moreover the band

$$I_A = \{ F \in (A')'_c : |F| \leq \widehat{x} \text{ for some } x \in A^+ \}$$

generated by $\widehat{A}$ is order dense in $(A')'_c$; i.e., for each $F \in (A')'_c$, there exists an upwards directed net $\{G_\lambda : \lambda \in \Lambda\}$ in $I_\widehat{A}$ such that $0 < G_\lambda \uparrow F$.

A bilinear operator $T : A \times B \to C$ is said to be order bounded if for all $(x, y) \in A^+ \times B^+$ we have

$${\{T(a, b) : 0 \leq a \leq x, 0 \leq b \leq y\}}$$

is order bounded. $T$ is positive if for all $x \in A^+$ and $y \in B^+$ we have $T(x, y) \in C^+$. Clearly every positive bilinear map is order bounded. Moreover if $T$ is positive, then so is $T^*$.

Let $0 \leq f \in B'$ and $x \in A^+$. Then the positive linear functional $^*T(x, f)$ in $A'$ defined by, for all $y \in A$,

$$^*T(x, f)(y) = f(T(y, x))$$

satisfies

$$T^{**}(\widehat{x}, f) = ^* T(x, f).$$

Indeed, for all $y \in A$,

$$T^{**}(\widehat{x}, f)(y) = \widehat{x}(T^*(f, y)) = T^*(f, y)(x) = f(T(y, x)) = ^* T(x, f)(y).$$

**Proposition 3.** Let $A, B$ be vector lattices and $T : A \times A \to B$ be a positive almost orthosymmetric bilinear map. If $x \in A^+$ and $0 \leq G, H \in (A')'_c$ satisfy $G, H \leq \widehat{x}$ and $G \wedge H = 0$, then $T^{**}(G, H) \wedge T^{**}(H, G) = 0$.

**Proof.** Let $T$ be positive almost orthosymmetric. Then clearly $T^{**}$ is positive.

Let $0 \leq f \in B'$ and $x \in A^+$. Then $0 \leq ^*T(x, f) + T^*(f, x) \in A'$, and so, by Corollary 1.2 of [1], there exist $g, h \in A'$ with $g \wedge h = 0$, and $G(g) = 0 = H(h)$ such that

$$^*T(x, f) + T^*(f, x) = g + h.$$ 

By the Riesz-Kontorovič Theorem ([1] Theorem 1.13]),

$$\inf\{g(y) + h(z) : x = y + z, y, z \in A^+\} = (g \wedge h)(x) = 0,$$

which implies that, for $\epsilon > 0$, there exist $y, z \in A^+$ such that $x = y + z$ and $g(y) < \epsilon$ and $h(z) < \epsilon$.

We now define the linear functionals $G_1$ and $H_1$ on $A'$ by

$$G_1 = G \wedge (y - y \wedge z) \quad \text{and} \quad H_1 = H \wedge (z - y \wedge z).$$
Clearly, \( 0 \leq G_1, H_1 \in (A')_p \) and the following inequalities hold.
\[
0 \leq H - H_1 = (H - (z - y \land z))^+ \leq (\bar{x} - (z - y \land z))^+
= (y + z - (z - y \land z))^+ = (y + y \land z)^+ \leq 2\bar{y},
\]
(1)
and similarly
\[
0 \leq G - G_1 \leq 2\bar{z}.
\]
(2)
Since \( T^{***} \) is positive and \( T^{***}(\hat{a}, \hat{b}) = \hat{T}(a, b) \) for all \( a, b \in A \), it follows that
\[
0 \leq T^{***}(G_1, H_1) \land T^{***}(H_1, G_1)
\leq T^{***}(y - y \land z, z - y \land z) \land T^{***}(z - y \land z, y - y \land z) = 0;
\]
i.e., \( T^{***}(G_1, H_1) \land T^{***}(H_1, G_1) = 0. \)
(3)
We next consider the elements
\[
0 \leq T^{***}(G - G_1, H), T^{***}(G_1, H - H_1), T^{***}(H - H_1, G), T^{***}(H_1, G - G_1)
\]
of \( (A')_p \). Then, by the positivity of \( T^{***} \) and [1],
\[
T^{***}(G - G_1, H)(f) \leq T^{***}(G - G_1, \bar{x})(f) = (G - G_1)(T^{**}(\bar{x}, f))
= (G - G_1)(T(x, f)) \leq (G - G_1)(T(x, f) + T^*(f, x))
= (G - G_1)(g + h) = (G - G_1)(g) + (G - G_1)(h)
\leq G(g) + (G - G_1)(h) \leq 0 + 2\bar{z}(h) = 2h(z)
\]
(4)
and, by [2],
\[
T^{***}(G_1, H - H_1)(f) \leq T^{***}(G, H - H_1)(f) \leq T^{***}(\bar{x}, H - H_1)(f)
= \bar{x}(T^{**}(H - H_1, f)) = T^{**}(H - H_1, f)(x)
= (H - H_1)(T^*(f, x))
\leq (H - H_1)(T^*(f, x) + T(x, f)) = (H - H_1)(g + h)
\leq (H - H_1)(g) + (H - H_1)(h) \leq H(g) + (H - H_1)(h)
\leq 0 + 2\bar{y}(g) = 2g(y).
\]
(5)
It follows by symmetry that
\[
T^{***}(H - H_1, G)(f) \leq 2g(y) \quad \text{and} \quad T^{***}(H_1, G - G_1)(f) \leq 2h(z).
\]
(6)
Using the fact that \((a + b) \land c \leq a \land c + b \land c \leq a + b \land c\) in vector lattices and [3], we find
\[
T^{***}(G, H) \land T^{***}(H, G) = (T^{***}(G - G_1, H) + T^{***}(G_1, H - H_1) + T^{***}(G_1, H_1))
\land (T^{***}(H - H_1, G) + T^{***}(H_1, G - G_1) + T^{***}(H_1, G_1))
\leq T^{***}(G - G_1, H) + T^{***}(G_1, H - H_1)
+ T^{***}(G, H_1) \land (T^{***}(H - H_1, G) + T^{***}(G_1, G - G_1)
+ T^{***}(H_1, G_1))
\leq T^{***}(G - G_1, H) + T^{***}(G_1, H - H_1)
\]
(7)
for all $H$, order continuity of

$$+T^{**}(G_1, H_1) \wedge T^{**}(H - H_1, G) + T^{**}(H_1, G - G_1)$$

$$+T^{**}(G_1, H_1) \wedge T^{**}(H_1, G_1)$$

$$\leq T^{**}(G - G_1, H) + T^{**}(G_1, H - H_1)$$

$$+T^{**}(H - H_1, G) + T^{**}(H_1, G - G_1).$$

Hence, by (4), (5) and (6),

$$0 \leq T^{**}(G, H) \wedge T^{**}(H, G)(f) \leq T^{**}(G - G_1, H)(f) + T^{**}(G_1, H - H_1)(f)$$

$$+T^{**}(H - H_1, G)(f) + T^{**}(H_1, G - G_1)(f)$$

$$\leq 2h(z) + 2g(y) + 2h(z) \leq 8\epsilon.$$

Since this holds for an arbitrary $\epsilon > 0$, we have $T^{**}(G, H) \wedge T^{**}(H, G)(f) = 0$ for all $0 \leq f \in B'$. It now follows that for all $f \in B'$


$$-T^{**}(G, H) \wedge T^{**}(H, G)(f^-)$$

$$= 0,$$

and so $T^{**}(G, H) \wedge T^{**}(H, G) = 0$, as required. \hfill \Box

We are in a position to prove the main result of this paper.

**Theorem 4.** Let $A, B$ be vector lattices and $T : A \times A \rightarrow B$ be a positive almost orthosymmetric bilinear map. Then the bilinear map $T^{**} : (A')_n \times (A')_n \rightarrow (B')_n$ is positive almost orthosymmetric.

**Proof.** In the preceding proposition we have proved that the restriction map $T^{**}|_{I_\delta \times I_\delta}$ is positive almost orthosymmetric whenever $T : A \times A \rightarrow B$ is so. We now extend the result to the whole $(A')_n \times (A')_n$. To do this, let $0 \leq G, H \in (A')_n$ such that $G \wedge H = 0$. We have to show that $T^{**}(G, H) \wedge T^{**}(H, G)(f) = 0$.

Since the band $I_\delta$ is order dense in $(A')_n$, there exist $G_\alpha, H_\beta \in I_\delta$ such that $0 \leq G_\alpha \uparrow G$ and $0 \leq H_\beta \uparrow H$ with $0 \leq G_\alpha \leq \hat{x}_\alpha$ and $0 \leq H_\beta \leq \hat{y}_\beta$ for some $x_\alpha, y_\beta \in A^+$. It follows from $G \wedge H = 0$ that $G_\alpha \wedge H_\beta = 0$ for all $\alpha, \beta$. Furthermore, $0 \leq G_\alpha, H_\beta \leq x_\alpha + y_\beta$. Hence, by above, we see that

$$T^{**}(G_\alpha, H_\beta) \wedge T^{**}(H_\beta, G_\alpha) = 0$$

(7)

for all $\alpha$ and $\beta$. Now let $0 \leq f \in B'$. It follows from $0 \leq H_\beta \uparrow H$ that $0 \leq H_\beta(T^{**}(f, x)) \uparrow H(T^{**}(f, x))$;

i.e., $0 \leq T^{**}(H_\beta, f)(x) \uparrow T^{**}(H, f)(x)$

for all $0 \leq x \in A$. This shows that $0 \leq T^{**}(H_\beta, f) \uparrow T^{**}(H, f)$. Hence, by the order continuity of $G_\alpha$ for each $\alpha$, $0 \leq G_\alpha(T^{**}(H_\beta, f)) \uparrow G_\alpha(T^{**}(H, f))$;

i.e., $0 \leq T^{**}(G_\alpha, H_\beta)(f) \uparrow T^{**}(G_\alpha, H)(f)$

which implies that, for each $\alpha$,

$$0 \leq T^{**}(G_\alpha, H_\beta) \uparrow T^{**}(G_\alpha, H).$$

(8)
Similarly, since $0 \leq G_\alpha \uparrow G$, we have $0 \leq G_\alpha(T^*(H, f)) \uparrow G(T^*(H, f))$; i.e., $0 \leq T^{***}(G_\alpha, H)(f) \uparrow T^{***}(G, H)(f)$ for all $0 \leq f \in B'$, and so
\begin{equation}
0 \leq T^{***}(G_\alpha, H) \uparrow T^{***}(G, H) \tag{9}
\end{equation}
In the same way, by the order continuity of $H_\beta$ for each $\beta$, we obtain
\begin{equation}
0 \leq T^{***}(H_\beta, G_\alpha) \uparrow T^{***}(H_\beta, G) \tag{10}
\end{equation}
leading to
\begin{equation}
0 \leq T^{***}(H_\beta, G_\alpha) \uparrow T^{***}(H, G). \tag{11}
\end{equation}
Now it follows from (9) and (10) that
\begin{equation}
0 \leq T^{***}(G_\alpha, H_\beta) \wedge T^{***}(H_\beta, G_\alpha) \uparrow T^{***}(G_\alpha, H) \wedge T^{***}(H_\beta, G), \tag{12}
\end{equation}
and so, by (7),
\begin{equation}
T^{***}(G_\alpha, H) \wedge T^{***}(H_\beta, G) = 0
\end{equation}
for all $\alpha, \beta$. On the other hand, from (9) and (11) we have
\begin{equation}
0 \leq T^{***}(G_\alpha, H) \wedge T^{***}(H_\beta, G) \uparrow T^{***}(G, H) \wedge T^{***}(H, G).
\end{equation}
It follows from (12) that
\begin{equation}
T^{***}(G, H) \wedge T^{***}(H, G) = 0,
\end{equation}
as required. \qed

As the Arens multiplications are separately order continuous and in a commutative algebra a pseudo-almost $f$-algebra and almost $f$-algebra coincide, we immediately obtain the following corollary.

**Corollary 5.**

1. The order continuous bidual of a pseudo-almost $f$-algebra is a Dedekind complete (and hence Archimedean) pseudo-almost $f$-algebra.
2. The order bidual of a commutative pseudo-almost $f$-algebra is a Dedekind complete pseudo-almost $f$-algebra.

Another way of obtaining the result of Proposition 3 is by means of the approximation by components ([11]). First we observe some notations: Let $A$ be a vector lattice and let $a$ be a fixed element of $A$. If $E := \{F \in (A')_+ : \exists \lambda > 0, |F| \leq \lambda \tilde{a}\}$-the ideal generated in $(A')_+$ by $\tilde{a}$. Consider the Boolean algebra $\mathcal{R}$ generated by the set of all band projections of $E$ onto principal bands generated by positive elements of $\tilde{A}$ in $E$. If we denote the band projection onto the band generated in $E$ by the element $F \in E$ by $P_F$, then $\mathcal{R}$ is generated by the set $\mathcal{G} := \{P_x : x \in A^+\}$-the set of all band projections onto the principal ideals generated by elements $\tilde{x}$ with $x \in A^+$. Also, $\tilde{G}\tilde{a} := \{P_x\tilde{a} : x \in A^+\}$. 

Proposition 6. Let $A, B$ be vector lattices and $T : A \times A \to B$ be a positive almost orthosymmetric bilinear map. If $x \in A^+$ and $0 \leq G, H \in (A^+)_n$ satisfy $G, H \leq \hat{x}$ and $G \wedge H = 0$ (that is, $G$ and $H$ are two disjoint elements of the band $I_\hat{A} = \{F \in (A^+)_n : |F| \leq \hat{x} \text{ for some } x \in A^+\}$) generated by $\hat{A}$, which is order dense in $(A^+)_n$, then $T^{**}(G, H) \wedge T^{**}(H, G) = 0$.

Proof. It is sufficient to prove that $T^{**}(P_C \hat{x}, P_H \hat{x}) \wedge T^{**}(P_H \hat{x}, P_G \hat{x}) = 0$ since $0 \leq G \leq P_G \hat{x}$ and $0 \leq H \leq P_H \hat{x}$. (Note that, as band projections are positive, $0 \leq G \wedge H = P_G G \wedge P_H H \leq P_G \hat{x} \wedge P_H \hat{x}$, and so $P_G \hat{x} \wedge P_H \hat{x} = 0$ implies $G \wedge H = 0$.) Hence $T^{**}(G, H) \wedge T^{**}(H, G) \leq T^{**}(P_C \hat{x}, P_H \hat{x}) \wedge T^{**}(P_H \hat{x}, P_G \hat{x})$ by the positivity of $T^{**}$.) But, to do this, it is sufficient to prove that $T^{**}((\hat{x} - F, F) \wedge T^{**}(F, \hat{x} - F) = 0$ for any component $F$ of $\hat{x}$; that is, $\hat{x} - F \wedge F = 0$.

The proof of this is in four steps, as follows.

Step 1. Let $F \in \mathcal{G}_\hat{a}$, say $F = P_{\hat{x}} \hat{x} = \sup_n (n\hat{a} \wedge \hat{x})$. Then it follows from
\[
\hat{x} - F = \hat{x} - \sup_n (n\hat{a} \wedge \hat{x}) = \inf_n (\hat{x} - n\hat{a} \wedge \hat{x}) = \inf_n (\hat{x} - n\hat{a})^+
\]
and that for each fixed $n$
\[
0 \leq T^{**}((\hat{x} - F, (n\hat{a} - \hat{x})^+) \wedge T^{**}((n\hat{a} - \hat{x})^+, \hat{x} - F)
\]
\[
\leq T^{**}((\hat{x} - n\hat{a})^+, (n\hat{a} - \hat{x})^+) \wedge T^{**}((\hat{x} - n\hat{a})^+, (n\hat{a} - \hat{x})^+)
\]
\[
= T((x - na)^+, (na - x)^+) \wedge T((x - na)^+, (na - x)^+)
\]
\[
= T((x - na)^+, (na - x)^+) \wedge T((x - na)^+, (na - x)^+)
\]
\[
= 0,
\]
as $(x - na)^+ \wedge (na - x)^+ = 0$ and $T$ is almost orthosymmetric (where we use the fact that $T^{**}(a, b) = T(a, b)$ for all $a, b \in A$). Hence
\[
T^{**}((\hat{x} - F, (n\hat{a} - \hat{x})^+) \wedge T^{**}((n\hat{a} - \hat{x})^+, \hat{x} - F) = 0,
\]
and so
\[
n(T^{**}((\hat{x} - F, (n\hat{a} - \hat{x})^+) \wedge T^{**}((n\hat{a} - \hat{x})^+, \hat{x} - F))) = 0.
\]
This implies that for each $n$
\[
T^{**}((\hat{x} - F, (\hat{a} - 1/n\hat{x})^+) \wedge T^{**}((\hat{a} - 1/n\hat{x})^+, \hat{x} - F) = 0.
\]
Therefore
\[
T^{**}((\hat{x} - F, \hat{a}) \wedge T^{**}(\hat{a}, \hat{x} - F) = 0, \text{ as } n \to \infty.
\]
It follows that for each $n$
\[
n(T^{**}((\hat{x} - F, \hat{a}) \wedge T^{**}(\hat{a}, \hat{x} - F))) = 0; \text{ i.e., } T^{**}((\hat{x} - F, n\hat{a}) \wedge T^{**}(n\hat{a}, \hat{x} - F) = 0.
\]
Hence,
\[
0 \leq T^{**}((\hat{x} - F, n\hat{a}) \wedge T^{**}(n\hat{a}, \hat{x} - F) \leq T^{**}((\hat{x} - F, n\hat{a}) \wedge T^{**}(n\hat{a}, \hat{x} - F) = 0;
\]
\[
i.e., T^{**}((\hat{x} - F, n\hat{a}) \wedge T^{**}(n\hat{a}, \hat{x} - F) = 0.
\]
Since this holds for each \( n \), we get
\[
\sup_n (T^{**}(\tilde{x} - F, n\hat{a} \land \tilde{x}) \land T^{**}(n\hat{a} \land \tilde{x}, \tilde{x} - F)) = 0,
\]
which leads that, by the separately order continuity of \( T^{**} \) (since \( T \) is positive, \( T \) is of order bounded variation, and so \( T^{**} \) is separately order continuous (see e.g. Theorem 2.1 in [7])),
\[
0 \leq T^{**}(\tilde{x} - F, F) \land T^{**}(F, \tilde{x} - F)
\]
\[
= T^{**}(\tilde{x} - F, \sup_n (n\hat{a} \land \tilde{x})) \land T^{**}(\sup_n (n\hat{a} \land \tilde{x}), \tilde{x} - F)
\]
\[
= \sup_n (T^{**}(\tilde{x} - F, n\hat{a} \land \tilde{x})) \land \sup_n (T^{**}(n\hat{a} \land \tilde{x}), \tilde{x} - F))
\]
\[
= \sup_n (T^{**}(\tilde{x} - F, n\hat{a} \land \tilde{x}) \land T^{**}(n\hat{a} \land \tilde{x}), \tilde{x} - F))
\]
\[
= 0;
\]
i.e., \( T^{**}(\tilde{x} - F, F) \land T^{**}(F, \tilde{x} - F) = 0. \)

Step 2. Let \( F = \bigwedge_{i=1}^m F_i \) where either \( F_i \in \mathcal{G}\hat{a} \) or \( \tilde{x} - F_i \in \mathcal{G}\hat{a} \). Then
\[
\tilde{x} - F = \bigvee_{i=1}^m (\tilde{x} - F_i),
\]
and so
\[
0 \leq T^{**}(\tilde{x} - F, F) \land T^{**}(F, \tilde{x} - F)
\]
\[
= T^{**}(\bigvee_{i=1}^m (\tilde{x} - F_i), \bigwedge_{i=1}^m F_i) \land T^{**}(\bigwedge_{i=1}^m F_i, \bigvee_{i=1}^m (\tilde{x} - F_i))
\]
\[
\leq T^{**}(\sum_{i=1}^m (\tilde{x} - F_i), F_i) \land T^{**}(F_i, \sum_{i=1}^m (\tilde{x} - F_i))
\]
\[
= \sum_{i=1}^m T^{**}((\tilde{x} - F_i), F_i) \land \sum_{i=1}^m T^{**}(F_i, \tilde{x} - F_i)
\]
\[
\leq \sum_{i=1}^m (T^{**}((\tilde{x} - F_i), F_i) \land T^{**}(F_i, \tilde{x} - F_i))
\]
\[
= 0 \quad \text{(by Step 1)};
\]
i.e., \( T^{**}(\tilde{x} - F, F) \land T^{**}(F, \tilde{x} - F) = 0. \)

Step 3. Let \( F = \bigvee_{i=1}^n F_i \) where each \( F_i \) is of the form \( F \) had in Step 1 (that is, \( F_i = \bigwedge_{j=1}^m F_{ij}, \forall i = 1, 2, \ldots, n \), and so \( F = \bigvee_{i=1}^n \bigwedge_{j=1}^m F_{ij} \)). Then, in the same way as Step 2,
\[
\tilde{x} - F = \bigwedge_{i=1}^m (\tilde{x} - F_i),
\]
and so

\[ 0 \leq T^{**}(\hat{x} - F, F) \land T^{**}(F, \hat{x} - F) \]

\[ = T^{**}(\bigcap_{i=1}^{m} (\hat{x} - F_i), \bigvee_{i=1}^{m} F_i) \land T^{**}(\hat{x} - F_i, \bigvee_{i=1}^{m} F_i) \]

\[ \leq T^{**}(\hat{x} - F_i, \bigvee_{i=1}^{m} F_i) \land T^{**}(\hat{x} - F_i, \bigvee_{i=1}^{m} F_i) \]

\[ \leq T^{**}(\hat{x} - F_i, \sum_{i=1}^{m} F_i) \land T^{**}(\sum_{i=1}^{m} F_i, \hat{x} - F_i) \]

\[ = \sum_{i=1}^{m} (T^{**}(\hat{x} - F_i, F_i)) \land \sum_{i=1}^{m} (T^{**}(F_i, \hat{x} - F_i)) \]

\[ \leq \sum_{i=1}^{m} (T^{**}(\hat{x} - F_i, F_i) \land T^{**}(F_i, \hat{x} - F_i)) \]

\[ = 0 \quad \text{(by Step 2)}; \]

i.e., \( T^{**}(\hat{x} - F, F) \land T^{**}(F, \hat{x} - F) = 0. \)

**Step 4.** Let \( F \in R\hat{x}. \) If \( F = \sup_{\alpha} F_{\alpha} \) or \( F = \inf_{\alpha} F_{\alpha} \) with each \( F_{\alpha} \) is a component of \( \hat{x} \) (that is, \( (\hat{x} - F_{\alpha}) \land F_{\alpha} = 0 \) for each \( \alpha \)) having the property that

\[ T^{**}(\hat{x} - F_{\alpha}, F_{\alpha}) \land T^{**}(F_{\alpha}, \hat{x} - F_{\alpha}) = 0, \]

then using the separately order continuity of \( T^{**} \) we show that \( F \) has the same property;

i.e., \( T^{**}(\hat{x} - F, F) \land T^{**}(F, \hat{x} - F) = 0. \)

Indeed, suppose that \( F = \sup_{\alpha} F_{\alpha} \). For each fixed \( \alpha \) and for all \( \beta \geq \alpha \) we have \( F_{\beta} \geq F_{\alpha} \), and so \( \hat{x} - F_{\beta} \leq \hat{x} - F_{\alpha} \). Hence, by the positivity of \( T^{**} \) and the hypothesis,

\[ 0 \leq T^{**}(\hat{x} - F_{\beta}, F_{\alpha}) \land T^{**}(F_{\alpha}, \hat{x} - F_{\beta}) \leq T^{**}(\hat{x} - F_{\alpha}, F_{\alpha}) \land T^{**}(F_{\alpha}, \hat{x} - F_{\alpha}) = 0; \]

i.e., \( T^{**}(\hat{x} - F_{\beta}, F_{\alpha}) \land T^{**}(F_{\alpha}, \hat{x} - F_{\beta}) = 0 \quad \forall \beta \geq \alpha. \)

Therefore

\[ \inf_{\beta \geq \alpha} (T^{**}(\hat{x} - F_{\beta}, F_{\alpha}) \land T^{**}(F_{\alpha}, \hat{x} - F_{\beta})) = 0, \]

and so, by the order continuity of lattice operations (\( x_{\tau} \downarrow x \) and \( y_{\tau} \downarrow y \) implies \( x_{\tau} \land y_{\tau} \downarrow x \land y \)),

\[ \inf_{\beta \geq \alpha} T^{**}(\hat{x} - F_{\beta}, F_{\alpha}) \land \inf_{\beta \geq \alpha} T^{**}(F_{\alpha}, \hat{x} - F_{\beta}) = 0. \]

Since \( T^{**} \) is a separately order continuous,

\[ T^{**}(\inf_{\beta \geq \alpha} (\hat{x} - F_{\beta}), F_{\alpha}) \land T^{**}(F_{\alpha}, \inf_{\beta \geq \alpha} (\hat{x} - F_{\beta})) = 0; \]

i.e., \( T^{**}(\hat{x} - F, F_{\alpha}) \land T^{**}(F_{\alpha}, \hat{x} - F) = 0. \)
Since this holds for all $\alpha$, 
$$\sup_{\alpha}(T^{**}(\hat{x} - F, F_\alpha) \land T^{**}(F_\alpha, \hat{x} - F)) = 0,$$
from which it follows that 
$$T^{**}(\hat{x} - F, F) \land T^{**}(F, \hat{x} - F) = 0,$$
by the order continuity of lattice operations (if $x\uparrow x$ and $y\uparrow y$, then $x\land y\uparrow x \land y$), as above.

In exactly the same way above we now show that if $F = \inf_\alpha F_\alpha$ such that 
$$(\hat{x} - F_\alpha) \land F_\alpha = 0$$
and 
$$T^{**}(\hat{x} - F_\alpha, F_\alpha) \land T^{**}(F_\alpha, \hat{x} - F_\alpha) = 0$$
for each $\alpha$, then 
$$T^{**}(\hat{x} - F, F) \land T^{**}(F, \hat{x} - F) = 0.$$

Let $\alpha$ be fixed. Then we have $F_\beta \leq F_\alpha$ for all $\beta \geq \alpha$. Hence, by the positivity of $T^{**}$ and the hypothesis,
$$0 \leq T^{**}(\hat{x} - F_\alpha, F_\beta) \land T^{**}(F_\beta, \hat{x} - F_\alpha) \leq T^{**}(\hat{x} - F_\alpha, F_\alpha) \land T^{**}(F_\alpha, \hat{x} - F_\alpha) = 0;$$
i.e., 
$$T^{**}(\hat{x} - F_\alpha, F_\alpha) \land T^{**}(F_\alpha, \hat{x} - F_\alpha) = 0,$$
$\forall \beta \geq \alpha$.
Therefore 
$$\inf_{\beta \geq \alpha}(T^{**}(\hat{x} - F_\alpha, F_\beta) \land T^{**}(F_\beta, \hat{x} - F_\alpha)) = 0,$$
and so,
$$\inf_{\beta \geq \alpha} T^{**}(\hat{x} - F_\alpha, F_\beta) \land \inf_{\beta \geq \alpha} T^{**}(F_\beta, \hat{x} - F_\alpha) = 0.$$
Since $T^{**}$ is a separately order continuous,
$$T^{**}(\hat{x} - F_\alpha, \inf_{\beta \geq \alpha} F_\beta) \land T^{**}(\inf_{\beta \geq \alpha} F_\beta, \hat{x} - F_\alpha) = 0.$$
i.e., 
$$T^{**}(\hat{x} - F_\alpha, F) \land T^{**}(F, \hat{x} - F_\alpha) = 0.$$
Since this holds for all $\alpha$, we get 
$$\sup_{\alpha}(T^{**}(\hat{x} - F_\alpha, F) \land T^{**}(F, \hat{x} - F_\alpha)) = 0.$$
Therefore 
$$T^{**}(\hat{x} - F, F) \land T^{**}(F, \hat{x} - F) = 0,$$
from which the result follows.

We conclude our work with the following important remark for further research.

Remark. The triadjoints on the whole order biduals is still an open problem. One has to obtain a way to handle the singular parts of order biduals, as the cases of orthosymmetric bilinear maps and bi-orthomorphisms [15], in order to prove that the triadjoint $T^{**} : A'' \times A'' \rightarrow B''$ of an almost orthosymmetric bilinear map $T : A \times A \rightarrow B$ is an almost orthosymmetric bilinear map.

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References


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